Fusion systems, groups, partial groups and simplicial sets

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Outline

Introduction

Fusion systems

Partial groups

Localities

Homotopy fixed points
Introduction

In a paper published in 2003, together with Ran Levi and Bob Oliver, we introduced the concept of $p$-local finite group.

For a fixed prime $p$, this is a triple $(S, F, L)$, where

- $S$ is a finite $p$-group
- $F$ is a saturated fusion system over $S$, and
- $L$ is an associated centric linking system.

This last is a category extending the fusion system and we define the classifying space of $(S, F, L)$ as the $p$-completed nerve $|L|^\wedge_p$

The aim of this talk is to describe the homotopy fixed point set

$$(|L|^\wedge_p)^{h\pi}$$

by the action of a finite $p$-group $\pi$ on $(S, F, L)$.

We describe new models for the classifying space due to Andy Chermak (2013) and point to a precise one, $L_?$, that carries an action of $\pi$, and for which we can prove

$$[(L_?)^\wedge_p]^{h\pi} \simeq \bigsqcup_{\sigma \in H^1(\pi, L_?)} (L_?^\sigma)^\wedge_p$$

Same result when $\pi$ acts on a finite group $G$ was shown by J. Lannes (1986).
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$$\left( (\mathbb{L}_?)_p \right)^{h\pi} \approx \bigcup_{\sigma \in H^1(\pi, \mathbb{L}_?)} (\mathbb{L}^\sigma_?)_p$$

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Fusion systems

Definition

Let $G$ be a finite group, $p$ a prime number, and $S \in \text{Syl}_p(G)$. The fusion system of $G$ consists of

- Objects: $P \leq S$, the subgroups of $S$, and
- Morphisms:

$$\text{Hom}_{\mathcal{F}_S(G)}(P, Q) = \{ \varphi: P \to Q \mid \exists g \in G, \varphi(x) = gxg^{-1} \}$$

$$\cong N_G(P, Q)/C_G(P)$$

that compose as group homomorphisms.
**Fusion systems**

**Definition (Puig)**

A fusion system $\mathcal{F}$ over a finite $p$-group $S$ consists of a set $\text{Hom}_\mathcal{F}(P, Q)$ for every pair $P, Q$ of subgroups of $S$ such that

$$\text{Hom}_S(P, Q) \subseteq \text{Hom}_\mathcal{F}(P, Q) \subseteq \text{Inj}(P, Q)$$

and form a category where every morphism decomposes as an isomorphism followed by an inclusion.

It is saturated if it satisfies some extra axioms.

- If $G$ is a finite group and $S \in \text{Syl}_p(G)$, then $\mathcal{F}_S(G)$ is a saturated fusion system.
- We will say that a saturated fusion system $\mathcal{F}$ is *exotic* if it is not of this form.

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There are exotic saturated fusion systems at each prime $p$, but only one family is known at the prime 2:

Solomon, 1974. There exists a well defined 2-local structure over the Sylow 2-subgroup of $\text{Spin}_7(q)$ ($q$ odd prime power) which contains a unique conjugacy class of involutions. But there is no finite group with such structure.

Benson, 1994. There is a space $B\text{Sol}(q)$, related to Dwyer-Wilkerson exotic 2-local finite loop space $\text{DI}(4)$, which supports the 2-local structure defined by Solomon: A classifying space for a non-existing group.

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Fusion systems show up while studying the homotopy type of $p$-completed classifying spaces of finite groups.

- **Martino-Priddy conjecture (1996):**

  $$BG^\wedge_p \simeq BH^\wedge_p \iff \mathcal{F}_p(G) \simeq \mathcal{F}_p(H)$$

Two sorts of objects are introduced in the arguments of the above results:

- Linking systems (B-Levi-Oliver)
- Partial groups and localities (Chermak)
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**Fusion systems**

**Definition**

Fix a saturated fusion system $\mathcal{F}$ over a finite $p$-group $S$. Let $\Delta$ be the family of $\mathcal{F}$-centric subgroups of $S$.

An associated centric linking system is a category defined with objects $\Delta$ and extending $\mathcal{F}$ in the sense that

$$\text{Mor}_\mathcal{F}(P,Q) = \text{Mor}_\mathcal{L}(P,Q)/Z(P).$$

Some axioms should be satisfied.

- $|\mathcal{L}|_p$, is called the classifying space of $(S, \mathcal{F}, \mathcal{L})$. 
Fusion systems

- If \( G \) is a finite group and \( S \in \text{Syl}_p(G) \), then we construct a centric linking system \( \mathcal{L}_S^c(G) \) associated to \( \mathcal{F}_S(G) \) as the category with
  - Objects: \( P \leq S \) such that \( C_G(P) = Z(P) \times C_G'(P) \) with \( (p, |C_G'(P)|) = 1 \)
  - Morphisms: \( \text{Mor}_{\mathcal{L}_S^c(G)}(P, G) = N_G(P, G)/C_G'(P) \)

- Then, \( (S, \mathcal{F}_S(G), \mathcal{L}_S^c(G)) \) is a \( p \)-local finite group with classifying space

\[
|\mathcal{L}_S^c(G)|_p^\wedge \simeq (BG)_p^\wedge
\]

- The Martino-Priddy conjecture is first reduced to showing that a fusion system \( \mathcal{F}_S(G) \) admits a unique associated centric linking system.

- Oliver (2004, 2006): M-P conjecture is true: \( \mathcal{L}_S(G) \) is the only possible centric linking system associated to \( \mathcal{F}_S(G) \). (The proof depends on the classification of finite simple groups.)

- Chermak (2013): any (abstract) fusion system admits a unique associated centric linking system. Still depending on CFSG,

- Oliver (2013): extends Chermak’s proof

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  - Objects: $P \leq S$ such that $C_G(P) = Z(P) \times C'_G(P)$ with $(p, |C'_G(P)|) = 1$
  - Morphisms: $\text{Mor}_{\mathcal{L}_S^c(G)}(P, G) = N_G(P, G)/C'_G(P)$
- Then, $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ is a $p$-local finite group with classifying space $|\mathcal{L}_S^c(G)|_p \simeq (BG)^\wedge_p$

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Partial groups

Definition

A partial group is a simplicial set $\mathbb{M}$ satisfying

(P1) $\mathbb{M}_0$ consists of a unique vertex

(P2) The spine operator $e^n: \mathbb{M}_n \to (\mathbb{M}_1)^n$ is injective for all $n \geq 1$.

(P3) There is an inversion $(-)^{-1}: \mathbb{M} \to \mathbb{M}$ such that for each $u \in \mathbb{M}_n$, $n \geq 1$:

(I1) There is a simplex $[\nu(u)|u] \in \mathbb{M}_{2n}$; and

(I2) $\Pi [\nu(u)|u] = 1$. 
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the sequence of edges joining the ordered vertices from 0 through $n$.

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- We write $\omega = [g_1|g_2|\ldots|g_n]$ if $\omega \in \mathbb{M}_n$ and $e^n(\omega) = (g_1,g_2,\ldots,g_n)$:

We write $1 = s_0(v)$ for $v \in \mathbb{M}_0$, and $\Pi[x_1|x_2|\ldots|x_n] = x_1 \cdot x_2 \cdot \ldots \cdot x_n$.

(P3) There is an inversion $(-)^{-1}:\mathbb{M} \rightarrow \mathbb{M}$ such that for each $u \in \mathbb{M}_n$, $n \geq 1$:

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- A homomorphism of partial groups is a simplicial map
- A extension of partial groups is a fibre bundle $\mathbb{M} \xrightarrow{i} \mathbb{E} \xrightarrow{\tau} \mathbb{H}$
- It turns out that if $\mathbb{H}$ and $\mathbb{M}$ are partial groups, then so is $\mathbb{E}$
Partial groups

We now concentrate in extensions

\[ \mathcal{M} \overset{i}{\longrightarrow} \mathcal{E} \overset{\tau}{\longrightarrow} BG \]

where \( G \) is a finite group.

- There is a fibration \( B^2Z(\mathcal{M}) \to B\text{aut}(\mathcal{M}) \to B\text{Out}(\mathcal{M}) \).
- The classification of extensions works exactly as in the case of finite groups.
- The extension is regular split extension if it admits a regular section. A regular section defines an action of \( G \) on \( \mathcal{M} \), and \( \mathcal{E} \) can be described as a semidirect product.
- For a regular split extension, \( H^1(G, \mathcal{M}) \) classifies equivalence classes of sections.
Localities

Definition

Let $\mathbb{L}$ be a partial group with finite $\mathbb{L}_1$, $S \leq \mathbb{L}$ a $p$-subgroup and $\Delta$ a family of subgroups of $S$. Then, $(\mathbb{L}, S)$ is a locality via $\Delta$ if the following holds

(O1) $[u_1|\ldots|u_n] \in \mathbb{L}_n$ if and only if there is a string of composable conjugation maps between objects of $\Delta$:

$$X_0 \overset{u_1}{\leftarrow} X_1 \overset{u_2}{\leftarrow} \ldots \overset{u_n}{\leftarrow} X_n, \quad X_i \in \Delta.$$ 

(O2) $\Delta$ is closed by overgroups. Also, if $X \in \Delta$ and $u \in \mathbb{L}_1$ are such that $uX \leq S$, then $uX \in \Delta$.

(L1) $S$ is maximal in the poset (ordered by inclusion) of $p$-subgroups of $\mathbb{L}$.

- Example: Let $G$ be a finite group and fix $S \in \text{Syl}_p(G)$, set $\mathcal{F} = \mathcal{F}_S(G)$, and let $\Gamma$ be a non-empty $\mathcal{F}$-invariant collection of subgroups of $S$, closed under taking overgroups. Then

$\mathbb{L}_\Gamma(G) = \{ [g_1|g_2|\ldots|g_n] \in BG \mid \exists P_0, P_1, \ldots, P_n \in \Gamma, \quad P_0 \overset{g_1}{\leftarrow} P_1 \overset{g_2}{\leftarrow} \ldots \overset{g_n}{\leftarrow} P_n \} \subseteq BG$

is a partial group, and $(\mathbb{L}_\Gamma(G), S)$ a locality via $\Gamma$. 
Localities

- The locality \( \mathcal{L}(\mathcal{L}) \) of a \( p \)-local finite group \((S, \mathcal{F}, \mathcal{L})\).

This is a construction due to Chermak.

Let \( \equiv \) be the equivalence relation defined on \( \text{Iso}(\mathcal{L}) \) such that \( f \equiv g \) if one is a restriction of the other. Define

\[
\mathcal{L}(\mathcal{L})_n = \left\{ [\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_n] \mid \exists P_0, P_1, \ldots, P_n \in \text{Ob}(\mathcal{L}) \right. , \\
P_0 \overset{f_1}{\leftarrow} P_1 \overset{f_2}{\leftarrow} \ldots \overset{f_n}{\leftarrow} P_n \right\}
\]

Then, \((\mathcal{L}(\mathcal{L}), S)\) is a locality via \( \Delta = \text{Ob}(\mathcal{L}) \)
Theorem (B-González)

The natural projection $|\mathcal{L}| \longrightarrow \mathbb{L}$ is a weak equivalence of simplicial sets.

Furthermore, it is $|\text{Aut}_{\text{typ}}(\mathbb{L})|$-equivariant and the action on $\mathbb{L}$ induces an isomorphism of simplicial groups

$$|\text{Aut}_{\text{typ}}(\mathcal{L})| \cong \text{aut}(\mathbb{L}, S).$$

(This last is the simplicial subgroup of $\text{aut}(\mathbb{L})$ that leaves $S$ stable.)

It follows $B \text{aut}(\mathcal{L}_p) \cong B|\text{Aut}_{\text{typ}}(\mathcal{L})| \cong B \text{aut}(\mathbb{L}, S).$

- Localities provide models for classifying spaces of $p$-local finite groups. Some questions arise:
  1. The homotopy type of $(\mathbb{L}, S)$ depends on the set of objects $\Delta$. We need to adjust $\Delta$ so that we get to the right homotopy type.
  2. We need a solid theory of extensions of localities.
  3. We need to construct new from old, e.g.: centralizers and mapping spaces.
Localities

**Theorem (B-González)**

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Localities

1. Concerning the homotopy type.

- If $(\mathcal{L}, S)$ is a locality via $\Delta$ and $H \in \Delta$, then
  \[ N_{\mathcal{L}}(H) = \{ u \in \mathcal{L}_1 | u^H = H \}. \]
  
  and
  \[ C_{\mathcal{L}}(H) = \{ u \in \mathcal{L}_1 | u^h = h, \ \forall h \in H \}. \]

  are subgroups of $\mathcal{L}$.

- This allows the association of fusion and linking systems, as we did for groups.

- A locality $(\mathcal{L}, S)$ via $\Delta$ is centric if it satisfies
  1. $C_{\mathcal{L}}(P)$ is a $p$-group for all $P \in \Delta$, and
  2. $\Delta$ contains all centric subgroups, that is, all $P \leq S$ such that $C_{\mathcal{L}}(P) = Z(P)$.

- Centric localities have the right homotopy type.
3. Centralizers and mapping spaces

- For a locality \((L, S)\) and an arbitrary subgroup \(T \leq S\), define the centralizer of \(T\) in \(L\) as the partial subgroup \(C_L(T) \leq L\) with \(n\)-simplices

\[
C_L(T)_n = \{ [u_1|u_2|\ldots|u_n] \in L \mid u_i h = h, \forall h \in H, \ i = 1, \ldots, n \}.
\]

**Proposition**

Let \((L, S)\) be a centric locality via \(\Delta\). If \(T \leq S\) is fully centralized, then

- \((C_L(T), C_S(T))\) is a centric locality via \(\Delta_T = \{ C_P(T) \mid T \leq P \in \Delta \}\).

And the adjoint of the product map provides a homotopy equivalence

\[
\xymatrix{ C_L(T)_p^\wedge \ar[r]^-{\cong} & \text{Map}(BT, L_p^\wedge)_{\text{incl}}}.
\]
Localities

2. Extensions

- Let \((\mathbb{L}, S)\) be a locality via \(\Delta\) and let \(G\) be a discrete group. An extension

\[
\mathbb{L} \longrightarrow \mathbb{E} \longrightarrow BG
\]

is called *isotypical* if the structural group is \(\text{aut}(\mathbb{L}; S)\).

- If this is the case, then \(\mathbb{E}\) is also a locality. More precisely:
  
  (a) There is an associated group extension

\[
BN_{\mathbb{L}}(S) \longrightarrow BN_{\mathbb{E}}(S) \longrightarrow BG
\]

\[
\begin{array}{c}
\downarrow \\
\downarrow \\
\mathbb{L} \longrightarrow \mathbb{E} \longrightarrow BG
\end{array}
\]

(b) Fix \(\tilde{S} \in \text{Syl}_p(N_{\mathbb{E}}(S))\). Then \((\mathbb{E}, \tilde{S})\) is a locality via \(\tilde{\Delta} = \{P \leq \tilde{S} \mid P \cap S \in \Delta\}\).

(c) If \(\mathbb{L}\) is a centric locality, then \(\mathcal{F}_{\tilde{S}}(\mathbb{E})\) is saturated and \(\tilde{\Delta}\) contains all of the \(\mathcal{F}_{\tilde{S}}(\mathbb{E})\)-centric \(\mathcal{F}_{\tilde{S}}(\mathbb{E})\)-radical subgroups of \(\tilde{S}\).
Localities

Definition
Let \((L, S)\) be a locality via \(\Delta\) and \(L \xrightarrow{\iota} E \xrightarrow{\tau} BG\) an isotypical extension, where \(G\) is a finite group.

A centric equivariant replacement for \((L, \Delta, S)\) with respect to the extension \(\tau\) is a partial group \(L_{eq}\) together with a map of extensions

\[
\begin{array}{ccc}
L & \xrightarrow{\iota} & E & \xrightarrow{\tau} & BG \\
\downarrow{j} & & \downarrow{} & & \downarrow{}
\end{array}
\]

\[
\begin{array}{ccc}
L_{eq} & \xrightarrow{} & E_{eq} & \xrightarrow{\tau_{eq}} & BG \\
\downarrow{} & & \downarrow{} & & \downarrow{}
\end{array}
\]

where \(j: L \to L_{eq}\) is a trivial cofibration and \(E_{eq}\) is a centric locality.

Theorem (B-González)
Let \((L, \Delta, S)\) be a centric locality. If \(\pi\) is a finite \(p\)-group and \(L \xrightarrow{\iota} E \xrightarrow{\tau} B\pi\) is an isotypical extension, then, there exists a centric equivariant replacement of \((L, \Delta, S)\) with respect to the extension \(\tau\).
**Localities**

**Definition**

Let \((L, S)\) be a locality via \(\Delta\) and \(L \xrightarrow{\iota} E \xrightarrow{\tau} BG\) an isotypical extension, where \(G\) is a finite group.

A centric equivariant replacement for \((L, \Delta, S)\) with respect to the extension \(\tau\) is a partial group \(L_{eq}\) together with a map of extensions

\[
\begin{array}{ccc}
L & \xrightarrow{\iota} & E \xrightarrow{\tau} BG \\
\downarrow{\jmath} & & \downarrow \quad \\
L_{eq} & \xrightarrow{\jmath_{eq}} & E_{eq} \xrightarrow{\tau_{eq}} BG
\end{array}
\]

where \(\jmath: L \to L_{eq}\) is a trivial cofibration and \(E_{eq}\) is a centric locality.

**Theorem (B-González)**

Let \((L, \Delta, S)\) be a centric locality.

If \(\pi\) is a finite \(p\)-group and \(L \to E \xrightarrow{\tau} B\pi\) is an isotypical extension, then, there exists a centric equivariant replacement of \((L, \Delta, S)\) with respect to the extension \(\tau\).
**Homotopy fixed points**

**Definition**

Let \((S, F, L)\) be a \(p\)-local finite group and \(\pi\) be a finite \(p\)-group. By an action of \(\pi\) on \((S, F, L)\) we understand an action of \(\pi\) on a classifying space \(X \simeq |L|^\wedge_p\).

- Borel construction gives a fibration \(X \longrightarrow X \times \pi E \pi \longrightarrow B \pi\).
- This fibration is the fibrewise \(p\)-completion of a fibre bundle \(|L| \longrightarrow Y \longrightarrow B \pi\) with structure group \(|\text{Aut}_\text{ty}_p(L)|\) (B-Levi-Oliver).
- Let \(\mathbb{L} = \mathbb{L}(L)\) be the locality of \((S, F, L)\), then the weak equivalence \(|L| \longrightarrow \mathbb{L}\) extends to a diagram of fibre bundles

\[
\begin{array}{ccc}
\mathbb{L} & \simeq_w & |L| \\
\downarrow & & \downarrow \\
\mathbb{E} & \simeq_w & Y \\
\downarrow & & \downarrow \\
B \pi & = & B \pi \\
\end{array}
\]

\[
\begin{array}{ccc}
|L| & \kappa_p & X \\
\downarrow & & \downarrow \\
Y & \kappa_p & X_{h \pi} \\
\downarrow & & \downarrow \\
B \pi & = & B \pi \\
\end{array}
\]
Homotopy fixed points

- The extension $\mathbb{L} \to \mathbb{E} \to B\pi$ admits a centric equivariant replacement:

\[
\begin{array}{ccc}
\mathbb{L}_{eq} & \xleftarrow{\simeq w} & \mathbb{L} & \xleftarrow{\simeq w} & |\mathcal{L}| & \xrightarrow{\kappa_p} & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{E}_{eq} & \xleftarrow{\simeq w} & \mathbb{E} & \xleftarrow{\simeq w} & Y & \xrightarrow{\kappa_p} & X_{h\pi} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B\pi & = & B\pi & = & B\pi & = & B\pi
\end{array}
\]

- It follows that $X^{h\pi} \simeq [(\mathbb{L}_{eq})^\wedge]^\wedge_{h\pi}$.

- There are bijections

\[H^1(\pi; \mathbb{L}) \cong H^1(\pi; \mathbb{L}_{eq}) \cong \pi_0((\mathbb{L}_{eq}^\wedge)^{h\pi}).\]

- Since $\mathbb{E}_{eq}$ is a centric locality, the adjoint of the evaluation provides a mod $p$ homotopy equivalence

\[\mathcal{C}_{\mathbb{E}_{eq}}(\sigma(\pi)) \xrightarrow{\simeq p} \text{Map}(B\pi, (\mathbb{E}_{eq})^\wedge)_{\sigma}\]
The extension $L \rightarrow E \rightarrow B\pi$ admits a centric equivariant replacement:

\[
\begin{array}{cccccc}
L_{eq} & \sim w & L & \sim w & |L| & \kappa_p \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
E_{eq} & \sim w & E & \sim w & Y & X_{h\pi} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
B\pi & \sim w & B\pi \sim w & B\pi \sim w & B\pi
\end{array}
\]

It follows that $X^{h\pi} \simeq [(L_{eq})^p]^{h\pi}$.

There are bijections

\[H^1(\pi; L) \simeq H^1(\pi; L_{eq}) \simeq \pi_0([(L^p)^h\pi]) \]

Since $E_{eq}$ is a centric locality, the adjoint of the evaluation provides a mod $p$ homotopy equivalence

\[
C_{E_{eq}}(\sigma(\pi)) \xrightarrow{\simeq p} \text{Map}(B\pi, (E_{eq})^p)_{\sigma}
\]
Homotopy fixed points

- The extension \( \mathbb{L} \rightarrow \mathcal{E} \rightarrow B\pi \) admits a centric equivariant replacement:

\[
\begin{array}{cccc}
\mathbb{L}_{eq} & \xleftarrow{\simeq_w} & \mathbb{L} & \xleftarrow{\simeq_w} & |\mathcal{L}| & \xrightarrow{\kappa_p} & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{E}_{eq} & \xleftarrow{\simeq_w} & \mathcal{E} & \xleftarrow{\simeq_w} & \mathcal{Y} & \xrightarrow{\kappa_p} & X_{h\pi} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B\pi & = & B\pi & = & B\pi & = & B\pi \\
\end{array}
\]

- It follows that \( X^{h\pi} \simeq \left( (\mathbb{L}_{eq})_p^\wedge \right)^{h\pi} \).

- There are bijections

\[
H^1(\pi; \mathbb{L}) \simeq H^1(\pi; \mathbb{L}_{eq}) \simeq \pi_0((\mathbb{L}_p^\wedge)^{h\pi})
\]

- Since \( \mathcal{E}_{eq} \) is a centric locality, the adjoint of the evaluation provides a mod \( p \) homotopy equivalence

\[
\mathcal{C}_{\mathcal{E}_{eq}}(\sigma(\pi)) \xrightarrow{simeq_p} \text{Map}(B\pi, (\mathcal{E}_{eq})_p^\wedge)_\sigma
\]
Homotopy fixed points

- The fixed points set \((\mathbb{L}^\sigma_{\text{eq}}, S^\sigma)\) is a centric locality and the above equivalence extends to

\[
\begin{array}{ccc}
\mathbb{L}^\sigma_{\text{eq}} & \xrightarrow{\sim_p} & (\mathbb{L}_{\text{eq}})^\wedge_{\mathbb{L}} h\pi
\\
\downarrow & & \downarrow
\\
\mathbb{C}_{\mathbb{L}_{\text{eq}}}(\sigma(\pi)) & \xrightarrow{\sim_p} & \text{Map}(B\pi, (\mathbb{L}_{\text{eq}})^\wedge_{\mathbb{L}})_\sigma
\\
\downarrow & & \downarrow
\\
B\mathbb{Z}(\pi) & \xrightarrow{\sim} & \text{Map}(B\pi, B\pi)_{\text{id}}
\end{array}
\]

Theorem (B-González)

Let \((S, \mathcal{F}, \mathcal{L})\) be a \(p\)-local finite group and \(\pi\) a finite \(p\)-group. Assume that \(\pi\) acts on a classifying space \(X \simeq |\mathcal{L}|_p^\wedge\). Then,

(a) \(\pi\) acts on the locality \(\mathbb{L}(\mathcal{L})\), and

(b) if \(\mathbb{L}_{\text{eq}}\) is a centric equivariant replacement for \(\mathbb{L}(\mathcal{L})\), then

\[
X^{h\pi} \simeq \bigsqcup_{\sigma \in H^1(\pi; \mathbb{L})} (\mathbb{L}_{\text{eq}})^\wedge_{\mathbb{L}}
\]
Homotopy fixed points

- The fixed points set \((\mathbb{L}_\sigma^\sigma, S^\sigma)\) is a centric locality and the above equivalence extends to

\[
\begin{align*}
\mathbb{L}_\text{eq} & \xrightarrow{\sim^p} [((\mathbb{L}_\text{eq})_p^\wedge)^h \pi^\sigma] \\
\mathbb{C}_{\mathbb{L}_\text{eq}}(\sigma(\pi)) & \xrightarrow{\sim^p} \text{Map}(B\pi, (\mathbb{L}_\text{eq})_p^\wedge)_\sigma \\
B\mathbb{Z}(\pi) & \xrightarrow{\sim} \text{Map}(B\pi, B\pi)_{\text{Id}}
\end{align*}
\]

**Theorem (B-González)**

Let \((S, \mathcal{F}, \mathcal{L})\) be a \(p\)-local finite group and \(\pi\) a finite \(p\)-group. Assume that \(\pi\) acts on a classifying space \(X \simeq |\mathcal{L}|_p^\wedge\). Then,

(a) \(\pi\) acts on the locality \(\mathbb{L}(\mathcal{L})\), and

(b) if \(\mathbb{L}_\text{eq}\) is a centric equivariant replacement for \(\mathbb{L}(\mathcal{L})\), then

\[
X^h\pi \simeq \bigsqcup_{\sigma \in H^1(\pi; \mathbb{L})} [\mathbb{L}_\text{eq}^\wedge]^\sigma_p
\]
Thank you for your attention
Saturation Axioms for fusion systems

Let $\mathcal{F}$ be a fusion system over a $p$-group $S$.

1. A subgroup $P \leq S$ is fully centralized in $\mathcal{F}$ if $|C_S(P)| \geq |C_S(P')|$ for all $P' \leq S$ which is $\mathcal{F}$-conjugate to $P$.

2. A subgroup $P \leq S$ is fully normalized in $\mathcal{F}$ if $|N_S(P)| \geq |N_S(P')|$ for all $P' \leq S$ which is $\mathcal{F}$-conjugate to $P$.

Definition

A fusion system $\mathcal{F}$ over a $p$-group $S$ is a saturated if the following two conditions hold:

(I) For all $P \leq S$ which is fully normalized in $\mathcal{F}$, $P$ is fully centralized in $\mathcal{F}$ and $\text{Aut}_S(P) \in \text{Syl}_p \text{Aut}_\mathcal{F}(P)$.

(II) If $P \leq S$ and $\varphi \in \text{Hom}_\mathcal{F}(P, S)$ are such that $\varphi P$ is fully centralized, and if we set

$$N_\varphi = \{ g \in N_S(P) | \varphi g \varphi^{-1} \in \text{Aut}_S(\varphi P) \},$$

then there is $\overline{\varphi} \in \text{Hom}_\mathcal{F}(N_\varphi, S)$ such that $\overline{\varphi}|_P = \varphi$. 

Return
Centric linking systems

Given a fusion system $\mathcal{F}$ over a finite $p$-group $S$ we say that
- $P$ is $\mathcal{F}$-conjugate to $P'$ if there is an isomorphism $\varphi: P \to P'$ in $\mathcal{F}$.
- $P \leq S$ is $\mathcal{F}$-centric if all $P'$ $\mathcal{F}$-conjugate to $P$ satisfies $C_S(P') = Z(P')$.

Let $\mathcal{F}$ be a fusion system over the $p$-group $S$. A centric linking system associated to $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of $S$, together with a functor

$$\pi: \mathcal{L} \to \mathcal{F},$$

and “distinguished” monomorphisms $P \xrightarrow{\delta_P} \text{Aut}_\mathcal{L}(P)$ for each $\mathcal{F}$-centric subgroup $P \leq S$, which satisfy the following conditions.
Centric linking systems

(A) $\pi$ is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $P, Q \in \mathcal{L}$, $Z(P)$ acts freely on $\text{Mor}_\mathcal{L}(P, Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \leq \text{Aut}_\mathcal{L}(P)$), and $\pi$ induces a bijection

$$\text{Mor}_\mathcal{L}(P, Q)/Z(P) \xrightarrow{\simeq} \text{Hom}_\mathcal{F}(P, Q).$$

(B) For each $\mathcal{F}$-centric subgroup $P \leq S$ and each $g \in P$, $\pi$ sends $\delta_P(g) \in \text{Aut}_\mathcal{L}(P)$ to $c_g \in \text{Aut}_\mathcal{F}(P)$.

(C) For each $f \in \text{Mor}_\mathcal{L}(P, Q)$ and each $g \in P$, the following square commutes in $\mathcal{L}$:

$$
\begin{array}{ccc}
P & \xrightarrow{f} & Q \\
\downarrow{\delta_P(g)} & & \downarrow{\delta_Q(\pi(f)(g))} \\
P & \xrightarrow{f} & Q
\end{array}
$$
Fusion systems

**Definition**

Let \((S, \mathcal{F}, \mathcal{L})\) be a \(p\)-local finite group. A self-equivalence of \(\mathcal{L}\) is called isotypical if it maps subgroups of \(S\) to isomorphic subgroups and inclusions to inclusions.

- \(\text{Aut}_{\text{typ}}(\mathcal{L})\) is the group of isotypical self-equivalences of \(\mathcal{L}\).
- \(\text{Out}_{\text{typ}}(\mathcal{L})\) is the group of isotypical self equivalences of \(\mathcal{L}\) modulo natural equivalence.
- \(\text{Aut}_{\text{typ}}(\mathcal{L})\) is the strict monoidal category with objects \(\text{Aut}_{\text{typ}}(\mathcal{L})\) and morphisms the natural equivalences

**Theorem**

The nerve \(\text{Nerve} \ | \text{Aut}_{\text{typ}}(\mathcal{L})|\) is a simplicial group that acts naturally on the nerve \(\text{Nerve} \ \mathcal{L}\), and

\[
\pi_i(\text{Nerve} \ \text{Aut}_{\text{typ}}(\mathcal{L})) = \begin{cases} 
\text{Out}_{\text{typ}}(\mathcal{L}) & i = 1, \\
\pi_i(\text{Nerve} \ \mathcal{L}) & i = 2 \\
0 & i \geq 3
\end{cases}
\]

Furthermore, \(\text{B}\text{Nerve} \ \text{Aut}_{\text{typ}}(\mathcal{L}) \simeq B\text{aut}(\mathcal{L}^\wedge_p)\).