On positive rational interpolatory quadrature formulas on the unit circle with prescribed nodes

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“To Pablo González Vera, our beloved master and friend.
In memóriam.”
ORTHOGONAL RATIONAL FUNCTIONS

ORTHOGONAL RATIONAL FUNCTIONS. Aim of this talk.

- **ORF**: generalization of orthogonal ordinary polynomials (poles at $\infty$) and orthogonal Laurent polynomials (poles at $\{0, \infty\}$).

**AIM OF THIS TALK:**

1. To present a research topic between people from
   - La Laguna University, Tenerife (P. González-Vera, C. Díaz-Mendoza, R. Orive, F. Perdomo Pío, R. Cruz-Barroso).
   - K.U. Leuven, Belgium (A. Bultheel).
   - U. Lille (B. Beckermann, K. Deckers).

2. More exactly: positive rational interpolatory quadrature formulas on the unit circle with prescribed nodes.

3. Survey and some recent results:
   - Connection with some recent results by B. Beckermann and K. Deckers.
   - Extension to ORF of some of the results presented in: *Orthogonal Polynomials and Special Functions - a Complex Analytic Perspective. Copenhagen, Denmark, June 11-15, 2012* (P. González-Vera, C. Díaz-Mendoza and F. Perdomo Pío, R. Cruz-Barroso).
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Some works


- More than 70 papers by A. Bultheel, P. González-Vera, E. Hendriksen and O. Njåstad!

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Rational quadrature formulas on the unit circle

- \( \mu \): positive Borel measure on \( \mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \} \).

- **Aim**: estimations of \( I_\mu(f) = \int_{-\pi}^{\pi} f(z) d\mu(\theta) \), with \( z = e^{i\theta} \), \( f \) continuous on \( \mathbb{T} \).

- **Quadrature formulas**:

  \[
  I_n(f) = \sum_{k=1}^{n} \lambda_k f(z_k) \quad z_j \neq z_k \text{ if } j \neq k.
  \]

  Election of the nodes and weights so that the rule is exact in a certain space of functions with dimension as large as possible.

  ~ Positive rule (positive weights): interest for convergence and stability.
Rational quadrature formulas on the unit circle

- Quadrature formulas for $l_{\mu}(f)$: $n$-point Szegő rules, positive and exact in $\text{span}\{z^k : -(n-1) \leq k \leq n-1\}$.
  $\sim$ One free parameter of modulus one.
  $\sim$ Nodal polynomial: para-orthogonal.

- Rational quadrature formulas for $l_{\mu}(f)$: rational $n$-point Szegő rules, positive and exact in spaces of rational functions.
  $\sim$ Rational quadratures: in many cases, $f$ is much better approximated by rational functions, instead of L-polynomials.
  e.g. $f$ meromorphic, with poles outside but maybe close to $\mathbb{T}$.
Rational generalization of well known concepts

- $\langle f, g \rangle_\mu = \int_{-\pi}^\pi f(z)\overline{g(z)}d\mu(\theta), \ z = e^{i\theta}$.
- Initial data: $\mu$ and $\alpha = \{\alpha_k\}_{k=1}^\infty \subset \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.
- Blaschke factor: $\zeta_i(z) = \frac{z - \alpha_i}{1 - \alpha_i z}$.
- Blaschke products: $B_0 \equiv 1, B_n = B_{n-1} \zeta_n, n \geq 1$.

$L_n = \text{span}\{B_k\}_{k=0}^n = \left\{f = \frac{P}{\pi_n} : P \in \mathbb{P}_n\right\}$ with

$\pi_n(z) = \prod_{j=1}^n 1 - \overline{\alpha_j}z$ and $\mathbb{P}_n = \text{span}\{1, z, \ldots, z^n\}$.

Set $\omega_n(z) = \prod_{j=1}^n (z - \alpha_j)$. Thus, $B_n(z) = \frac{\omega_n(z)}{\pi_n(z)}$.

Substar conjugate of $f$: $f^*(z) = \overline{f(1/\overline{z})}$. We can define:

$L_{n^*} = \{f : f^* \in L_n\} = \left\{f = \frac{Q}{\omega_n} : Q \in \mathbb{P}_n\right\}$. 
Rational generalization of well known concepts

- For $p, q$ non-negative integers, rational (multipoint) analog of the space of Laurent polynomials:

$$\mathcal{R}_{-p,q} = \mathcal{L}_{p^*} + \mathcal{L}_q = \text{span}\{B_k : k = -p, \ldots, -1, 0, 1, \ldots, q\}$$

$$= \left\{ f = \frac{P}{\omega_p\pi_q} : P \in \mathbb{P}_{p+q} \right\},$$

where $B_{-k}(z) = B_{k^*}(z) = \frac{1}{B_k(z)}$.

Functions in $\mathcal{R}_{-p,q}$ have their poles in $\{\alpha_k\}_{k=1}^p \cup \{1/\alpha_l\}_{l=1}^q$.

- Super-star conjugation: $f^*(z) = B_n(z)f^*_*(z)$, for $f \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$.

- $\mathcal{L}_n(\beta) = \{ f \in \mathcal{L}_n : f(\beta) = 0 \}$.

- Blaschke products are useful for computations:
  - $B_{-k}(z) = B_{k^*}(z) = \overline{B_k(z)}$ for $z \in \mathbb{T}$. $B_n^* \equiv 1$.
  - $f = p_n/\pi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$, then $f^*_* = p_n^*/\pi_n^*$ and $f^* = p_n^*/\pi_n$.
  - ...
Aim: estimations of $I_{\mu}(f) = \int_{-\pi}^{\pi} f(z) d\mu(\theta)$, with $z = e^{i\theta}$ by $I_n(f) = \sum_{k=1}^{n} \lambda_k f(z_k)$.

Problem: find nodes and weights such that $I_{\mu}(f) = I_n(f)$ for all $f \in \mathcal{R}_{-p(n),q(n)}$, with $p(n) + q(n)$ as large as possible.

Basic requirement: rational moments $\mu_k = \int_{-\pi}^{\pi} B_k(z) d\mu(\theta)$ with $z = e^{i\theta}$ must exist $\forall k$. Notice: $\mu_{-k} = \overline{\mu_k}$.

Gram-Schmidt process to the basis $B_0, B_1, \ldots, B_n$:

$\{\rho_j\}_{j=0}^{n}$ orthonormal basis for $\mathcal{L}_n$.

Repeated for all $n$: orthonormal system $\{\rho_j\}_{j=0}^{\infty}$. ORF.

$\Rightarrow$ not unique, we can always multiply with a unimodular constant. Normalization: choosing leading coefficient $\kappa_n = \rho_n^*(\alpha_n)$ of $\rho_n$ to be real and positive.
Para-orthogonal rational functions, PORF.

As in the ordinary polynomial situation, set
\[ \chi_n(z, \tau_n) = C_n [\rho_n(z) + \tau_n \rho_n^*(z)], \]
with \( C_n \neq 0 \) and \( \tau_n \in \mathbb{T} \). Then:

- \( \chi_n^*(z, \tau_n) = \kappa \chi_n(z, \tau_n) \), for \( \kappa = \frac{C_n}{\overline{C_n} \overline{\tau_n}} \in \mathbb{T} \) and \( z \in \mathbb{C} \setminus \{1/\alpha_1, \ldots, 1/\alpha_n\} \). (invariance)

- \( \langle \chi_n, f \rangle_\mu = 0 \) for all \( f \in \mathcal{L}_{n-1}(\alpha_n) \) and \( \langle \chi_n, 1 \rangle_\mu \cdot \langle \chi_n, B_n \rangle_\mu \neq 0 \).

- \( \chi_n \) has exactly \( n \) distinct zeros on \( \{z_j\}^{n}_{j=1} \subset \mathbb{T} \).

- Taking such zeros as nodes, there exist positive numbers \( \{\lambda_j\}^{n}_{j=1} \) s.t. \( I_n(f) = I_\mu(f) \), for all \( f \in \mathcal{R}_{-(n-1),n-1} \).

Rational Szegő quadrature formula
Maximal domain of exactness since there cannot exist a positive \( n \)-point rule with nodes on \( T \) that is exact neither \( R_{-n,n-1} \) or \( R_{-(n-1),n} \).

The weights \( \{\lambda_k\}_{k=1}^n \) are positive and can be computed by
\[
\lambda_k^{-1} = \sum_{j=0}^{n-1} |\rho_j(z_k)|^2 > 0, \; k = 1, \ldots, n.
\]
Rational Szegő recurrence:

\[ \rho_0(z) \equiv 1 \text{ and for } n \geq 1, \]
\[
\begin{pmatrix}
\rho_n(z) \\
\rho^*_n(z)
\end{pmatrix}
= M_n(z)
\begin{pmatrix}
\rho_{n-1}(z) \\
\rho^*_{n-1}(z)
\end{pmatrix}
\]
with
\[ M_n(z) = e_n \frac{1-\alpha_{n-1}z}{1-\alpha_n z} \begin{pmatrix} 1 & \delta_n \\ \bar{\delta}_n & 1 \end{pmatrix} \begin{pmatrix} \zeta_{n-1}(z) & 0 \\ 0 & 1 \end{pmatrix} \]

where
\[ \delta_n = \frac{\rho_n(\alpha_{n-1})}{\rho^*_n(\alpha_{n-1})} = - \frac{\langle \frac{z-\alpha_{n-1}}{1-\alpha_n z}, \rho_{n-1}(z), \rho_k(z) \rangle_{\hat{\mu}}}{\langle \frac{1-\alpha_{n-1}z}{1-\alpha_n z}, \rho^*_{n-1}(z), \rho_k(z) \rangle_{\hat{\mu}}}, \quad \forall k = 0, 1, \ldots, n-1. \]

(\( \delta_n \): Rational Verblunsky coefficients) and
\[ 0 < e_n = \sqrt{\frac{1-|\alpha_n|^2}{1-|\alpha_{n-1}|^2} \frac{1}{1-|\delta_n|^2}}. \]
Para-orthogonal rational functions (PORF)

- **PORF**: by applying the rational Szegő recurrence for $\rho_n$ and $\rho_n^*$, $\chi_n$ can be expressed in terms of $\rho_{n-1}$ and $\rho_{n-1}^*$:

$$\chi_n(z, \tau_n) = \rho_n(z) + \tau_n \rho_n^*(z)$$

$$= C_n \frac{1 - \alpha_n - 1 z}{1 - \alpha_n z} \left[ \zeta_{n-1}(z) \rho_{n-1}(z) + \tau_n^* \rho_{n-1}^*(z) \right]$$

with $C_n \neq 0$ and $\tau_n^* \in \mathbb{T}$.

- **Thus**: it will be suffice to compute ORF of degree $n - 1$. The nodes of the rational Szegő rule are the zeros of

$$\chi_n(z, \tau_n^*) = (z - \alpha_{n-1}) \rho_{n-1}(z) + \tau_n^* (1 - \alpha_{n-1} z) \rho_{n-1}^*(z),$$

for some $\tau_n^* \in \mathbb{T}$. 
Fixing nodes in advance in the quadrature formula

- **Rational Szegő**: always exist, depend on $\tau_n^* \in \mathbb{T}$ (arbitrary), positive and exact in $\mathcal{R}_{-(n-1),n-1}$.

- **Rational Szegő-Radau**: always exist, positive, exact in $\mathcal{R}_{-(n-1),n-1}$ and it is unique.

To fix a node $u \in \mathbb{T}$, take $\tau_n^* = \frac{(\alpha_{n-1} - u)\rho_{n-1}(u)}{(1 - \alpha_{n-1}u)\rho_{n-1}^*(u)} \in \mathbb{T}$. 
Fixing nodes in advance in the quadrature formula

- **Rational Szegő-Lobatto:** To fix two distinct nodes \( u, v \in \mathbb{T} \).


Modify the last rational Verblunsky parameter:
\[
\delta_{n-1} \sim \tilde{\delta}_{n-1} \in \mathbb{D}
\]
and to take an appropriate parameter \( \tilde{\tau}_n^* \)

s.t. the \( n \)-th para-orthogonal rational function has \( u, v \) among it zeros.

**Problem:** given \( \hat{\mu}, \{\alpha\} \in \mathbb{D} \) and \( u, v \in \mathbb{T} \), to find a positive q.f. with \( u, v \) prescribed nodes and a minimal number of other nodes, mutually distinct, exact in \( \mathcal{R}_{-(n-1),n-1} \).

**Three possible situations:**
Fixing nodes in advance in the quadrature formula

Three possible situations:

1. $u, v$ happen to be among the zeros of a para-orthogonal rational function of degree $n$. Exactness in $\mathcal{R}_{-(n-1),n-1}$.

2. $u, v$ happen to be among the zeros of a para-orthogonal rational function of degree $n+1$. Exactness in $\mathcal{R}_{-n,n} \supset \mathcal{R}_{-(n-1),n-1}$.
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Fixing nodes in advance in the quadrature formula

Three possible situations:

1. **Generic case**: there exist a proper \((n + 1)\)-point rational Szegő-Lobatto rule. Exactness in \(\mathcal{R}_{-(n-1),n-1}\).

Set:

\[
\xi_1 = \zeta_n(u) \in \mathbb{T}, \quad \xi_2 = \zeta_n(v) \in \mathbb{T}, \\
\tau_1 = -\frac{\zeta_{n-1}(u)\rho_{n-1}(u)}{\rho^*_{n-1}(u)} \in \mathbb{T}, \quad \tau_2 = -\frac{\zeta_{n-1}(v)\rho_{n-1}(v)}{\rho^*_{n-1}(v)} \in \mathbb{T}.
\]

\(\leadsto\) Modified \(\delta_n \rightarrow \tilde{\delta}_n \in C \cap \mathbb{D} \neq \emptyset\), with \(C\) the circle with center \(c = \frac{\xi_1 - \xi_2}{\tau_1 \xi_1 - \tau_2 \xi_2}\) and radius \(r = |1 - c\tau_2|\).

\(\leadsto\) It holds \(C \cap \mathbb{T} = \{\tau_1, \tau_2\}\).

\(\leadsto\) If the center and the radius are infinite, then the circle becomes a straight line.

\(\leadsto\) Particular choice of \(\tilde{\tau}^*_n = \xi_1 \frac{\tau_1 - \tilde{\delta}_n}{1 - \tau_1 \tilde{\delta}_n}\).
Fixing nodes in advance in the quadrature formula
Some references


A first new result: three prescribed nodes

- **General open problem (hard!):** to characterize a polynomial and rational quadrature formula on the unit circle with an arbitrary number of prescribed nodes and maximal domain of exactness.

- **Next step:** to fix three nodes.

- **The solution now is completely different!:** the rule may not exist.

- **A characterization can be obtained:** a positive quadrature formula exist under conditions on $\tilde{\mu}$ and $\{\alpha\} \subset \mathbb{D}$.

- **Idea:** in the rational Szegő-Lobatto rule, to fix the free rational Verblunsky parameter $\tilde{\delta}_n$ on $C \cap \mathbb{D}$ s.t. the PORF $\chi_{n+1}(z, \tilde{\tau}_{n+1}^*)$ has $u, v, w$ among its zeros.
A first new result: three prescribed nodes

- **Theorem:** this \((n + 1)\)-point rule exists (positive, unique and exact in \(R_{-(n-1),n-1}\), iff

\[
\tilde{\delta}_n = \text{(long expression depending only on } u, v, w, \mu, \{\alpha\}\text{)} \in C.
\]

**Problem!**: this parameter does not always satisfy \(|\tilde{\delta}_n| < 1\).

- **Idea of the proof:**
  \(\sim\) Rational Szegő recurrence: \(\tilde{\chi}_{n+1}(z, \tilde{\tau}_{n+1}^*)\) is written in terms of \(\{\rho_{n-1}, \rho_{n-1}^*, \tilde{\delta}_n, \tilde{\tau}_{n+1}^*\}\).

  \(\sim\) Construct Rational Szegő-Lobatto rule to fix \(u\) and \(v \rightarrow \tilde{\delta}_n \in C \cap D\) (free) and \(\tilde{\tau}_{n+1}^*\) uniquely determined from \(\tilde{\delta}_n\).

  \(\sim\) To introduce the expression for \(\tilde{\tau}_{n+1}^*\) + expressions for the center and radius + \(\tilde{\delta}_n \in C \cap D\) + to fix \(w\) + tedious calculations (and some luck!) \(\rightarrow\) explicit expression for \(\tilde{\delta}_n\).
An example

- \( \mu: \) Lebesgue measure on the unit circle.

- ORF: Takenaka-Malmquist basis,
  \[ \{1\} \cup \left\{ \frac{(1-|\alpha_{k+1}|)^{1/2}z}{1-\alpha_{k+1}z} B_k(z) \right\}_{k=0}^{n-1}. \]

- Two nodes \( u \) and \( v \) prescribed (green triangles). A third node \( w \) moving around the unit circle.

- Take \( n = 10 \).

- Blue region: accepted rational quadrature rule (existence and positive).
An example
An example

![Graph showing an example of positive rational quadrature formulas on the unit circle.](image)
A second new result (in progress)

- Joukowsky transform: \( x = \frac{1}{2} \left( z + \frac{1}{z} \right) \), \( x = J(z) \). Maps \( \mathbb{D} \) onto the cut Riemann sphere \( \mathbb{C} \setminus [-1, 1] \) and \( \mathbb{T} \) onto \( [-1, 1] \).

- Connection between quadrature formulas on the interval and the unit circle:

- Gaussian quadrature formulas on the interval with a prescribed node anywhere:
A second new result (in progress)

- **Next step**: quadrature formulas on the interval with a prescribed node inside the interval of integration and possibly one or both endpoints.
- **Rational extension**: A first approach has been done by K. Deckers (in progress, not submitted yet!), by considering quasi-orthogonal rational functions on the interval:
  \[ \phi_n(z) = \rho_n(z) + a\rho_{n-1}(z) \].
- **Alternative approach (in progress)**: to use Joukowsky transform and symmetric rational Szegő-Lobatto rules.
A second new result (in progress)

**Theorem:** Suppose that \( \{\alpha_1, \ldots, \alpha_{n-1}\} \) are real or appear in complex conjugate pairs and that \( \mu \) is symmetric. Set

\[
\nu_n = \prod_{j=1}^{n} \eta_j \in \{\pm 1\} \quad \text{with} \quad \eta_j = \begin{cases} 
-\frac{\bar{\alpha}_j}{|\alpha_j|} & \text{if} \quad \alpha_j \neq 0, \\
1 & \text{if} \quad \alpha_j = 0.
\end{cases}
\]

Then,

1. The zeros of \( \chi_n(z, \tau_n) \) appear in complex conjugate pairs iff \( \tau_n = \pm 1 \).
2. \( \chi_n(z, \tau_n) \) has a zero in
   1. \( z = 1 \) iff \( \tau_n = -\nu_n \).
   2. \( z = -1 \) iff \( \tau_n = (-1)^{n+1}\nu_n \).
3. In a rational Szegő rule, the weights associated to two complex conjugate nodes are equal, iff, \( \tau_n = \pm 1 \).
A second new result (in progress)

- By considering symmetric rational Szegő-Lobatto quadrature formulas ($u$ and $\bar{u}$ prescribed) and connecting with the interval by Joukowsky transform $\rightsquigarrow$
- Characterization of rational Gaussian quadrature formulas with a node prescribed inside and possibly one or both endpoints.

**Computation:**

- **Rational Gaussian formulas:** generalized eigenvalue problem
  (A. Bultheel and J. Van Deun, Numer. Algor., 2007)

- **Rational Szegő formulas:**
  Operator Möbius transformations of Hessenberg or CMV matrices (L. Velázquez, J. Funct. Anal., 2008) and
A second new result (in progress)

- An **algorithm** for the computation of Gaussian rules with a prescribed node by passing to the unit circle in the **polynomial situation** is presented in the PhD Thesis of F. Perdomo-Píó (2013): key fact:

  - Geronimus relations (computation of Jacobi matrices from Verblunsky coefficients for two measures related by the Joukowsky transform)


- Extension to the **rational case**: ?
A second new result (in progress)

Thank You!