Bredon homology of wreath products

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Métodos categóricos y homotópicos en Álgebra, Geometría y Topología

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Goal

The main goal of this talk is to show how to compute the Bredon homology (with respect to the family of finite subgroups) of some integer extensions of locally finite groups, with coefficients in the complex representation ring.

This will allow us to describe the topological side of the Baum-Connes conjecture for these groups.

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This will allow us to describe the topological side of the Baum-Connes conjecture for these groups.

Outline

- Lamplighter groups.
- Classifying spaces.
- Bredon homology.
- Martínez spectral sequence.
- The Baum-Connes conjecture.

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Lamplighter groups

The lamplighter group is the group L given by the following presentation:

$$\langle a, t | a^2, [t^m a t^{-m}, t^n a t^{-n}]; m, n \in \mathbb{Z} \rangle.$$

This group can be expressed as a (restricted) wreath product

$$L = B \wr \mathbb{Z}$$

being $B = \bigoplus_{\mathbb{Z}} C_2$ and the integers acting over the kernel by translation. Its study was the first motivation of our work.

Any group that can be obtained changing C_2 by a finite group F in the previous extension will be called a lamplighter group of finite groups in the sequel.

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Every element $(a, t) \in L$ with $t \neq 0$ has infinite order, while the elements $(b, 0)$ have exponent 2. Moreover, all the torsion subgroups of L are elementary abelian 2-subgroups.

The locally finite group B is identified with the subgroup of the elements $(b, 0)$, while $\mathbb{Z}$ is identified with the subgroup generated by $(0, 1)$. Observe that $(b, t) = (b, 0)(0, t)$.

It is also remarkable that the conjugation $(0, t)(b, 0)(0, -t)$ produces an element $(b', 0)$ where b' has all the entries of b shifted t positions to the right.

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Classifying space for proper actions

Definition

Given a discrete group G, a classifying space for proper actions of G is a G-CW-complex $\underline{E}G$ such that the fixed-point set $\underline{E}G^H$ is contractible for every finite subgroup $H < G$, and empty otherwise.

These spaces were defined by Baum-Connes-Higson, and are the subject of the Baum-Connes conjecture, that we will describe afterwards.

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- The classifying space $\underline{E}G$ is unique up to G-homotopy equivalence.
- The stabilizers of the G-action on $\underline{E}G$ are finite, and in fact $\underline{E}G$ classifies spaces with proper actions, i.e., actions with finite stabilizers.
- As the trivial group is finite, $\underline{E}G$ is always contractible.

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Classifying spaces for integer extensions

For certain integer extensions $H \to G \to \mathbb{Z}$, Martin Fluch has developed a method to construct a model of $\underline{E}G$ out of a model of $\underline{E}H$, by means of a cylinder construction.

Theorem

Let G be a discrete group such that $G = B \rtimes \mathbb{Z}$, and such that every finite subgroup of G is contained in B. Then if there is a model for $\underline{E}B$ of dimension n, there exists a model for $\underline{E}G$ of dimension $n+1$.

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Classifying space for the lamplighter group

Let us explain Fluch construction in the case of the lamplighter group.

First, it is worth to recall that a tree is a well-known model for the classifying space for proper actions of a locally finite group.

Let B be a locally finite group, $B_1 \subset B_2 \subset \ldots$ a nested sequence of finite groups such that $B = \bigcup_{i \geq 1} B_i$.

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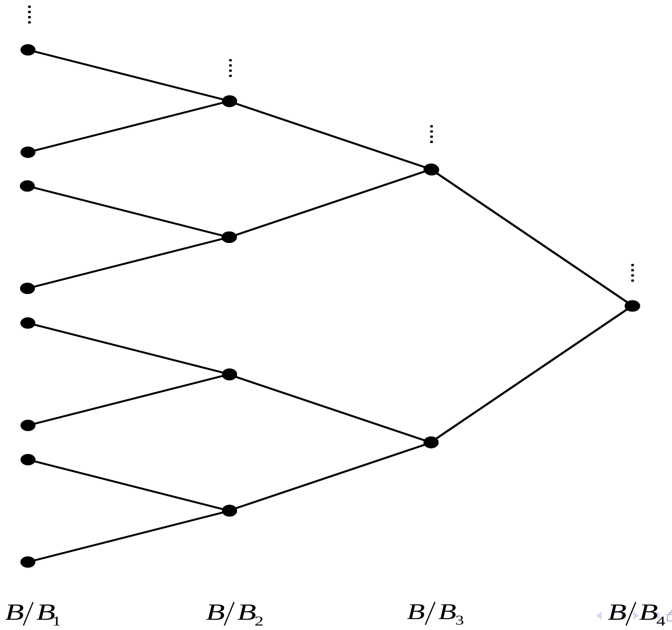
Classifying spaces for locally finite groups



Fluch model

Consider the automorphism ϕ of B given by right translation of one index. Let X be the tree model for $\underline{E}B$, and consider copies X_k of X indexed by $k \in \mathbb{Z}$.

For every k, there is an action of B over X_k given by $g * x = \phi^{-k}(g)x$. With this action, X_k is again a model for $\underline{E}B$.

Now for every k and for these actions, there exists a continuous cellular B-map $f_k : X_k \to X_{k+1}$, as the X_k are models for $\underline{E}B$.

We denote by X_∞ the infinite cylinder $X_\infty = \coprod_{k\in\mathbb{Z}} X_k \times [0,1]$, quotient by the relation $(x,1) \sim (f_k(x),0)$ for every $x \in X_k$, for every $k \in \mathbb{Z}$.

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Let (x, t) be a point of $X_k \times [0, 1]$. There is an action of L over X_∞ given by $(g, r)(x, t) = (\phi^{-(k+r)}(g)x, t) \in X_{k+r} \times [0, 1]$.

Proposition (based on Fluch work)

The space X_∞, endowed with the previous L-action, is a 2-dimensional model for $\underline{E}L$.

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Given a discrete group G, denote by $\mathcal{F}$ the family of finite subgroups of G.

Definition

The orbit category $\mathfrak{O}_{\mathcal{F}}G$ is the category whose objects are the homogeneous G-spaces G/K, for K finite, and whose morphisms are the G-maps.

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Definition

A $\mathfrak{O}_{\mathcal{F}}G$-module is a functor from $\mathfrak{O}_{\mathcal{F}}G$ to the category of abelian groups.

The category whose objects are the covariant (resp. contravariant) $\mathfrak{O}_{\mathcal{F}}G$-modules and whose morphisms are the natural transformations is denoted by $G\text{-}\mathrm{Mod}_{\mathcal{F}}$ (resp. $\mathrm{Mod}_{\mathcal{F}}\text{-}G$).

Modules over the orbit category

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Now we define $\mathbf{Tor}_i(-, N)$ as the i-th left derived functor of the tensor product functor:

$$- \bigotimes_{\mathcal{F}} N : \mathrm{Mod}_{\mathcal{F}}\text{-}G \to \mathfrak{Ab}.$$

The Bredon homology groups of G with coefficients in $N \in G\text{-}\mathrm{Mod}_{\mathcal{F}}$ are the groups

$$H_i^{\mathcal{F}}(G, N) = \mathbf{Tor}_i(\underline{\mathbb{Z}}, N),$$

being $\underline{\mathbb{Z}}$ the constant functor with value $\mathbb{Z}$.

We will be interested in the case in which N is the complex representation ring functor.

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If V is a finite dimensional complex vector space, a complex representation of V is a group homomorphism $V \to GL(V)$.

The formal differences of isomorphism classes of finite dimensional complex representations of V is an abelian group with the direct sum operation. If we endow this group with the product of representations, we obtain the complex representation ring $R_{\mathbb{C}}G$ of G.

We denote by $R_{\mathbb{C}}$ the (covariant) associated module over the orbit category of G.

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Martínez spectral sequence

One of the main tools that can be used to compute Bredon homology of an extension is the version of Lyndon-Hochschild-Serre spectral sequence developed by C. Martínez. We review it in a particular version.

Let $N \to G \to \bar{G}$ be a group extension, and we denote by $Fin(G)$ and $Fin(\bar{G})$ the corresponding families of finite subgroups.

In this way it is defined a first quadrant spectral sequence such that $E^2_{p,q} = H_p^{Fin(\bar{G})}(\bar{G}, \overline{H_q^{Fin(G)\cap -}(-, D)})$, and converging to $E^\infty_{p,q} = H_{p+q}^{Fin(G)}(G, D)$.

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- In the page E_2, $\overline{H_q^{Fin(G)\cap -}(-, D)}$ is a module in $G\text{-}\mathrm{Mod}_{Fin(\bar{G})}$.

- The values of $\overline{H_q^{Fin(G)\cap -}(-, D)}$ are computed in the following way: first take an element $\bar{V} < \bar{G}$ in $Fin(\bar{G})$, and consider its preimage V in G. Then consider the family $\mathfrak{F}_V$ of the finite subgroups of V.

- Now the value of the functor $\overline{H_q^{Fin(G)\cap -}(-, D)}$ over $\bar{V}$ is $H_q^{\mathfrak{F}_V}(V, D)$.

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Invariants and co-invariants

Recall that given a G-module M, the G-invariants are the fixed elements under the action of G. The submodule of G-invariants is usually denoted M^G.

Similarly, the module $M_G = M/\langle m - gm : m \in M, g \in G \rangle$ are the G-coinvariants.

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Bredon homology of integer extensions

Lemma

Let $G = B \rtimes \mathbb{Z}$ be a lamplighter group of finite groups, $\mathcal{F}$ the family of finite subgroups of G. Then $H_0^{\mathcal{F}}(G; R_{\mathbb{C}})$ is the module of co-invariants of the action of $\mathbb{Z}$ on $H_0^{\mathcal{F}}(B; R_{\mathbb{C}})$, and $H_1^{\mathcal{F}}(G; R_{\mathbb{C}})$ are the invariants.

This is true because the groups $H_i^{\mathcal{F}}(G; R_{\mathbb{C}})$, $i = 0, 1$, can be respectively identified with ordinary homology groups $H_i(\mathbb{Z}; H_0^{\mathcal{F}}(B; R_{\mathbb{C}}))$.

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Now we may present our main result:

Theorem

Let $G = B \rtimes \mathbb{Z}$ be a lamplighter group of finite groups, $\mathcal{F}$ the family of finite subgroups of G. Then:

- $H_0^{\mathcal{F}}(G; R_{\mathbb{C}})$ is a free abelian group over a countable base.
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Sketch proof

The existence of 1-dimensional models for $\underline{E}B$ and $E\mathbb{Z}$ guarantee that the upper homology groups vanish and Martínez spectral sequence collapses in the E^2-term.

Then, $H_0^{\mathcal{F}}(G; R_{\mathbb{C}})$ and $H_1^{\mathcal{F}}(G; R_{\mathbb{C}})$, can be identified from their previous description as coinvariants and invariants, taking into account that the induced action of $\mathbb{Z}$ over the 0-th Bredon homology group of B is induced by the shift.

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The Baum-Connes conjecture establishes, for $n = 0, 1$, an isomorphism

$$K_n^G(\underline{E}G) \simeq K_n^{top}(C_r^*(G))$$

being K_*^G the equivariant K-homology groups, an algebraic-topological invariant, and K_*^{top} the K-theory of Banach algebras, whose nature is essentially analytic.

The left-hand side

If $\mathcal{F}$ is the family of finite subgroups of a group G, a $\mathfrak{D}_{\mathcal{F}}G$-spectrum is a covariant functor from $\mathfrak{D}_{\mathcal{F}}G$ to the category of spectra.

If X is a G-CW-complex and $\mathbf{S}$ is a $\mathfrak{D}_{\mathcal{F}}G$-spectrum, one defines the spectrum $X_+ \otimes_G \mathbf{S}$ as the disjoint union $\coprod_{G/H \in \mathfrak{D}_{\mathcal{F}}G} (X_+^H \wedge \mathbf{S}(G/H))$, quotient by a certain equivalence relation.

When $\mathbf{S}$ is the non-connective topological K-theory spectrum, the homotopy groups $\pi_i(\underline{E}G_+ \otimes_G \mathbf{S})$, $i = 0, 1$, are by definition the equivariant K-homology groups that appear in the left-hand side of Baum-Connes conjecture.

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The left-hand side

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Given a discrete group G, the Hilbert space of square summable functions $f : G \to \mathbb{C}$ is denoted by $l_2(G)$, and the Banach algebra $B(l_2(G))$ of bounded operators $l_2(G) \to l_2(G)$ contains $\mathbb{C}[G]$ as a subalgebra.

The reduced C^*-algebra $C_r^*(G)$ is defined as the norm closure of $\mathbb{C}[G]$ inside $B(l_2(G))$.

With this notation, $K_0^{top}(C_r^*(G))$ is defined as the projective class group of $C_r^*(G)$, while $K_1^{top}(C_r^*(G))$ is $\pi_0(\mathrm{colim}_n GL_n(C_r^*(G)))$.

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Examples

The Baum-Connes conjecture is known to be true for a big number of families of groups, as for example:

- Amenable groups (which include soluble groups).
- Finite groups
- Free groups
- One-relator groups
- Coxeter groups
- Braid groups

The simplest example for which Baum-Connes is not known to hold is $SL_3(\mathbb{Z})$.

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Theorem. Let G a discrete group such that $\dim \underline{E}G \leq 2$. Then there is a natural exact sequence:

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K-homology of integer extensions

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Corollary

Let F be a finite group, $B = \bigoplus_{\mathbb{Z}} F$, $G = B \rtimes \mathbb{Z}$, with $\mathbb{Z}$ acting by the shift. Then:

$$K_0^G(\underline{E}G) = \bigoplus_{\mathbb{Z}} \mathbb{Z} \text{ and } K_1^G(\underline{E}G) = \mathbb{Z}.$$

As Baum-Connes conjecture holds for these groups, we also have $K_0^{top} C_r^*(G) = \bigoplus_{\mathbb{Z}} \mathbb{Z}$ and $K_1^{top} C_r^*(G) = \mathbb{Z}$.

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THANK YOU!!!