Complex orthogonal polynomials and Gaussian quadrature

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Outline

1. Gaussian quadrature
2. Complex Gaussian quadrature. Motivation
3. Complex OPs. Existence and asymptotics
4. Examples
5. Tools
6. Related problems

Warm up. Gaussian quadrature

Gaussian quadrature is a discretization method for integration

$$I[f] = \int_a^b f(x)w(x)\mathrm{d}x \approx \sum_{k=1}^{n} \alpha_i f(x_i) = Q[f],$$

where x_i are **nodes** and α_i are **weights**. Here $w(x)$ is a weight function, assumed integrable and positive in $[a,b]$ and such that

$$\mu_k = \int_a^b x^k w(x)\mathrm{d}x < \infty, \qquad k = 0, 1, 2, \ldots$$

Warm up. Gaussian quadrature

Important properties:

- With n nodes, Gaussian quadrature is exact for polynomials of degree up to $2n-1$ (optimal).
- The nodes are the zeros of the n-th **orthogonal polynomial** $P_n(x)$ with respect to $w(x)$:

$$\int_a^b P_n(x)x^k w(x)\mathrm{d}x = 0, \qquad k = 0, 1, \ldots, n-1.$$

- The nodes x_i are real, simple and lie in $[a, b]$.

Warm up. Gaussian quadrature

How to compute the nodes and weights?

- Golub–Welsch method. Eigenvalue problem for the Jacobi matrix corresponding to the recurrence relation:

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n^2 P_{n-1}(x),$$

with $P_{-1}(x) = 0$, $P_0(x) = 1$. In matrix form

$$x\mathbf{P} = J\mathbf{P},$$

where $\mathbf{P} = [P_0, P_1, \ldots]^T$ and $J = \mathrm{diag}(\gamma_{i+1}, \beta_i, 1)$, for $i = 0, 1, \ldots$

Warm up. Gaussian quadrature

- Methods based on Newton iteration plus recurrence relation, after Glasier, Liu, Rokhlin (2007).
- Methods based on Newton iteration plus known asymptotics: Hale, Townsend (2013).
- Fixed point methods: Gil, Segura (2004–...).
- What to do in non–classical cases? For example, Freud weights?

A problem in oscillatory integrals

Consider the oscillatory integral

$$I[f] = \int_a^b f(x)h(x,\omega x)\mathrm{d}x,$$

where $f(x)$ is smooth and $h(x,\omega x)$ is an oscillatory kernel depending on $\omega \gg 1$. For example

$$h(x,\omega x) = \mathrm{e}^{\mathrm{i}\omega g(x)}, \qquad h(x,\omega x) = H_0^{(1)}(\omega x).$$

Some applications: wave scattering, solution of oscillatory ODEs and oscillatory integral equations.

Oscillatory problems

How to compute this type of integral?

Standard quadrature (Gauss, Newton–Cotes) is inefficient because of oscillation (the number of nodes scales with ω).

But asymptotic information for large ω is readily available! It follows from the principle of stationary phase. For example:

$$\int_a^b f(x)e^{i\omega x}dx = \frac{1}{i\omega}\left[f(b)e^{i\omega b} - f(a)e^{i\omega a}\right] + \mathcal{O}(\omega^{-2}).$$

$\rightarrow$ information coming from endpoints, stationary points and singularities.

Oscillatory problems

Some ideas proposed in the literature are:

- Filon quadrature (designed for oscillatory problems).
- Levin methods (based on converting the integral into an ODE for h).
- Deform the path into $\mathbb{C}$ and use steepest descent analysis.
- Superinterpolation: combination of Filon and steepest descent.

Oscillatory problems

If steepest descent can be used, we get

$$\int_a^b f(z)\mathrm{e}^{\mathrm{i}\omega g(x)}\mathrm{d}x = \sum_i \int_{\Gamma_i} f(z)\mathrm{e}^{\mathrm{i}\omega g(z)}\mathrm{d}z,$$

where Γ_i are suitable contours in $\mathbb{C}$, along which $\mathrm{Im}\, g(z)$ increases.

An attractive discretization procedure for evaluating this kind of integrals is **Gaussian quadrature**, with weight function

$$w(z) = \mathrm{e}^{-\omega \mathrm{Im}\, g(z)}.$$

So far so good, although the weight and/or the path might be complicated.

Oscillatory integrals

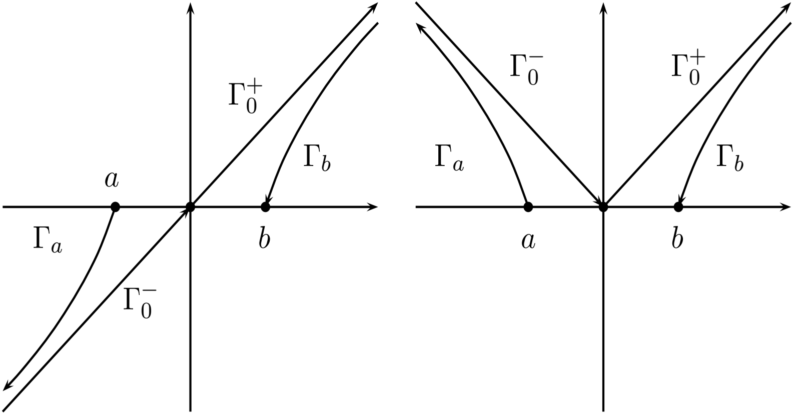
An important example is given by

$$I[f] = \int_a^b f(x) \mathrm{e}^{\mathrm{i}\omega x^r} \mathrm{d}x, \qquad a < 0 < b, \qquad r \geq 2.$$

Deformation into the complex plane, subject to analytic extension of $f(x)$, leads to

$$I[f] = \left(\int_{\Gamma_a} + \int_{\Gamma_0^-} + \int_{\Gamma_0^+} + \int_{\Gamma_b} \right) f(z) \mathrm{e}^{\mathrm{i}\omega z^r} \mathrm{d}z.$$

Oscillatory integrals



Paths of steepest descent for even r (left) and odd r (right).

Complex Gaussian quadrature

In general, consider the following problem: look for polynomials $\{P_n(z)\}_{n\geq 0}$ such that

$$\int_\Gamma P_n(z) z^k w(z) \mathrm{d}z = 0, \qquad k = 0, 1, \ldots, n-1,$$

Here

- Γ is a suitable curve in $\mathbb{C}$ such that the integral is well defined.
- $w(z)$ is a weight function. Important examples are

$$w(z) = \mathrm{e}^{-nV(z)}, \qquad w(z) = J_\nu(z),$$

  where $V(z)$ is a polynomial.

Problem: In the complex case, $w(z)$ need not be real or positive, so $P_n(z)$ might not exist!

Complex Gaussian quadrature

In general, we seek the following information:

- Existence of $P_n(z)$, at least asymptotically for large n.
- Limit zero distribution:

$$\nu_n = \frac{1}{n}\sum_{i=1}^{n}\delta(z-z_i) \to \mathrm{d}\mu, \qquad n\to\infty,$$

  where z_i are the zeros of $P_n(z)$ (possibly scaled).
- Asymptotic information about recurrence coefficients, kernel polynomials, leading coefficients, ..., as $n\to\infty$.

Example. The cubic case

Consider

$$w(z) = \mathrm{e}^{n\frac{iz^3}{3}},$$

after D., Huybrechs, Kuijlaars (2010).

We look for polynomials $P_n(z)$ orthogonal in the following sense:

$$\int_\Gamma P_n(z) z^k \mathrm{e}^{n\frac{iz^3}{3}} \mathrm{d}z = 0, \qquad k = 0, \ldots, n-1,$$

where Γ connects the sectors

$$\frac{5\pi}{6} < \arg\, z < \pi, \qquad 0 < \arg\, z < \frac{\pi}{6}.$$

No existence guaranteed! However...

Example. The cubic case

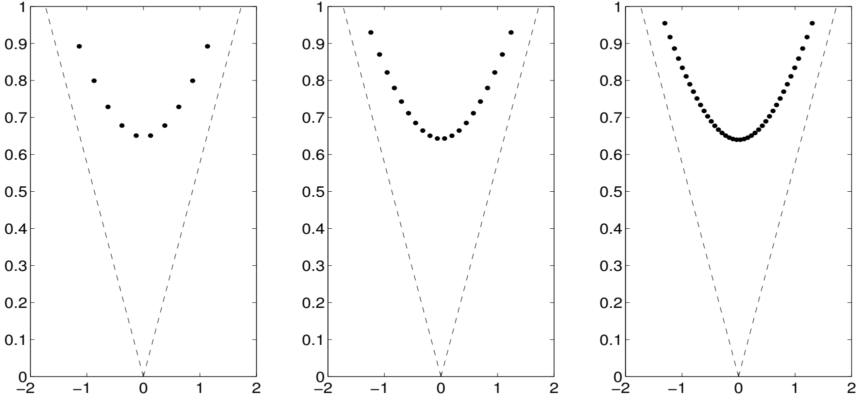


Figure : Location of the quadrature nodes for $r = 3$ on $[-1, 1]$, corresponding to $n = 10$ (left), $n = 20$ (center) and $n = 40$ (right). In dashed line, the paths of steepest descent from the origin.

Example. The cubic case

What is this limit curve in $\mathbb{C}$? Here we need ideas from

- Logarithmic potential theory in the complex plane.
- Symmetry properties of the curve (S-property, after Gonchar, Rakhmanov, Stahl...).

Example. The cubic case

Given $V(z) = -\mathrm{i}z^3/3$, consider the polynomial

$$Q(z) = \frac{1}{4}(z+\mathrm{i})^2(z^2 - 2\mathrm{i}z - 3).$$

Theorem

There exists an analytic arc γ that joins the simple zeros of $Q(z)$, and that can be extended to an unbounded contour Γ with the S-property in the external field $Re\, V(z)$. The equilibrium measure is given by

$$d\mu(s) = \frac{1}{\pi i} Q_+^{1/2}(s)\, ds.$$

For $V(z)$ of degree $2r+1$, $Q(z)$ would be a polynomial of degree $4r$, but the analysis is more complicated...

Our contribution

Some weight functions under study are:

- General monomial of odd degree: $V(z) = z^{2n+1}$.
- Cubic plus linear term: $V(z) = z^3 + tz$, $t \in \mathbb{R}$. Connection with random matrix theory. Joint and ongoing work with P. M. Bleher.
- OPs with respect to Airy functions.

Some other examples

- OPs with respect to Bessel functions. Consider $P_n(x)$ such that

$$\int_0^\infty P_n(x) x^m J_\nu(x) \mathrm{d}x = 0, \qquad m = 0, 1, \ldots, n-1, \quad \nu \geq 0,$$

defined as the limit $P_n(x) = \lim_{s \to 0^+} P_{n,s}(x)$, where

$$\int_0^\infty P_{n,s}(x) x^m J_\nu(x) \mathrm{e}^{-sx} \mathrm{d}x = 0, \qquad m = 0, 1, \ldots, n-1, \quad \nu \geq 0.$$

See Asheim & Huybrechs 2011. Joint work with A. B. J. Kuijlaars and P. Román (in preparation).

Zeros of other exotic OPs (after A. Asheim)

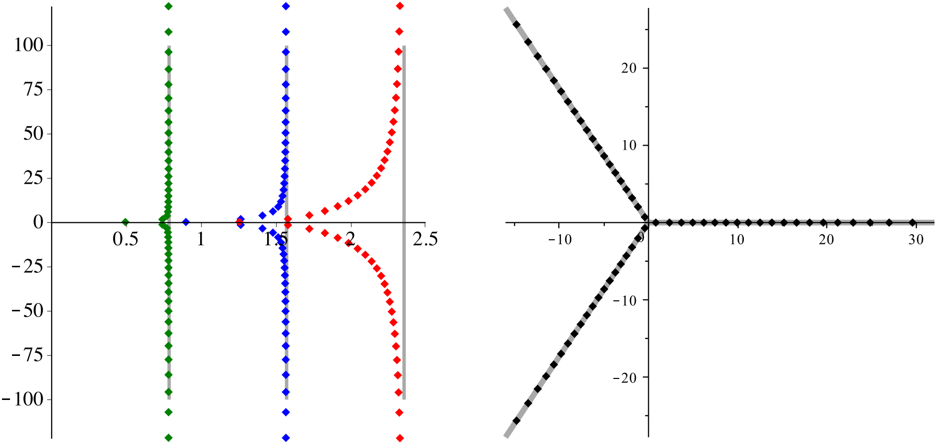


Figure : Left: quadrature nodes for OPs w.r.t. $J_{1/2}(x)$ (green), $J_1(x)$ (blue), $J_{3/2}(x)$ (red). Right: quadrature nodes for OPs w.r.t. Airy function.

Tools

The main tool is... the Riemann–Hilbert formulation for OPs!

Why Riemann–Hilbert?

If successful, Riemann–Hilbert + nonlinear steepest descent gives **asymptotic results** for large n:

- existence of $P_n(z)$,
- behaviour of $P_n(z)$, uniformly in $\mathbb{C}$,
- limit zero distribution of $P_n(z)$,
- recurrence coefficients γ_n^2, β_n,
- kernel polynomials, Hankel determinants, leading coefficients...

In classical cases, other asymptotic methods (integral representations, differential equations) are simpler, but RH is robust in non–standard situations.

Additional questions

Apart from technical Riemann–Hilbert work:

- Is it possible to use the asymptotic information (e.g. given by Riemann–Hilbert) for computations?
- Are these complex rules competitive vs. the real case?
- Do the complex rules have good numerical properties?
- Other theoretical/computational problems associated with complex OPs?

That's all for now...

Thanks for your attention!