On the order of indeterminate moment problems based on the recurrence coefficients

Christian Berg
University of Copenhagen, Denmark

Legroño, February 22, 2013

Based on joint work with
Ryszard Szwarc
University of Wrocław, Poland


door mat

Overview

1. Introduction to indeterminate moment problems

2. Growth properties of functions: order and type; logarithmic order and type

3. The order and logarithmic order of moment problems

4. How can these numbers be determined from the three term recurrence or the moments?

5. Some answers to 4 by B. and Szwarc

6. References

The basics

A system of orthonormal polynomials is given by a three term recurrence relation

$$zP_n(z) = b_n P_{n+1}(z) + a_n P_n(z) + b_{n-1} P_{n-1}(z),\ n \geq 0$$

where $P_0 = 1, P_{-1} = 0$ and $a_n \in \mathbb{R}$, $b_n > 0$. Equivalently P_n is a polynomial of degree n and there exists a probability measure μ on $\mathbb{R}$ such that

$$\int P_n P_m \, d\mu = \delta_{n,m}.$$

We need the coefficients of the orthonormal polynomials

$$P_n(x) = \sum_{k=0}^{n} b_{k,n} x^k,$$

P_n is uniquely determined by the orthonormality if we assume the leading coefficient $b_{n,n} > 0$ and we have

$$b_{n,n} = 1/(b_0 b_1 \cdots b_{n-1}).$$

The basics

A system of orthonormal polynomials is given by a three term recurrence relation

$$zP_n(z) = b_n P_{n+1}(z) + a_n P_n(z) + b_{n-1} P_{n-1}(z),\ n \geq 0$$

where $P_0 = 1, P_{-1} = 0$ and $a_n \in \mathbb{R}$, $b_n > 0$. Equivalently P_n is a polynomial of degree n and there exists a probability measure μ on $\mathbb{R}$ such that

$$\int P_n P_m \, d\mu = \delta_{n,m}.$$

We need the coefficients of the orthonormal polynomials

$$P_n(x) = \sum_{k=0}^{n} b_{k,n} x^k,$$

P_n is uniquely determined by the orthonormality if we assume the leading coefficient $b_{n,n} > 0$ and we have

$$b_{n,n} = 1/(b_0 b_1 \cdots b_{n-1}).$$

The basics

A system of orthonormal polynomials is given by a three term recurrence relation

$$zP_n(z) = b_n P_{n+1}(z) + a_n P_n(z) + b_{n-1} P_{n-1}(z),\ \ n \ge 0$$

where $P_0 = 1, P_{-1} = 0$ and $a_n \in \mathbb{R},\ b_n > 0$. Equivalently P_n is a polynomial of degree n and there exists a probability measure μ on $\mathbb{R}$ such that

$$\int P_n P_m \, d\mu = \delta_{n,m}.$$

We need the coefficients of the orthonormal polynomials

$$P_n(x) = \sum_{k=0}^{n} b_{k,n} x^k,$$

P_n is uniquely determined by the orthonormality if we assume the leading coefficient $b_{n,n} > 0$ and we have

$$b_{n,n} = 1/(b_0 b_1 \cdots b_{n-1}).$$

Determinacy/indeterminacy

The measure μ need not be uniquely determined, but all possible orthonormality measures have the same moments and vice versa

$$s_n = \int_{-\infty}^{\infty} x^n \, d\mu(x), \quad n \geq 0.$$

Dichotomy:

The determinate case: μ is uniquely determined

The indeterminate case: There are several (and then infinitely many) μ. Stieltjes was the first to observe this dichotomy around 1892 in correspondence with his mentor Hermite. Letter 325 from Stieltjes to Hermite dated January 30, 1892:

L'existence de ces fonctions $\varphi(u)$ qui, sans être nulles, sont telles que

$$\int_0^\infty u^k \varphi(u) \, du = 0 \ , \quad (k = 0, 1, 2, 3, \cdots)$$

me paraît très remarquable

Determinacy/indeterminacy

The measure μ need not be uniquely determined, but all possible orthonormality measures have the same moments and vice versa

$$s_n = \int_{-\infty}^{\infty} x^n \, d\mu(x), \quad n \geq 0.$$

Dichotomy:
The determinate case: μ is uniquely determined
The indeterminate case: There are several (and then infinitely many) μ. Stieltjes was the first to observe this dichotomy around 1892 in correspondence with his mentor Hermite. Letter 325 from Stieltjes to Hermite dated January 30, 1892:
L'existence de ces fonctions $\varphi(u)$ qui, sans être nulles, sont telles que

$$\int_0^{\infty} u^k \varphi(u) \, du = 0 \,, \quad (k = 0, 1, 2, 3, \cdots)$$

me paraît très remarquable

Determinacy/indeterminacy

The measure μ need not be uniquely determined, but all possible orthonormality measures have the same moments and vice versa

$$s_n = \int_{-\infty}^{\infty} x^n \, d\mu(x), \quad n \geq 0.$$

Dichotomy:
The determinate case: μ is uniquely determined
The indeterminate case: There are several (and then infinitely many) μ. Stieltjes was the first to observe this dichotomy around 1892 in correspondence with his mentor Hermite. Letter 325 from Stieltjes to Hermite dated January 30, 1892:

L'existence de ces fonctions $\varphi(u)$ qui, sans être nulles, sont telles que

$$\int_0^{\infty} u^k \varphi(u) \, du = 0 \ , \quad (k = 0, 1, 2, 3, \cdots)$$

me paraît très remarquable

Figure : Jan Thomas Stieltjes, 1856-1894

Stieltjes' example

Let $0 < q < 1$ be given by $q = e^{-\sigma^2}, \sigma > 0$.

$$d_\sigma(x) = (2\pi\sigma^2)^{-\frac{1}{2}} x^{-1} \exp\left(-\frac{(\log x)^2}{2\sigma^2}\right)$$

is a probability density: lognormal density. The moment sequence is

$$s_n(d_\sigma) = q^{-\frac{1}{2}n^2}, \quad n \geq 0.$$

Stieltjes showed that these moments belong to an indeterminate moment problem by pointing out that all the densities ($s \in [-1, 1]$)

$$d_\sigma(x)\left(1 + s\sin(\frac{2\pi}{\sigma^2}\log x)\right)$$

have the same moments.

Stieltjes' example

Let $0 < q < 1$ be given by $q = e^{-\sigma^2}, \sigma > 0$.

$$d_\sigma(x) = (2\pi\sigma^2)^{-\frac{1}{2}} x^{-1} \exp\left(-\frac{(\log x)^2}{2\sigma^2}\right)$$

is a probability density: lognormal density. The moment sequence is

$$s_n(d_\sigma) = q^{-\frac{1}{2}n^2}\ , \quad n \geq 0.$$

Stieltjes showed that these moments belong to an indeterminate moment problem by pointing out that all the densities ($s \in [-1, 1]$)

$$d_\sigma(x)\left(1 + s\sin(\frac{2\pi}{\sigma^2}\log x)\right)$$

have the same moments.

Chihara-Leipnik's discrete example

Chihara(1970) and later Leipnik(1981) gave the following family of discrete measures with these moments. For $a > 0$ define the discrete probability

$$\lambda_a = \frac{1}{L(a)} \sum_{k=-\infty}^{\infty} a^k q^{\frac{1}{2}k^2} \delta_{aq^k},$$

where

$$L(a) = \sum_{k=-\infty}^{\infty} a^k q^{\frac{1}{2}k^2} = (q, -\sqrt{q}\, a, -\sqrt{q}/a; q)_\infty.$$

It is easy to calculate the moments of λ_a using the translation invariance of $\sum_{-\infty}^{\infty}$. In fact

$$\begin{aligned} s_n(\lambda_a) &= \frac{1}{L(a)} \sum_{k=-\infty}^{\infty} a^k q^{\frac{1}{2}k^2} (aq^k)^n = \frac{q^{-\frac{1}{2}n^2}}{L(a)} \sum_{k=-\infty}^{\infty} a^{k+n} q^{\frac{1}{2}(k+n)^2} \\ &= q^{-\frac{1}{2}n^2}. \end{aligned}$$

Chihara-Leipnik's discrete example

Chihara(1970) and later Leipnik(1981) gave the following family of discrete measures with these moments. For $a > 0$ define the discrete probability

$$\lambda_a = \frac{1}{L(a)} \sum_{k=-\infty}^{\infty} a^k q^{\frac{1}{2}k^2} \delta_{aq^k},$$

where

$$L(a) = \sum_{k=-\infty}^{\infty} a^k q^{\frac{1}{2}k^2} = (q, -\sqrt{q}\, a, -\sqrt{q}/a; q)_\infty.$$

It is easy to calculate the moments of λ_a using the translation invariance of $\sum_{-\infty}^{\infty}$. In fact

$$\begin{aligned} s_n(\lambda_a) &= \frac{1}{L(a)} \sum_{k=-\infty}^{\infty} a^k q^{\frac{1}{2}k^2} (aq^k)^n = \frac{q^{-\frac{1}{2}n^2}}{L(a)} \sum_{k=-\infty}^{\infty} a^{k+n} q^{\frac{1}{2}(k+n)^2} \\ &= q^{-\frac{1}{2}n^2}. \end{aligned}$$

Pakes-Christiansen's examples

Pakes(1996) and later Christiansen(2003) proved the following: Any positive finite measure ν on $(q,1]$ can be extended to a positive finite measure μ on the half-line such that

$$\frac{1}{\mu([0,\infty[)}\int_0^\infty x^n\,d\mu(x) = q^{-\frac{1}{2}n^2},\ n \geq 0.$$

In this way one can construct solutions to the log-normal moment problem with very special properties. If e.g. ν is chosen continuous singular then so is μ.

A general result about indeterminate moment problems

Theorem (B-Christensen(1981))

Let V be the compact convex set of solutions to an arbitrary indeterminate moment problem. Then each of the following three subsets are dense in V:

(i) $V_1 = \{f(x)dx \in V \mid f \in C^\infty(\mathbb{R})\}$,

(ii) $V_2 = \{\mu \in V \mid \mu \text{ is discrete }\}$,

(iii) $V_3 = \{\mu \in V \mid \mu \text{ is continuous singular }\}$

Characterizations of indeterminate moment problems

Let Q_n be the polynomials of the second kind

$$Q_n(x) = \int \frac{P_n(x) - P_n(y)}{x - y}\, d\mu(y).$$

Classic The following conditions are equivalent

(i) μ is indeterminate,

(ii) $P_n(0), Q_n(0) \in \ell^2$,

(iii) $\sum_{n=0}^{\infty} |P_n(i)|^2 < \infty$

(iv) $\sum_{n=0}^{\infty} |P_n(z)|^2$ converges locally uniformly for $z \in \mathbb{C}$,

(v) The operator $T(p) = xp(x)$ in $\overline{\mathbb{C}[x]}^{L^2(\mu)}$ has deficiency indices $(1, 1)$.

In the indeterminate case consider:

$$P(z) = \left(\sum_{n=0}^{\infty} |P_n(z)|^2\right)^{1/2}, \quad Q(z) = \left(\sum_{n=0}^{\infty} |Q_n(z)|^2\right)^{1/2}.$$

(Note: also $\sum |Q_n(z)|^2$ converges locally uniformly in $\mathbb{C}$)

Characterizations of indeterminate moment problems

Let Q_n be the polynomials of the second kind

$$Q_n(x) = \int \frac{P_n(x) - P_n(y)}{x - y}\, d\mu(y).$$

Classic The following conditions are equivalent

(i) μ is indeterminate,

(ii) $P_n(0), Q_n(0) \in \ell^2$,

(iii) $\sum_{n=0}^{\infty} |P_n(i)|^2 < \infty$

(iv) $\sum_{n=0}^{\infty} |P_n(z)|^2$ converges locally uniformly for $z \in \mathbb{C}$,

(v) The operator $T(p) = xp(x)$ in $\overline{\mathbb{C}[x]}^{L^2(\mu)}$ has deficiency indices $(1,1)$.

In the indeterminate case consider:

$$P(z) = \left(\sum_{n=0}^{\infty} |P_n(z)|^2\right)^{1/2}, \quad Q(z) = \left(\sum_{n=0}^{\infty} |Q_n(z)|^2\right)^{1/2}.$$

(Note: also $\sum |Q_n(z)|^2$ converges locally uniformly in $\mathbb{C}$)

Indeterminacy in terms of coefficients a_n, b_n

It is well known that the moment problem is determinate with a measure μ of compact support iff a_n, b_n are bounded.

Carleman's criterion(1926): Indeterminacy $\Rightarrow \sum(1/b_n) < \infty$.
Warning: There exist determinate problems with $\sum(1/b_n) < \infty$.

Lemma

Let $b_n > 0$ satisfy $\sum(1/b_n) < \infty$ and that b_n is either eventually log-convex (i.e. $b_n^2 \le b_{n-1}b_{n+1}, n \ge n_0$) or eventually log-concave (i.e. $b_n^2 \ge b_{n-1}b_{n+1}, n \ge n_0$), then b_n is eventually strictly increasing to infinity.

Indeterminacy in terms of coefficients a_n, b_n

It is well known that the moment problem is determinate with a measure μ of compact support iff a_n, b_n are bounded.

Carleman's criterion(1926): Indeterminacy $\Rightarrow \sum(1/b_n) < \infty$.
Warning: There exist determinate problems with $\sum(1/b_n) < \infty$.

Lemma

Let $b_n > 0$ satisfy $\sum(1/b_n) < \infty$ and that b_n is either eventually log-convex (i.e. $b_n^2 \le b_{n-1}b_{n+1}, n \ge n_0$) or eventually log-concave (i.e. $b_n^2 \ge b_{n-1}b_{n+1}, n \ge n_0$), then b_n is eventually strictly increasing to infinity.

Indeterminacy in terms of coefficients a_n, b_n

It is well known that the moment problem is determinate with a measure μ of compact support iff a_n, b_n are bounded.

Carleman's criterion(1926): Indeterminacy $\Rightarrow \sum(1/b_n) < \infty$.
Warning: There exist determinate problems with $\sum(1/b_n) < \infty$.

Lemma

Let $b_n > 0$ satisfy $\sum(1/b_n) < \infty$ and that b_n is either eventually log-convex (i.e. $b_n^2 \le b_{n-1}b_{n+1}, n \ge n_0$) or eventually log-concave (i.e. $b_n^2 \ge b_{n-1}b_{n+1}, n \ge n_0$), then b_n is eventually strictly increasing to infinity.

Carleman plus "extra condition" implies indeterminacy

Theorem (1. B.-Szwarc(2012))

Assume that the coefficients a_n, b_n satisfy

$$\sum_{n=1}^{\infty} \frac{1+|a_n|}{\sqrt{b_n b_{n-1}}} < \infty,$$

and that b_n is either eventually log-convex or eventually log-concave. Then for a constant c independent of z

$$\sqrt{b_{n-1}}|P_n(z)| \le c\,\Pi(|z|), \quad \Pi(z) = \prod_{k=0}^{\infty}\left(1+\frac{z}{b_{k-1}}\right), \quad n \ge 0.$$

The moment problem is indeterminate.

This extends a result of Berezanskiĭ(1956).

Some comments

Under the conditions of the theorem one can further obtain

$$P_n^2(0) = O(1/b_{n-1}), \quad 1/b_n = o(1/n)$$

and

$$\frac{K}{b_{n+1}} \le |P_n(z)|^2 + |P_{n+1}(z)|^2 \le \frac{L}{b_{n-1}}$$

for suitable constants K, L depending on z (but not on n).
Similar results are true for Q_n.

In the symmetric case $a_n = 0$ the sum condition is equivalent to $\sum(1/b_n) < \infty$.

Some comments

Under the conditions of the theorem one can further obtain

$$P_n^2(0) = O(1/b_{n-1}), \quad 1/b_n = o(1/n)$$

and

$$\frac{K}{b_{n+1}} \le |P_n(z)|^2 + |P_{n+1}(z)|^2 \le \frac{L}{b_{n-1}}$$

for suitable constants K, L depending on z (but not on n). Similar results are true for Q_n.

In the symmetric case $a_n = 0$ the sum condition is equivalent to $\sum(1/b_n) < \infty$.

The Nevanlinna matrix and parametrization

From now on consider only indeterminate moment problems.
Consider the four functions

$$\begin{aligned} A(z) &= z\sum_{k=0}^{\infty} Q_k(0)Q_k(z), \\ B(z) &= -1 + z\sum_{k=0}^{\infty} Q_k(0)P_k(z), \\ C(z) &= 1 + z\sum_{k=0}^{\infty} P_k(0)Q_k(z), \\ D(z) &= z\sum_{k=0}^{\infty} P_k(0)P_k(z). \end{aligned}$$

Entire functions of minimal exponential type (M. Riesz(1923)).

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 1.$$

They can be used to describe the set V of all solutions to the moment problem as $V = \{\mu_\varphi \mid \varphi \in \mathcal{P}^*\}$, with

$$\int \frac{d\mu_\varphi(x)}{x-z} = -\frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Nevanlinna parametrization

The Nevanlinna matrix and parametrization

From now on consider only indeterminate moment problems.
Consider the four functions

$$\begin{aligned} A(z) &= z\sum_{k=0}^{\infty} Q_k(0)Q_k(z), \\ B(z) &= -1 + z\sum_{k=0}^{\infty} Q_k(0)P_k(z), \\ C(z) &= 1 + z\sum_{k=0}^{\infty} P_k(0)Q_k(z), \\ D(z) &= z\sum_{k=0}^{\infty} P_k(0)P_k(z). \end{aligned}$$

Entire functions of minimal exponential type (M. Riesz(1923)).

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 1.$$

They can be used to describe the set V of all solutions to the moment problem as $V = \{\mu_\varphi \mid \varphi \in \mathcal{P}^*\}$, with

$$\int \frac{d\mu_\varphi(x)}{x-z} = -\frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

Nevanlinna parametrization

Pick functions

Here $\mathcal{P}$ is the set of holomorphic functions (Pick functions)

$$\varphi : \{\Im(z) > 0\} \to \{\Im(w) \geq 0\}$$

and $\mathcal{P}^* = \mathcal{P} \cup \{\infty\}$.

If $\varphi(z) = i$ then

$$\mu_\varphi = \frac{dx}{\pi(B^2(x) + D^2(x))}$$

if $\varphi(z) = t \in \mathbb{R}$ then

$$\mu_\varphi = \sum_{\lambda \in \Lambda_t} (1/P^2(\lambda))\delta_\lambda, \quad \Lambda_t = \{x \mid B(x)t - D(x) = 0\}.$$

Pick functions

Here $\mathcal{P}$ is the set of holomorphic functions (Pick functions)

$$\varphi : \{\Im(z) > 0\} \to \{\Im(w) \geq 0\}$$

and $\mathcal{P}^* = \mathcal{P} \cup \{\infty\}$.

If $\varphi(z) = i$ then

$$\mu_\varphi = \frac{dx}{\pi(B^2(x) + D^2(x))}$$

if $\varphi(z) = t \in \mathbb{R}$ then

$$\mu_\varphi = \sum_{\lambda \in \Lambda_t} (1/P^2(\lambda))\delta_\lambda, \quad \Lambda_t = \{x \mid B(x)t - D(x) = 0\}.$$

Pick functions

Here $\mathcal{P}$ is the set of holomorphic functions (Pick functions)

$$\varphi : \{\Im(z) > 0\} \to \{\Im(w) \geq 0\}$$

and $\mathcal{P}^* = \mathcal{P} \cup \{\infty\}$.

If $\varphi(z) = i$ then

$$\mu_\varphi = \frac{dx}{\pi(B^2(x) + D^2(x))}$$

if $\varphi(z) = t \in \mathbb{R}$ then

$$\mu_\varphi = \sum_{\lambda \in \Lambda_t} (1/P^2(\lambda))\delta_\lambda, \quad \Lambda_t = \{x \mid B(x)t - D(x) = 0\}.$$

Order and type of moment problems

B.-Pedersen(1994) proved that the functions A, B, C, D, P, Q have the same order and type which we call the order ρ and type τ of the moment problem.

In case the order $\rho = 0$, B.-Pedersen(2005) proved that these functions have the same logarithmic order and type called logarithmic order $\rho^{[1]}$ and logarithmic type $\tau^{[1]}$ of the moment problem.

By Marcel Riesz: A, B, C, D are of minimal exponential type, hence $0 \le \rho \le 1$ and if $\rho = 1$ then $\tau = 0$.

Main question: How can we determine these numbers from the three term recurrence equation or from the moments?
In a manuscript with R. Szwarc we have some answers–to be explained here.

Order and type of moment problems

B.-Pedersen(1994) proved that the functions A, B, C, D, P, Q have the same order and type which we call the order ρ and type τ of the moment problem.
In case the order $\rho = 0$, B.-Pedersen(2005) proved that these functions have the same logarithmic order and type called logarithmic order $\rho^{[1]}$ and logarithmic type $\tau^{[1]}$ of the moment problem.

By Marcel Riesz: A, B, C, D are of minimal exponential type, hence $0 \leq \rho \leq 1$ and if $\rho = 1$ then $\tau = 0$.

Main question: How can we determine these numbers from the three term recurrence equation or from the moments?
In a manuscript with R. Szwarc we have some answers–to be explained here.

Order and type of moment problems

B.-Pedersen(1994) proved that the functions A, B, C, D, P, Q have the same order and type which we call the order ρ and type τ of the moment problem.
In case the order $\rho = 0$, B.-Pedersen(2005) proved that these functions have the same logarithmic order and type called logarithmic order $\rho^{[1]}$ and logarithmic type $\tau^{[1]}$ of the moment problem.

By Marcel Riesz: A, B, C, D are of minimal exponential type, hence $0 \leq \rho \leq 1$ and if $\rho = 1$ then $\tau = 0$.

Main question: How can we determine these numbers from the three term recurrence equation or from the moments?
In a manuscript with R. Szwarc we have some answers–to be explained here.

Order and type of moment problems

B.-Pedersen(1994) proved that the functions A, B, C, D, P, Q have the same order and type which we call the order ρ and type τ of the moment problem.
In case the order $\rho = 0$, B.-Pedersen(2005) proved that these functions have the same logarithmic order and type called logarithmic order $\rho^{[1]}$ and logarithmic type $\tau^{[1]}$ of the moment problem.

By Marcel Riesz: A, B, C, D are of minimal exponential type, hence $0 \le \rho \le 1$ and if $\rho = 1$ then $\tau = 0$.

Main question: How can we determine these numbers from the three term recurrence equation or from the moments?
In a manuscript with R. Szwarc we have some answers–to be explained here.

Order of functions

For a continuous unbounded function $f : \mathbb{C} \to \mathbb{C}$ define maximum modulus

$$M_f(r) = \max_{|z| \le r} |f(z)|, \quad r \ge 0.$$

Note that $\log M_f(r) > 0$ for r sufficiently large.

The order ρ_f of f is defined as the infimum of the numbers $\alpha > 0$ for which there exists a majorization of the form

$$\log M_f(r) \le_{\text{as}} r^{\alpha},$$

where the notation means that the inequality holds for r sufficiently large.

In other words: For all $\varepsilon > 0$ we have an estimate

$$|f(z)| \le \exp(r^{\rho_f + \varepsilon}), |z| \le r \text{ for } r = r(\varepsilon) \text{ sufficiently large}$$

but such an estimate cannot hold for any $\varepsilon < 0$.

$$\rho_f = \limsup_{r \to \infty} \frac{\log \log M_f(r)}{\log r}.$$

Order of functions

For a continuous unbounded function $f : \mathbb{C} \to \mathbb{C}$ define maximum modulus

$$M_f(r) = \max_{|z| \le r} |f(z)|, \quad r \ge 0.$$

Note that $\log M_f(r) > 0$ for r sufficiently large.
The order ρ_f of f is defined as the infimum of the numbers $\alpha > 0$ for which there exists a majorization of the form

$$\log M_f(r) \le_{\text{as}} r^{\alpha},$$

where the notation means that the inequality holds for r sufficiently large.

In other words: For all $\varepsilon > 0$ we have an estimate

$$|f(z)| \le \exp(r^{\rho_f + \varepsilon}), |z| \le r \text{ for } r = r(\varepsilon) \text{ sufficiently large}$$

but such an estimate cannot hold for any $\varepsilon < 0$.

$$\rho_f = \limsup_{r \to \infty} \frac{\log \log M_f(r)}{\log r}.$$

Order of functions

For a continuous unbounded function $f : \mathbb{C} \to \mathbb{C}$ define maximum modulus

$$M_f(r) = \max_{|z| \le r} |f(z)|, \quad r \ge 0.$$

Note that $\log M_f(r) > 0$ for r sufficiently large.
The order ρ_f of f is defined as the infimum of the numbers $\alpha > 0$ for which there exists a majorization of the form

$$\log M_f(r) \le_{\text{as}} r^{\alpha},$$

where the notation means that the inequality holds for r sufficiently large.
In other words: For all $\varepsilon > 0$ we have an estimate

$$|f(z)| \le \exp(r^{\rho_f + \varepsilon}), |z| \le r \text{ for } r = r(\varepsilon) \text{ sufficiently large}$$

but such an estimate cannot hold for any $\varepsilon < 0$.

$$\rho_f = \limsup_{r \to \infty} \frac{\log \log M_f(r)}{\log r}.$$

Type of functions, examples

If $0 < \rho_f < \infty$ we define the type τ_f of f as

$$\tau_f = \inf\{c > 0 \mid \log M_f(r) \leq_{\text{as}} cr^{\rho_f}\},$$

and we have

$$\tau_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}}.$$

$\exp(z), \sin(z), \cos(z)$ have order 1 and type 1.
$1/\Gamma(z)$ has order 1 and type ∞.
$\cos(\sqrt{z})$ has order $\frac{1}{2}$ and type 1.
The moment problem studied by B.-Valent (1994) has order and type

$$\rho = \tfrac{1}{4}, \quad \tau = \int_0^1 \frac{du}{\sqrt{1-u^4}}.$$

The coefficients of the three term recurrence relation are given by $a_n = \lambda_n + \mu_n,\ b_n = \sqrt{\lambda_n \mu_{n+1}}$ with

$$\lambda_n = (4n+1)(4n+2)^2(4n+3), \quad \mu_n = (4n-1)(4n)^2(4n+1).$$

Type of functions, examples

If $0 < \rho_f < \infty$ we define the type τ_f of f as

$$\tau_f = \inf\{c > 0 \mid \log M_f(r) \leq_{\text{as}} cr^{\rho_f}\},$$

and we have

$$\tau_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}}.$$

$\exp(z), \sin(z), \cos(z)$ have order 1 and type 1.
$1/\Gamma(z)$ has order 1 and type ∞.
$\cos(\sqrt{z})$ has order $\frac{1}{2}$ and type 1.
The moment problem studied by B.-Valent (1994) has order and type

$$\rho = \tfrac{1}{4}, \quad \tau = \int_0^1 \frac{du}{\sqrt{1-u^4}}.$$

The coefficients of the three term recurrence relation are given by $a_n = \lambda_n + \mu_n,\ b_n = \sqrt{\lambda_n \mu_{n+1}}$ with

$$\lambda_n = (4n+1)(4n+2)^2(4n+3), \quad \mu_n = (4n-1)(4n)^2(4n+1).$$

Type of functions, examples

If $0 < \rho_f < \infty$ we define the type τ_f of f as

$$\tau_f = \inf\{c > 0 \mid \log M_f(r) \leq_{\text{as}} cr^{\rho_f}\},$$

and we have

$$\tau_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}}.$$

$\exp(z), \sin(z), \cos(z)$ have order 1 and type 1.
$1/\Gamma(z)$ has order 1 and type ∞.
$\cos(\sqrt{z})$ has order $\frac{1}{2}$ and type 1.
The moment problem studied by B.-Valent (1994) has order and type

$$\rho = \tfrac{1}{4}, \quad \tau = \int_0^1 \frac{du}{\sqrt{1-u^4}}.$$

The coefficients of the three term recurrence relation are given by $a_n = \lambda_n + \mu_n,\ b_n = \sqrt{\lambda_n \mu_{n+1}}$ with

$$\lambda_n = (4n+1)(4n+2)^2(4n+3), \quad \mu_n = (4n-1)(4n)^2(4n+1).$$

Type of functions, examples

If $0 < \rho_f < \infty$ we define the type τ_f of f as

$$\tau_f = \inf\{c > 0 \,|\, \log M_f(r) \leq_{\text{as}} cr^{\rho_f}\},$$

and we have

$$\tau_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}}.$$

$\exp(z), \sin(z), \cos(z)$ have order 1 and type 1.
$1/\Gamma(z)$ has order 1 and type ∞.
$\cos(\sqrt{z})$ has order $\frac{1}{2}$ and type 1.
The moment problem studied by B.-Valent (1994) has order and type

$$\rho = \tfrac{1}{4}, \quad \tau = \int_0^1 \frac{du}{\sqrt{1-u^4}}.$$

The coefficients of the three term recurrence relation are given by $a_n = \lambda_n + \mu_n,\ b_n = \sqrt{\lambda_n \mu_{n+1}}$ with

$$\lambda_n = (4n+1)(4n+2)^2(4n+3), \quad \mu_n = (4n-1)(4n)^2(4n+1).$$

Type of functions, examples

If $0 < \rho_f < \infty$ we define the type τ_f of f as

$$\tau_f = \inf\{c > 0 \,|\, \log M_f(r) \leq_{\text{as}} cr^{\rho_f}\},$$

and we have

$$\tau_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}}.$$

$\exp(z), \sin(z), \cos(z)$ have order 1 and type 1.
$1/\Gamma(z)$ has order 1 and type ∞.
$\cos(\sqrt{z})$ has order $\frac{1}{2}$ and type 1.
The moment problem studied by B.-Valent (1994) has order and type

$$\rho = \tfrac{1}{4}, \quad \tau = \int_0^1 \frac{du}{\sqrt{1-u^4}}.$$

The coefficients of the three term recurrence relation are given by $a_n = \lambda_n + \mu_n,\ b_n = \sqrt{\lambda_n \mu_{n+1}}$ with

$$\lambda_n = (4n+1)(4n+2)^2(4n+3), \quad \mu_n = (4n-1)(4n)^2(4n+1).$$

Logarithmic order and type

Suppose the order of f is zero. Define the logarithmic order

$$\rho_f^{[1]} = \inf\{\alpha > 0 \mid M_f(r) \leq_{\text{as}} r^{(\log r)^\alpha}\}.$$

$$\rho_f^{[1]} = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log\log r} - 1.$$

When $\rho_f^{[1]} < \infty$ we define the logarithmic type $\tau_f^{[1]}$ as

$$\tau_f^{[1]} = \inf\{c > 0 \mid M_f(r) \leq_{\text{as}} r^{c(\log r)^{\rho_f^{[1]}}}\},$$

$$\tau_f^{[1]} = \limsup_{r\to\infty} \frac{\log M_f(r)}{(\log r)^{\rho_f^{[1]}+1}}.$$

An entire function f satisfying $\rho_f^{[1]} = 0$ and $\tau_f^{[1]} < \infty$ is necessarily a polynomial of degree $\leq \tau_f^{[1]}$.

Logarithmic order and type

Suppose the order of f is zero. Define the logarithmic order

$$\rho_f^{[1]} = \inf\{\alpha > 0 \,|\, M_f(r) \leq_{\text{as}} r^{(\log r)^\alpha}\}.$$

$$\rho_f^{[1]} = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log\log r} - 1.$$

When $\rho_f^{[1]} < \infty$ we define the logarithmic type $\tau_f^{[1]}$ as

$$\tau_f^{[1]} = \inf\{c > 0 \,|\, M_f(r) \leq_{\text{as}} r^{c(\log r)^{\rho_f^{[1]}}}\},$$

$$\tau_f^{[1]} = \limsup_{r\to\infty} \frac{\log M_f(r)}{(\log r)^{\rho_f^{[1]}+1}}.$$

An entire function f satisfying $\rho_f^{[1]} = 0$ and $\tau_f^{[1]} < \infty$ is necessarily a polynomial of degree $\leq \tau_f^{[1]}$.

Logarithmic order and type

Suppose the order of f is zero. Define the logarithmic order

$$\rho_f^{[1]} = \inf\{\alpha > 0 \mid M_f(r) \leq_{\text{as}} r^{(\log r)^\alpha}\}.$$

$$\rho_f^{[1]} = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log\log r} - 1.$$

When $\rho_f^{[1]} < \infty$ we define the logarithmic type $\tau_f^{[1]}$ as

$$\tau_f^{[1]} = \inf\{c > 0 \mid M_f(r) \leq_{\text{as}} r^{c(\log r)^{\rho_f^{[1]}}}\},$$

$$\tau_f^{[1]} = \limsup_{r\to\infty} \frac{\log M_f(r)}{(\log r)^{\rho_f^{[1]}+1}}.$$

An entire function f satisfying $\rho_f^{[1]} = 0$ and $\tau_f^{[1]} < \infty$ is necessarily a polynomial of degree $\leq \tau_f^{[1]}$.

A sharpening from ℓ^2 to ℓ^α

Theorem (2. B.-Szwarc(2012))

For a moment problem and $0 < \alpha \leq 1$ *the following conditions are equivalent:*

(i) $(P_n^2(0)), (Q_n^2(0)) \in \ell^\alpha$,

(ii) $(P_n^2(z)), (Q_n^2(z)) \in \ell^\alpha$ *for all* $z \in \mathbb{C}$.

If the conditions are satisfied, the moment problem is indeterminate and the two series indicated in (ii) converge uniformly on compact subsets of $\mathbb{C}$. *Furthermore*

$$(1/b_n) \in \ell^\alpha, \quad P(z) \leq C \exp(K|z|^\alpha) \text{ with}$$

$$C^2 = \sum_{n=0}^{\infty} (P_n^2(0) + Q_n^2(0)), \quad K = \frac{1}{\alpha} \sum_{n=0}^{\infty} (|P_n(0)|^{2\alpha} + |Q_n(0)|^{2\alpha}).$$

In particular the moment problem has order $\rho \leq \alpha$*, and if the order is* α*, then the type* $\tau \leq K$*.*

A partial connverse of Theorem 2

Theorem (3. B.-Szwarc(2012))

Assume that a_n, b_n satisfy

$$\sum_{n=1}^{\infty} \frac{1+|a_n|}{\sqrt{b_n b_{n-1}}} < \infty,$$

and the logconvex/concave conditions. Assume in addition that P satisfies

$$P(z) \leq C \exp(K|z|^{\alpha})$$

for some α such that $0 < \alpha < 1$ and suitable constants $C, K > 0$. Then

$$1/b_n, P_n^2(0), Q_n^2(0) = O(n^{-1/\alpha}),$$

so in particular $(1/b_n), (P_n^2(0)), (Q_n^2(0)) \in \ell^{\alpha+\varepsilon}$ for any $\varepsilon > 0$.

Exponent of convergence for a sequence

For a sequence (z_n) of complex numbers for which $|z_n| \to \infty$, we introduce the exponent of convergence

$$\mathcal{E}(z_n) = \inf \left\{ \alpha > 0 \mid \sum_{n=n^*}^{\infty} \frac{1}{|z_n|^\alpha} < \infty \right\},$$

where $n^* \in \mathbb{N}$ is such that $|z_n| > 0$ for $n \geq n^*$.
The counting function of (z_n) is defined as

$$n(r) = \#\{n \mid |z_n| \leq r\}.$$

The following result is well-known

Lemma

$$\mathcal{E}(z_n) = \limsup_{r \to \infty} \frac{\log n(r)}{\log r}.$$

Order and logarithmic order given by b_n

Theorem (4. B.-Szwarc(2012))

Assume that a_n, b_n satisfy

$$\sum_{n=1}^{\infty} \frac{1+|a_n|}{\sqrt{b_n b_{n-1}}} < \infty,$$

and that the logconvex/logconcave condition holds.
Then the order ρ of the moment problem is given by $\rho = \mathcal{E}(b_n)$.
If $\rho = 0$ then the logarithmic order $\rho^{[1]}$ of the moment problem is given by $\rho^{[1]} = \mathcal{E}(\log b_n)$.

Some examples

1. For $\alpha > 1$ let $b_n = (n+1)^\alpha, a_n = 0, n \geq 0$.
The three-term recurrence relation with these coefficients determine the orthonormal polynomials of a symmetric indeterminate moment problem satisfying the logconcavity condition and $\sum 1/b_n < \infty$. By Theorem 3 the order of the moment problem is $1/\alpha$.

2. $b_n = (n+1)\log^\alpha(n+2), a_n = 0$ lead for $\alpha > 1$ to a symmetric indeterminate moment problem of order 1 and type 0.

3. For $a > 1, \alpha > 0$ let

$$b_n = a^{n^{1/\alpha}}, \quad |a_n| \leq a^{cn^{1/\alpha}} \text{ where } 0 < c < 1.$$

$$b_n^2 \left\{ \begin{array}{c} = \\ < \\ > \end{array} \right\} b_{n-1} b_{n+1} \Leftrightarrow \left\{ \begin{array}{l} \alpha = 1 \\ \alpha < 1 \\ \alpha > 1 \end{array} \right. .$$

We find $\mathcal{E}(b_n) = 0$ and $\mathcal{E}(\log b_n) = \alpha$, so the moment problem has order 0 and logarithmic order $\rho^{[1]} = \alpha$.

Some examples

1. For $\alpha > 1$ let $b_n = (n+1)^\alpha, a_n = 0, n \geq 0$.
The three-term recurrence relation with these coefficients determine the orthonormal polynomials of a symmetric indeterminate moment problem satisfying the logconcavity condition and $\sum 1/b_n < \infty$. By Theorem 3 the order of the moment problem is $1/\alpha$.

2. $b_n = (n+1)\log^\alpha(n+2), a_n = 0$ lead for $\alpha > 1$ to a symmetric indeterminate moment problem of order 1 and type 0.

3. For $a > 1, \alpha > 0$ let

$$b_n = a^{n^{1/\alpha}}, \quad |a_n| \leq a^{cn^{1/\alpha}} \text{ where } 0 < c < 1.$$

$$b_n^2 \left\{ \begin{array}{c} = \\ < \\ > \end{array} \right\} b_{n-1}b_{n+1} \Leftrightarrow \left\{ \begin{array}{l} \alpha = 1 \\ \alpha < 1 \\ \alpha > 1 \end{array} \right. .$$

We find $\mathcal{E}(b_n) = 0$ and $\mathcal{E}(\log b_n) = \alpha$, so the moment problem has order 0 and logarithmic order $\rho^{[1]} = \alpha$.

Some examples

1. For $\alpha > 1$ let $b_n = (n+1)^\alpha, a_n = 0, n \geq 0$.
The three-term recurrence relation with these coefficients determine the orthonormal polynomials of a symmetric indeterminate moment problem satisfying the logconcavity condition and $\sum 1/b_n < \infty$. By Theorem 3 the order of the moment problem is $1/\alpha$.
2. $b_n = (n+1)\log^\alpha(n+2), a_n = 0$ lead for $\alpha > 1$ to a symmetric indeterminate moment problem of order 1 and type 0.
3. For $a > 1, \alpha > 0$ let

$$b_n = a^{n^{1/\alpha}}, \quad |a_n| \leq a^{cn^{1/\alpha}} \text{ where } 0 < c < 1.$$

$$b_n^2 \left\{ \begin{array}{c} = \\ < \\ > \end{array} \right\} b_{n-1}b_{n+1} \Leftrightarrow \left\{ \begin{array}{l} \alpha = 1 \\ \alpha < 1 \\ \alpha > 1 \end{array} \right. .$$

We find $\mathcal{E}(b_n) = 0$ and $\mathcal{E}(\log b_n) = \alpha$, so the moment problem has order 0 and logarithmic order $\rho^{[1]} = \alpha$.

Double logarithmic order and type

4. For $a, b > 1$ let

$$b_n = a^{b^n}, \quad |a_n| \leq a^{cb^n} \text{ with } bc < 1.$$

(b_n) is logconvex, and the coefficients lead to an indeterminate moment problem with order as well as logarithmic order equal to 0.
This motivates a refined growth scale: For an unbounded continuous function f we define the double logarithmic order $\rho_f^{[2]}$

$$\rho_f^{[2]} = \inf\{\alpha > 0 \,|\, M_f(r) \leq_{\text{as}} r^{(\log\log r)^\alpha}\}.$$

Of course $\rho_f^{[2]} < \infty$ is only possible if $\rho_f^{[1]} = 0$.
In case $0 < \rho^{[2]} = \rho_f^{[2]} < \infty$ we define the double logarithmic type

$$\tau_f^{[2]} = \inf\{c > 0 \,|\, M_f(r) \leq_{\text{as}} r^{c(\log\log r)^{\rho^{[2]}}}\}.$$

Double logarithmic order and type

4. For $a, b > 1$ let

$$b_n = a^{b^n}, \quad |a_n| \leq a^{cb^n} \text{ with } bc < 1.$$

(b_n) is logconvex, and the coefficients lead to an indeterminate moment problem with order as well as logarithmic order equal to 0. This motivates a refined growth scale: For an unbounded continuous function f we define the double logarithmic order $\rho_f^{[2]}$

$$\rho_f^{[2]} = \inf\{\alpha > 0 \mid M_f(r) \leq_{\text{as}} r^{(\log\log r)^\alpha}\}.$$

Of course $\rho_f^{[2]} < \infty$ is only possible if $\rho_f^{[1]} = 0$.

In case $0 < \rho^{[2]} = \rho_f^{[2]} < \infty$ we define the double logarithmic type

$$\tau_f^{[2]} = \inf\{c > 0 \mid M_f(r) \leq_{\text{as}} r^{c(\log\log r)^{\rho^{[2]}}}\}.$$

Double logarithmic order and type

4. For $a, b > 1$ let

$$b_n = a^{b^n}, \quad |a_n| \le a^{cb^n} \text{ with } bc < 1.$$

(b_n) is logconvex, and the coefficients lead to an indeterminate moment problem with order as well as logarithmic order equal to 0. This motivates a refined growth scale: For an unbounded continuous function f we define the double logarithmic order $\rho_f^{[2]}$

$$\rho_f^{[2]} = \inf\{\alpha > 0 \,|\, M_f(r) \le_{\text{as}} r^{(\log\log r)^\alpha}\}.$$

Of course $\rho_f^{[2]} < \infty$ is only possible if $\rho_f^{[1]} = 0$.
In case $0 < \rho^{[2]} = \rho_f^{[2]} < \infty$ we define the double logarithmic type

$$\tau_f^{[2]} = \inf\{c > 0 \,|\, M_f(r) \le_{\text{as}} r^{c(\log\log r)^{\rho^{[2]}}}\}.$$

Double logarithmic order and type of moment problems

Theorem (5. (B.-Szwarc(2012))

For an indeterminate moment problem of logarithmic order zero the functions A, B, C, D, P, Q have the same double logarithmic order $\rho^{[2]}$ and type $\tau^{[2]}$ called the double logarithmic order and type of the moment problem.
Under the sum condition and the logconvex/logconcave condition

$$\rho^{[2]} = \mathcal{E}(\log\log b_n).$$

Example **5.** $b_n = \exp(e^{n^{1/\alpha}})$ is eventually log-convex because $\exp(x^{1/\alpha})$ is convex for $x > (\alpha-1)^\alpha$ when $\alpha > 1$ and convex for $x > 0$ when $0 < \alpha \leq 1$. The indeterminate moment problem with recurrence coefficients $a_n = 0$ and b_n as above has double logarithmic order equal to $\mathcal{E}(\log\log b_n) = \alpha$.

Double logarithmic order and type of moment problems

Theorem (5. (B.-Szwarc(2012))

For an indeterminate moment problem of logarithmic order zero the functions A, B, C, D, P, Q have the same double logarithmic order $\rho^{[2]}$ and type $\tau^{[2]}$ called the double logarithmic order and type of the moment problem.
Under the sum condition and the logconvex/logconcave condition

$$\rho^{[2]} = \mathcal{E}(\log\log b_n).$$

Example **5.** $b_n = \exp(e^{n^{1/\alpha}})$ is eventually log-convex because $\exp(x^{1/\alpha})$ is convex for $x > (\alpha - 1)^\alpha$ when $\alpha > 1$ and convex for $x > 0$ when $0 < \alpha \leq 1$. The indeterminate moment problem with recurrence coefficients $a_n = 0$ and b_n as above has double logarithmic order equal to $\mathcal{E}(\log\log b_n) = \alpha$.

Livšic's function

For an indeterminate moment sequence (s_n) Livšic(1939) considered the function

$$L(z) = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{s_{2n}}}.$$

It is entire of minimal exponential type.

Livšic proved $\rho_L \leq \rho$, where ρ is the order of the moment problem.

It is interesting to know whether the equality sign holds.

In fact, we do not know any example with $\rho_L < \rho$.

Consider also the following entire functions

$$G(z) = \sum_{n=0}^{\infty} \frac{z^n}{b_n^n}, \quad H(z) = \sum_{n=0}^{\infty} b_{n,n} z^n.$$

Livšic's function

For an indeterminate moment sequence (s_n) Livšic(1939) considered the function

$$L(z)=\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{s_{2n}}}.$$

It is entire of minimal exponential type.
Livšic proved $\rho_L \le \rho$, where ρ is the order of the moment problem.
It is interesting to know whether the equality sign holds.
In fact, we do not know any example with $\rho_L < \rho$.
Consider also the following entire functions

$$G(z)=\sum_{n=0}^{\infty}\frac{z^n}{b_n^n},\quad H(z)=\sum_{n=0}^{\infty}b_{n,n}z^n.$$

Livšic's function

For an indeterminate moment sequence (s_n) Livšic(1939) considered the function

$$L(z) = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{s_{2n}}}.$$

It is entire of minimal exponential type.
Livšic proved $\rho_L \leq \rho$, where ρ is the order of the moment problem.
It is interesting to know whether the equality sign holds.
In fact, we do not know any example with $\rho_L < \rho$.
Consider also the following entire functions

$$G(z) = \sum_{n=0}^{\infty} \frac{z^n}{b_n^n}, \quad H(z) = \sum_{n=0}^{\infty} b_{n,n} z^n.$$

A final result

Theorem (6. (B.-Szwarc(2012))

Given an (indeterminate) moment problem where

$$\sum_{n=1}^{\infty} \frac{1+|a_n|}{\sqrt{b_n b_{n-1}}} < \infty,$$

and the logconvex/logconcave condition holds.
Then

(i) $\rho = \rho_G = \rho_H = \rho_L = \mathcal{E}(b_n)$.
If $\rho = 0$ *then*

(ii) $\rho^{[1]} = \rho_G^{[1]} = \rho_H^{[1]} = \rho_L^{[1]} = \mathcal{E}(\log b_n)$.
If $\rho^{[1]} = 0$ *then*

(iii) $\rho^{[2]} = \rho_G^{[2]} = \rho_H^{[2]} = \rho_L^{[2]} = \mathcal{E}(\log\log b_n)$.

References 1

N. I. Akhiezer, *The Classical Moment Problem and Some Related Questions in Analysis*. English translation, Oliver and Boyd, Edinburgh, 1965.

C. Berg, J. P. R. Christensen, *Density questions in the classical theory of moments*, Ann. Inst. Fourier, Grenoble **31**,3 (1981), 99–114.

C. Berg, R. Szwarc, *On the order of indeterminate moment problems*, Manuscript

C. Berg, G. Valent, *The Nevanlinna parametrization for some indeterminate Stieltjes moment problems associated with birth and death processes*, Methods and Applications of Analysis **1** (1994), 169–209.

Yu. M. Berezanskiĭ, *Expansion according to eigenfunction of a partial difference equation of order two*, Trudy Moskov. Mat. Obšč. **5** (1956), 203–268. (In Russian).

References 2

- J. S. Christiansen, *The moment problem associated with the Stieltjes–Wigert polynomials*, J. Math. Anal. Appl. **277** (2003), 218–245.
- J. Shohat, J. D. Tamarkin, *The Problem of Moments.* Revised edition, American Mathematical Society, Providence, 1950.
- B. Simon, *The classical moment problem as a self-adjoint finite difference operator*, Adv. Math. **137**(1998), 82–203.
- T.J. Stieltjes, *Recherches sur les fractions continues*, Annales de la Faculté des Sciences de Toulouse, **8** (1894), 1–122; **9** (1895), 5–47. English translation in Thomas Jan Stieltjes, *Collected papers*, Vol. II, pp. 609–745. Springer-Verlag, Berlin, Heidelberg. New York, 1993.

Muchas gracias por la atención