On the order of indeterminate moment problems based on the recurrence coefficients

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Based on joint work with
Ryszard Szwarc
University of Wrocław, Poland
Figure: OSCA, Copenhagen, June 11-15, 2012

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Indeterminate moment problems
1. Introduction to indeterminate moment problems
2. Growth properties of functions: order and type; logarithmic order and type
3. The order and logarithmic order of moment problems
4. How can these numbers be determined from the three term recurrence or the moments?
5. Some answers to 4 by B. and Szwarc
6. References
A system of orthonormal polynomials is given by a three term recurrence relation

\[ zP_n(z) = b_nP_{n+1}(z) + a_nP_n(z) + b_{n-1}P_{n-1}(z), \quad n \geq 0 \]

where \( P_0 = 1, P_{-1} = 0 \) and \( a_n \in \mathbb{R}, b_n > 0 \). Equivalently \( P_n \) is a polynomial of degree \( n \) and there exists a probability measure \( \mu \) on \( \mathbb{R} \) such that

\[ \int P_n P_m \, d\mu = \delta_{n,m}. \]

We need the coefficients of the orthonormal polynomials

\[ P_n(x) = \sum_{k=0}^{n} b_{k,n} x^k, \]

\( P_n \) is uniquely determined by the orthonormality if we assume the leading coefficient \( b_{n,n} > 0 \) and we have

\[ b_{n,n} = \frac{1}{(b_0 b_1 \cdots b_{n-1})}. \]
A system of orthonormal polynomials is given by a \textbf{three term recurrence relation}

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The measure $\mu$ need not be uniquely determined, but all possible orthonormality measures have the same moments and vice versa

$$ s_n = \int_{-\infty}^{\infty} x^n \, d\mu(x), \quad n \geq 0. $$

Dichotomy:
The determinate case: $\mu$ is uniquely determined
The indeterminate case: There are several (and then infinitely many) $\mu$. Stieltjes was the first to observe this dichotomy around 1892 in correspondence with his mentor Hermite. Letter 325 from Stieltjes to Hermite dated January 30, 1892:
L’existence de ces fonctions $\varphi(u)$ qui, sans être nulles, sont telles que

$$ \int_{0}^{\infty} u^k \varphi(u) \, du = 0, \quad (k = 0, 1, 2, 3, \cdots) $$

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Figure: Jan Thomas Stieltjes, 1856-1894
Stieltjes’ example

Let $0 < q < 1$ be given by $q = e^{-\sigma^2}, \sigma > 0$.

$$d_\sigma(x) = (2\pi\sigma^2)^{-\frac{1}{2}}x^{-1}\exp\left(-\frac{(\log x)^2}{2\sigma^2}\right)$$

is a probability density: lognormal density. The moment sequence is

$$s_n(d_\sigma) = q^{-\frac{1}{2}n^2}, \quad n \geq 0.$$  

Stieltjes showed that these moments belong to an indeterminate moment problem by pointing out that all the densities ($s \in [-1, 1]$)

$$d_\sigma(x) \left(1 + s \sin\left(\frac{2\pi}{\sigma^2} \log x\right)\right)$$

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Chihara-Leipnik’s discrete example

Chihara(1970) and later Leipnik(1981) gave the following family of discrete measures with these moments. For \( a > 0 \) define the discrete probability

\[
\lambda_a = \frac{1}{L(a)} \sum_{k=-\infty}^{\infty} a^k q^{\frac{1}{2}k^2} \delta_{aq^k},
\]

where

\[
L(a) = \sum_{k=-\infty}^{\infty} a^k q^{\frac{1}{2}k^2} = (q, -\sqrt{q} a, -\sqrt{q}/a; q)_{\infty}.
\]

It is easy to calculate the moments of \( \lambda_a \) using the translation invariance of \( \sum_{-\infty}^{\infty} \). In fact

\[
s_n(\lambda_a) = \frac{1}{L(a)} \sum_{k=-\infty}^{\infty} a^k q^{\frac{1}{2}k^2} (aq^k)^n = \frac{q^{-\frac{1}{2}n^2}}{L(a)} \sum_{k=-\infty}^{\infty} a^{k+n} q^{\frac{1}{2}(k+n)^2}
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Pakes(1996) and later Christiansen(2003) proved the following:

Any positive finite measure $\nu$ on $(q, 1]$ can be extended to a positive finite measure $\mu$ on the half-line such that

$$\frac{1}{\mu([0, \infty[)} \int_0^\infty x^n \, d\mu(x) = q^{-\frac{1}{2}n^2}, \quad n \geq 0.$$ 

In this way one can construct solutions to the log-normal moment problem with very special properties. If e.g. $\nu$ is chosen continuous singular then so is $\mu$. 
A general result about indeterminate moment problems

Theorem (B-Christensen(1981))

Let $V$ be the compact convex set of solutions to an arbitrary indeterminate moment problem. Then each of the following three subsets are dense in $V$:

(i) $V_1 = \{ f(x)dx \in V \mid f \in C^\infty(\mathbb{R}) \}$,
(ii) $V_2 = \{ \mu \in V \mid \mu \text{ is discrete} \}$,
(iii) $V_3 = \{ \mu \in V \mid \mu \text{ is continuous singular} \}$
Let $Q_n$ be the polynomials of the second kind

$$Q_n(x) = \int \frac{P_n(x) - P_n(y)}{x - y} \, d\mu(y).$$

**Classic** The following conditions are equivalent

(i) $\mu$ is indeterminate,
(ii) $P_n(0), Q_n(0) \in \ell^2$,
(iii) $\sum_{n=0}^{\infty} |P_n(i)|^2 < \infty$
(iv) $\sum_{n=0}^{\infty} |P_n(z)|^2$ converges locally uniformly for $z \in C$,
(v) The operator $T(p) = xp(x)$ in $\overline{C[x]}^{L^2(\mu)}$ has deficiency indices $(1, 1)$.

In the indeterminate case consider:

$$P(z) = \left(\sum_{n=0}^{\infty} |P_n(z)|^2\right)^{1/2}, \quad Q(z) = \left(\sum_{n=0}^{\infty} |Q_n(z)|^2\right)^{1/2}.$$

(Note: also $\sum |Q_n(z)|^2$ converges locally uniformly in $C$)

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Indeterminate moment problems
Characterizations of indeterminate moment problems

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(Note: also $\sum |Q_n(z)|^2$ converges locally uniformly in $\mathbb{C}$)
It is well known that the moment problem is determinate with a measure $\mu$ of compact support iff $a_n, b_n$ are bounded.

Carleman’s criterion (1926): Indeterminacy $\Rightarrow \sum (1/b_n) < \infty$.

Warning: There exist determinate problems with $\sum (1/b_n) < \infty$.

**Lemma**

Let $b_n > 0$ satisfy $\sum (1/b_n) < \infty$ and that $b_n$ is either eventually log-convex (i.e. $b_n^2 \leq b_{n-1}b_{n+1}, n \geq n_0$) or eventually log-concave (i.e. $b_n^2 \geq b_{n-1}b_{n+1}, n \geq n_0$), then $b_n$ is eventually strictly increasing to infinity.
Indeterminacy in terms of coefficients $a_n, b_n$

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Carleman plus “extra condition” implies indeterminacy

**Theorem (1. B.-Szwarc(2012))**

Assume that the coefficients $a_n, b_n$ satisfy

$$
\sum_{n=1}^{\infty} \frac{1 + |a_n|}{\sqrt{b_n b_{n-1}}} < \infty,
$$

and that $b_n$ is either eventually log-convex or eventually log-concave. Then for a constant $c$ independent of $z$

$$
\sqrt{b_{n-1}} |P_n(z)| \leq c \Pi(|z|), \quad \Pi(z) = \prod_{k=0}^{\infty} \left(1 + \frac{z}{b_{k-1}}\right), \quad n \geq 0.
$$

The moment problem is indeterminate.

This extends a result of Berezanskiĭ(1956).
Some comments

Under the conditions of the theorem one can further obtain

\[ P_n^2(0) = O(1/b_{n-1}), \quad 1/b_n = o(1/n) \]

and

\[ \frac{K}{b_{n+1}} \leq |P_n(z)|^2 + |P_{n+1}(z)|^2 \leq \frac{L}{b_{n-1}} \]

for suitable constants \( K, L \) depending on \( z \) (but not on \( n \)).

Similar results are true for \( Q_n \).

In the symmetric case \( a_n = 0 \) the sum condition is equivalent to
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From now on consider only indeterminate moment problems. Consider the four functions

\[
A(z) = z \sum_{k=0}^{\infty} Q_k(0) Q_k(z),
\]
\[
B(z) = -1 + z \sum_{k=0}^{\infty} Q_k(0) P_k(z),
\]
\[
C(z) = 1 + z \sum_{k=0}^{\infty} P_k(0) Q_k(z),
\]
\[
D(z) = z \sum_{k=0}^{\infty} P_k(0) P_k(z).
\]

Entire functions of minimal exponential type (M. Riesz(1923)).

\[
\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 1.
\]

They can be used to describe the set \( V \) of all solutions to the moment problem as

\[
V = \{ \mu_\varphi \mid \varphi \in \mathcal{P}^* \},
\]

with

\[
\int \frac{d\mu_\varphi(x)}{x - z} = -\frac{A(z)\varphi(z) - C(z)}{B(z)\varphi(z) - D(z)}, \quad z \in \mathbb{C} \setminus \mathbb{R}.
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The Nevanlinna matrix and parametrization

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Nevanlinna parametrization
Here $\mathcal{P}$ is the set of holomorphic functions (Pick functions)

$$\varphi : \{\Im(z) > 0\} \rightarrow \{\Im(w) \geq 0\}$$

and $\mathcal{P}^* = \mathcal{P} \cup \{\infty\}$.

If $\varphi(z) = i$ then

$$\mu_{\varphi} = \frac{dx}{\pi (B^2(x) + D^2(x))}$$

if $\varphi(z) = t \in \mathbb{R}$ then

$$\mu_{\varphi} = \sum_{\lambda \in \Lambda_t} \frac{1}{P^2(\lambda)} \delta_{\lambda}, \quad \Lambda_t = \{x \mid B(x)t - D(x) = 0\}.$$
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B.-Pedersen(1994) proved that the functions $A, B, C, D, P, Q$ have the same order and type which we call the order $\rho$ and type $\tau$ of the moment problem.

In case the order $\rho = 0$, B.-Pedersen(2005) proved that these functions have the same logarithmic order and type called logarithmic order $\rho^{[1]}$ and logarithmic type $\tau^{[1]}$ of the moment problem.

By Marcel Riesz: $A, B, C, D$ are of minimal exponential type, hence $0 \leq \rho \leq 1$ and if $\rho = 1$ then $\tau = 0$.

**Main question:** How can we determine these numbers from the three term recurrence equation or from the moments?

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Order of functions

For a continuous unbounded function $f : \mathbb{C} \rightarrow \mathbb{C}$ define maximum modulus

$$M_f(r) = \max_{|z| \leq r} |f(z)|, \quad r \geq 0.$$ 

Note that $\log M_f(r) > 0$ for $r$ sufficiently large.

The order $\rho_f$ of $f$ is defined as the infimum of the numbers $\alpha > 0$ for which there exists a majorization of the form

$$\log M_f(r) \leq_{\text{as}} r^\alpha,$$

where the notation means that the inequality holds for $r$ sufficiently large.

In other words: For all $\varepsilon > 0$ we have an estimate

$$|f(z)| \leq \exp(r^{\rho_f + \varepsilon}), |z| \leq r \text{ for } r = r(\varepsilon) \text{ sufficiently large}$$

but such an estimate cannot hold for any $\varepsilon < 0$.

$$\rho_f = \limsup_{r \rightarrow \infty} \frac{\log \log M_f(r)}{\log r}.$$
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Type of functions, examples

If $0 < \rho_f < \infty$ we define the type $\tau_f$ of $f$ as

$$\tau_f = \inf\{ c > 0 \mid \log M_f(r) \leq_{\text{as}} cr^\rho_f \},$$

and we have

$$\tau_f = \limsup_{r \to \infty} \frac{\log M_f(r)}{r^\rho_f}.$$ 

$\exp(z), \sin(z), \cos(z)$ have order 1 and type 1.

$1/\Gamma(z)$ has order 1 and type $\infty$.

$\cos(\sqrt{z})$ has order $\frac{1}{2}$ and type 1.

The moment problem studied by B.-Valent (1994) has order and type

$$\rho = \frac{1}{4}, \quad \tau = \int_0^1 \frac{du}{\sqrt{1 - u^4}}.$$

The coefficients of the three term recurrence relation are given by

$$a_n = \lambda_n + \mu_n, \quad b_n = \sqrt{\lambda_n \mu_{n+1}}$$ 

with

$$\lambda_n = (4n+1)(4n+2)^2(4n+3), \quad \mu_n = (4n-1)(4n)^2(4n+1).$$
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$\exp(z), \sin(z), \cos(z)$ have order 1 and type 1. $1/\Gamma(z)$ has order 1 and type $\infty$. $\cos(\sqrt{z})$ has order $\frac{1}{2}$ and type 1.

The moment problem studied by B.-Valent (1994) has order and type

$$\rho = \frac{1}{4}, \quad \tau = \int_0^1 \frac{du}{\sqrt{1 - u^4}}.$$

The coefficients of the three term recurrence relation are given by

$$a_n = \lambda_n + \mu_n, \quad b_n = \sqrt{\lambda_n \mu_{n+1}}$$

with

$$\lambda_n = (4n + 1)(4n + 2)^2(4n + 3), \quad \mu_n = (4n - 1)(4n)^2(4n + 1).$$
If $0 < \rho_f < \infty$ we define the type $\tau_f$ of $f$ as

$$\tau_f = \inf\{c > 0 \mid \log M_f(r) \leq_{as} cr^{\rho_f}\},$$

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Logarithmic order and type

Suppose the order of $f$ is zero. Define the logarithmic order

$$\rho^{[1]}_f = \inf\{\alpha > 0 \mid M_f(r) \leq_{as} r^{(\log r)^{\alpha}}\}.$$ 

$$\rho^{[1]}_f = \limsup_{r \to \infty} \frac{\log \log M_f(r)}{\log \log r} - 1.$$ 

When $\rho^{[1]}_f < \infty$ we define the logarithmic type $\tau^{[1]}_f$ as

$$\tau^{[1]}_f = \inf\{c > 0 \mid M_f(r) \leq_{as} r^{c(\log r)^{\rho^{[1]}_f}}\},$$

$$\tau^{[1]}_f = \limsup_{r \to \infty} \frac{\log M_f(r)}{(\log r)^{\rho^{[1]}_f}}.$$ 

An entire function $f$ satisfying $\rho^{[1]}_f = 0$ and $\tau^{[1]}_f < \infty$ is necessarily a polynomial of degree $\leq \tau^{[1]}_f$. 

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Indeterminate moment problems
Logarithmic order and type

Suppose the order of $f$ is zero. Define the logarithmic order

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\rho_f^{[1]} = \inf \{ \alpha > 0 \mid M_f(r) \leq_{\text{as}} r^{(\log r)^\alpha} \}.
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\tau_f^{[1]} = \limsup_{r \to \infty} \frac{\log M_f(r)}{(\log r)^{\rho_f^{[1]}}+1}.
\]

An entire function \( f \) satisfying \( \rho_f^{[1]} = 0 \) and \( \tau_f^{[1]} < \infty \) is necessarily a polynomial of degree \( \leq \tau_f^{[1]} \).
Theorem (2. B.-Szwarc(2012))

For a moment problem and $0 < \alpha \leq 1$ the following conditions are equivalent:

(i) $(P_n^2(0)), (Q_n^2(0)) \in \ell^\alpha$,

(ii) $(P_n^2(z)), (Q_n^2(z)) \in \ell^\alpha$ for all $z \in \mathbb{C}$.

If the conditions are satisfied, the moment problem is indeterminate and the two series indicated in (ii) converge uniformly on compact subsets of $\mathbb{C}$. Furthermore

$$(1/b_n) \in \ell^\alpha, \quad P(z) \leq C \exp(K|z|^\alpha)$$

with

$$C^2 = \sum_{n=0}^{\infty} (P_n^2(0) + Q_n^2(0)), \quad K = \frac{1}{\alpha} \sum_{n=0}^{\infty} (|P_n(0)|^{2\alpha} + |Q_n(0)|^{2\alpha}).$$

In particular the moment problem has order $\rho \leq \alpha$, and if the order is $\alpha$, then the type $\tau \leq K$. 

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Indeterminate moment problems
A partial converse of Theorem 2

Theorem (3. B.-Szwarc(2012))

Assume that $a_n, b_n$ satisfy

$$
\sum_{n=1}^{\infty} \frac{1 + |a_n|}{\sqrt{b_n b_{n-1}}} < \infty,
$$

and the logconvex/concave conditions. Assume in addition that $P$ satisfies

$$
P(z) \leq C \exp(K|z|^\alpha)
$$

for some $\alpha$ such that $0 < \alpha < 1$ and suitable constants $C, K > 0$. Then

$$
1/b_n, P_n^2(0), Q_n^2(0) = O(n^{-1/\alpha}),
$$

so in particular $(1/b_n), (P_n^2(0)), (Q_n^2(0)) \in \ell^{\alpha+\varepsilon}$ for any $\varepsilon > 0$. 
Exponent of convergence for a sequence

For a sequence \((z_n)\) of complex numbers for which \(|z_n| \to \infty\), we introduce the exponent of convergence

\[
\mathcal{E}(z_n) = \inf \left\{ \alpha > 0 \mid \sum_{n=n^*}^{\infty} \frac{1}{|z_n|^\alpha} < \infty \right\},
\]

where \(n^* \in \mathbb{N}\) is such that \(|z_n| > 0\) for \(n \geq n^*\).

The counting function of \((z_n)\) is defined as

\[
n(r) = #\{n \mid |z_n| \leq r\}.
\]

The following result is well-known

**Lemma**

\[
\mathcal{E}(z_n) = \limsup_{r \to \infty} \frac{\log n(r)}{\log r}.
\]
Assume that $a_n, b_n$ satisfy

$$\sum_{n=1}^{\infty} \frac{1 + |a_n|}{\sqrt{b_n b_{n-1}}} < \infty,$$

and that the logconvex/logconcave condition holds.

Then the order $\rho$ of the moment problem is given by $\rho = \mathcal{E}(b_n)$.

If $\rho = 0$ then the logarithmic order $\rho^{[1]}$ of the moment problem is given by $\rho^{[1]} = \mathcal{E}(\log b_n)$. 
Some examples

1. For $\alpha > 1$ let $b_n = (n + 1)^\alpha$, $a_n = 0$, $n \geq 0$.
   The three-term recurrence relation with these coefficients determine the orthonormal polynomials of a symmetric indeterminate moment problem satisfying the logconcavity condition and $\sum 1/b_n < \infty$. By Theorem 3 the order of the moment problem is $1/\alpha$.

2. $b_n = (n + 1) \log^\alpha(n + 2)$, $a_n = 0$ lead for $\alpha > 1$ to a symmetric indeterminate moment problem of order 1 and type 0.

3. For $a > 1$, $\alpha > 0$ let

\[
b_n = a^{n^{1/\alpha}}, \quad |a_n| \leq a^{cn^{1/\alpha}} \quad \text{where} \quad 0 < c < 1.
\]

\[
b_n^2 \begin{cases} = \\ < \\ > \end{cases} b_{n-1}b_{n+1} \iff \begin{cases} \alpha = 1 \\ \alpha < 1 \\ \alpha > 1 \end{cases}.
\]

We find $\mathcal{E}(b_n) = 0$ and $\mathcal{E}(\log b_n) = \alpha$, so the moment problem has order 0 and logarithmic order $\rho^{[1]} = \alpha$. 

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Indeterminate moment problems
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   \[ b_n = a^{n^{1/\alpha}}, \quad |a_n| \leq a^{cn^{1/\alpha}} \text{ where } 0 < c < 1. \]
   \[
   b_n^2 \begin{cases} = & \alpha = 1 \\ < & \alpha < 1 \\ > & \alpha > 1 \end{cases} b_{n-1}b_{n+1} \Leftrightarrow \begin{cases} \alpha = 1 \\ \alpha < 1 \\ \alpha > 1 \end{cases}.
   \]
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Some examples

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Indeterminate moment problems
4. For $a, b > 1$ let

$$b_n = a^{b^n}, \quad |a_n| \leq a^{cb^n} \text{ with } bc < 1.$$  

$(b_n)$ is logconvex, and the coefficients lead to an indeterminate moment problem with order as well as logarithmic order equal to 0. This motivates a refined growth scale: For an unbounded continuous function $f$ we define the double logarithmic order $\rho_f^{[2]}$

$$\rho_f^{[2]} = \inf\{ \alpha > 0 \mid M_f(r) \leq_{\text{as}} r^{(\log \log r)^{\alpha}} \}.$$  

Of course $\rho_f^{[2]} < \infty$ is only possible if $\rho_f^{[1]} = 0$. In case $0 < \rho_f^{[2]} = \rho_f^{[2]} < \infty$ we define the double logarithmic type $\tau_f^{[2]}$

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Double logarithmic order and type

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Theorem (5. (B.-Szwarc(2012)))

For an indeterminate moment problem of logarithmic order zero the functions \( A, B, C, D, P, Q \) have the same double logarithmic order \( \rho^2 \) and type \( \tau^2 \) called the double logarithmic order and type of the moment problem. Under the sum condition and the logconvex/logconcave condition

\[
\rho^2 = E(\log \log b_n).
\]

Example 5. \( b_n = \exp(e^{n^{1/\alpha}}) \) is eventually log-convex because \( \exp(x^{1/\alpha}) \) is convex for \( x > (\alpha - 1)^\alpha \) when \( \alpha > 1 \) and convex for \( x > 0 \) when \( 0 < \alpha \leq 1 \). The indeterminate moment problem with recurrence coefficients \( a_n = 0 \) and \( b_n \) as above has double logarithmic order equal to \( E(\log \log b_n) = \alpha \).
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Livšic’s function

For an indeterminate moment sequence \((s_n)\) Livšic(1939) considered the function

\[
L(z) = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{s_{2n}}}. 
\]

It is entire of minimal exponential type.

Livšic proved \(\rho_L \leq \rho\), where \(\rho\) is the order of the moment problem. It is interesting to know whether the equality sign holds. In fact, we do not know any example with \(\rho_L < \rho\).

Consider also the following entire functions

\[
G(z) = \sum_{n=0}^{\infty} \frac{z^n}{b_n^n}, \quad H(z) = \sum_{n=0}^{\infty} b_{n,n} z^n. 
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\]
Theorem (6. (B.-Szwarc(2012)))

Given an (indeterminate) moment problem where

\[ \sum_{n=1}^{\infty} \frac{1 + |a_n|}{\sqrt{b_n b_{n-1}}} < \infty, \]

and the logconvex/logconcave condition holds.

Then

(i) \( \rho = \rho_G = \rho_H = \rho_L = \mathcal{E}(b_n). \)
   If \( \rho = 0 \) then

(ii) \( \rho^{[1]} = \rho_G^{[1]} = \rho_H^{[1]} = \rho_L^{[1]} = \mathcal{E}(\log b_n). \)
     If \( \rho^{[1]} = 0 \) then

(iii) \( \rho^{[2]} = \rho_G^{[2]} = \rho_H^{[2]} = \rho_L^{[2]} = \mathcal{E}(\log \log b_n). \)
References


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