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Solutions

1. *Find all positive integers n such that $17^{n-1}+19^{n-1}$ divides 17^n+19^n.*

Solution. We have that $17(17^{n-1}+19^{n-1}) < 17^n+19^n < 19(17^{n-1}+19^{n-1})$ as can be easily checked. Since $17^{n-1}+19^{n-1}$ divides 17^n+19^n, then $17^n+19^n = 18(17^{n-1}+19^{n-1}) = (17+1)17^{n-1}+(19-1)19^{n-1} = 17^n+17^{n-1}+19^n-19^{n-1}$ from which follows $17^{n-1}=19^{n-1}$.The preceding is only possible for $n=1$ for which $17^{n-1}+19^{n-1}=2$ that divides $17^n+19^n=36$, and we are done. $\square$

2. *Prove that*

$$\sum_{k=1}^{n}\sqrt{\frac{k-\sqrt{k^2-1}}{\sqrt{k(k+1)}}} \le \sqrt[4]{\frac{n^3}{n+1}},$$

where n is a positive integer.

Solution. Squaring both sides, yields

$$\left(\sum_{k=1}^{n}\sqrt{\frac{k-\sqrt{k^2-1}}{\sqrt{k(k+1)}}}\right)^2 \le n\sqrt{\frac{n}{n+1}}$$

Applying CBS inequality to the vectors $\vec{u}=(1,1,\ldots,1)$ and

$$\vec{v}=\left(\sqrt{\frac{1}{\sqrt{1\cdot 2}}},\sqrt{\frac{2-\sqrt{3}}{\sqrt{2\cdot 3}}},\ldots,\sqrt{\frac{n-\sqrt{n^2-1}}{\sqrt{n(n+1)}}}\right),$$

we get

$$\left(\sum_{k=1}^{n}1\cdot\sqrt{\frac{k-\sqrt{k^2-1}}{\sqrt{k(k+1)}}}\right)^2 \le \left(\sum_{k=1}^{n}1\right)^2\left(\sum_{k=1}^{n}\frac{k-\sqrt{k-1}\sqrt{k+1}}{\sqrt{k}\sqrt{k+1}}\right)$$
$$= n\sum_{k=1}^{n}\left(\sqrt{\frac{k}{k+1}}-\sqrt{\frac{k-1}{k}}\right) = n\sqrt{\frac{n}{n+1}}$$

and we are done.

$\square$

3. *Find all functions $f:\mathbb{Q}^+\to\mathbb{Q}^+$ such that for all $x,y\in\mathbb{Q}^+$, satisfy*

$$y=\frac{1}{2}\left[f\left(x+\frac{y}{x}\right)-\left(f(x)+\frac{f(y)}{f(x)}\right)\right],$$

where $\mathbb{Q}^+$ represents the set of positive rational numbers.

Solution. Rearranging terms the equation claimed may be written in the form

$$f\left(x+\frac{y}{x}\right)=f(x)+\frac{f(y)}{f(x)}+2y$$

Setting $(x,y)=(1,1),(x,y)=(1,2)$ and $(x,y)=(2,2)$, we obtain $f(2)=f(1)+3$, $f(3)=f(1)+\dfrac{f(2)}{f(1)}+4$ and $f(3)=f(2)+5$, respectively. Form the preceding we obtain $f(1)=1, f(2)=4$ and $f(3)=9$. Now we claim that for all positive integer n is $f(n)=n^2$. Indeed, putting $x=y=n$ we get $f(n+1)=f(n)+2n+1$ which is satisfied when $f(n)=n^2$. It can be easily proven using mathematical induction. Now, we assume that the only solution to our functional equation is $f(x)=x^2$ for all $x\in\mathbb{Q}^+$. To prove it we consider the pairs $(x,y)=(n,m)$ and $(x,y)=(m/n,m)$ where $n,m\in\mathbb{N}$. We get

$$\begin{aligned} f\left(n+\frac{m}{n}\right) &= f(n)+\frac{f(m)}{f(n)}+2m=n^2+\frac{m^2}{n^2}+2m,\\ f\left(\frac{m}{n}+m\right) &= f\left(\frac{m}{n}\right)+\frac{f(m)}{f(m/n)}+2m=f\left(\frac{m}{n}\right)+\frac{m^2}{f(m/n)}+2m. \end{aligned}$$

From the preceding immediately follows

$$\begin{aligned} f\left(\frac{m}{n}\right)-\frac{m^2}{n^2}-n^2+\frac{m^2}{f(m/n)} &= f\left(\frac{m}{n}\right)-\frac{m^2}{n^2}-\frac{n^2}{f(m/n)}\left(f\left(\frac{m}{n}\right)-\frac{m^2}{n^2}\right)\\ &=\left(f\left(\frac{m}{n}\right)-\frac{m^2}{n^2}\right)\left(1-\frac{n^2}{f(m/n)}\right)=0 \end{aligned}$$

Let $\dfrac{a}{b}\in\mathbb{Q}^+$, $(a,b)=1$. Then, we distinguish two cases:

- If $1-\dfrac{b^2}{f(a/b)}\neq 0$, then $f\left(\dfrac{a}{b}\right)-\dfrac{a^2}{b^2}=0$. That is, $f\left(\dfrac{a}{b}\right)=\left(\dfrac{a}{b}\right)^2$.
- If $1-\dfrac{b^2}{f(a/b)}=0$, then $\dfrac{f(2b)}{f(2a/2b)}=\dfrac{4b^2}{f(a/b)}\neq\dfrac{b^2}{f(a/b)}=1$. Thus, $\dfrac{f(2b)}{f(2a/2b)}\neq 1$, and then putting $n=2b$ and $m=2a$ into the above equality, yields again

$$f\left(\frac{a}{b}\right)=f\left(\frac{2a}{2b}\right)=\frac{(2b)^2}{(2a)^2}=\left(\frac{a}{b}\right)^2$$

So, the only solution is $f(x)=x^2$ as can be easily checked and we are done. $\square$

4. *Find all real solutions of the equation*

$$3\cdot 1331^x+4\cdot 363^x=34\cdot 99^x+77\cdot 27^x$$

Solution. Dividing both sides of the given equation by 27^x we obtain

$$3\left(\frac{1331}{27}\right)^x+4\left(\frac{363}{27}\right)^x=34\left(\frac{99}{27}\right)^x+77$$

or

$$3\left(\frac{11}{3}\right)^{3x}+4\left(\frac{11}{3}\right)^{2x}=34\left(\frac{11}{3}\right)^{x}+77$$

Putting $t=(11/3)^x$ in the preceding equation, we get

$$3t^3+4t^2-34t-77=0$$

Factoring it yields

$$(3t-11)(t^2+5t+7)=0$$

which only real root is $t=11/3$. Therefore, we have

$$\left(\frac{11}{3}\right)^{x}=\frac{11}{3}$$

from which follows that $x=1$ is the unique real root of the equation to be solved, and we are done.

$\square$

5. *Let n be an odd positive integer and let p be a prime number of the form $3n+2$. Prove that if*

$$\frac{a}{b}=\sum_{i=1}^{2n+1}\frac{(-1)^{i+1}}{i},$$

then p divides a.

Solution. We have

$$\begin{aligned}
\frac{a}{b} &= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots-\frac{1}{2n}+\frac{1}{2n+1}\\
&= \left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2n+1}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2n}\right)\\
&= \left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2n+1}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)\\
&= \underbrace{\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}\ldots+\frac{1}{2n-1}+\frac{1}{2n}+\frac{1}{2n+1}}_{\text{even number of terms}}\\
&= \left(\frac{1}{n+1}+\frac{1}{2n+1}\right)+\left(\frac{1}{n+2}+\frac{1}{2n}\right)+\ldots+\left(\frac{1}{(3n+1)/2}+\frac{1}{(3n+3)/2}\right)\\
&= \sum_{r+s=3n+2}\left(\frac{1}{r}+\frac{1}{s}\right)=(3n+2)\sum_{r+s=3n+2}\frac{1}{rs}=(3n+2)\,\frac{c}{d},
\end{aligned}$$

where c is a positive integer and $d=(n+1)(n+2)\ldots(2n+1)$. Since $3n+2$ is prime and every factor of d is smaller then $3n+2$, then $(d,3n+2)=1$. Thus,

$$\frac{a}{b}=\frac{(3n+2)c}{d}\Leftrightarrow ad=(3n+2)bc$$

So, $(3n+2)|a$ and we are done.

$\square$

6. *Let a,b,c and α be positive real numbers such that $a^3+b^3+c^3<\alpha abc$. Find the values of α for which a,b,c are the length of the sides of a triangle.*

Solution. First, we observe that applying AM-GM inequality we have

$$\alpha abc > a^3 + b^3 + c^3 \geq 3abc$$

Therefore, $\alpha > 3$. To find the range of values of α that verify the statement we argue by contradiction. Supposing that $c \geq a + b$, then exist $x \geq 0$ such that $c = a + b + x$ and the condition $a^3+b^3+c^3 < \alpha abc$ becomes $a^3+b^3+(a+b+x)^3 < \alpha ab(a+b+x)$, or equivalently,

$$x^3 + 3x^2(a+b) + 3x(a^2+b^2) + abx(6-\alpha) + (a+b)\Big(2a^2 + (1-\alpha)ab + 2b^2\Big) < 0$$

Now we have that $2a^2 + (1-\alpha)ab + 2b^2 \geq 0$ when its discriminant

$$\Delta = \Big((1-\alpha)^2 - 16\Big)b^2 \leq 0$$

which occurs when $-3 \leq \alpha \leq 5$. So, for $3 < \alpha \leq 5$ we have

$$x^3 + 3x^2(a+b) + 3x(a^2+b^2) + abx(6-\alpha) > 0$$

From the preceding follows that

$$x^3 + 3x^2(a+b) + 3x(a^2+b^2) + abx(6-\alpha) + (a+b)\Big(2a^2 + (1-\alpha)ab + 2b^2\Big) > 0$$

Contradiction. Therefore, a, b, c are the length of the sides of a triangle. We conclude that for $3 < \alpha \leq 5$ the statement holds and we are done. $\square$