Intersection homotopy type of complex varieties with isolated singularities

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Introduction

Objective: Use techniques of homotopical algebra to study topological invariants for singular spaces (and in particular, for complex algebraic varieties).

We will consider the combination of three “enhancements” of cohomology:

- **Intersection cohomology** (Goresky-MacPherson)
  - Restores Poincaré duality for singular spaces.
  - Defined for any topological pseudomanifold $X \mapsto IH^\bullet_\bullet(X; \mathbb{Q})$.
  - Topological invariant (not homotopy invariant!).

- **Rational homotopy** (Quillen, Sullivan)
  - $X \in \text{Top} \mapsto A^\bullet_{pl}(X) \in \text{CDGA}_\mathbb{Q}$ such that $H^\bullet(A^\bullet_{pl}(X)) \cong H^\bullet(X; \mathbb{Q})$.
  - Contains more information than $H^\bullet(X; \mathbb{Q})$ (e.g. Massey products).

- **Mixed Hodge theory** (Deligne)
  - If $X$ is a complex algebraic variety, then $H^k(X; \mathbb{Q})$ has a filtration $W$ such that $W_m/W_{m-1}$ “looks like” the cohomology of a compact Kähler manifold.
  - The weight filtration $W$ is an algebraic invariant (not a topological invariant!).
The Kähler package

The rational cohomology $H^*(X; \mathbb{Q})$ of a compact Kähler manifold $X$ (or a smooth projective variety) satisfies the Kähler package:

- Poincaré duality.
- Weak and Hard Lefschetz Theorems.
- Hodge decomposition.
- Hodge signature Theorem.

All the previous “enhancements” of $H^*(X; \mathbb{Q})$ are trivial in this case:

- $IH^*(X; \mathbb{Q}) \cong H^*(X; \mathbb{Q})$.

(Deligne-Griffiths-Morgan-Sullivan) Every Kähler manifold is formal:

$$\mathcal{A}_{pl}^*(X) \leftarrow \cdots \leftarrow H^*(X; \mathbb{Q}).$$

In particular, higher order Massey products vanish.

- The weight filtration $W$ is pure: $0 = W_{k-1} \subset W_k = H^k(X; \mathbb{Q})$. 
A question of Goresky

**Theorem (Goresky-MacPherson, Beilinson-Bernstein-Deligne, Saito, ...)**

If $X$ = complex projective variety $\Rightarrow IH^*_m(X; \mathbb{Q})$ satisfies the Kähler package.

“It remains as open question whether there is an intersection homology analogue to the rational homotopy theory of Sullivan. For example, one would like to know when Massey triple products are defined in intersection homology and whether they always vanish on a projective algebraic variety” (Goresky 84’).

- **Intersection homotopy theory** (Chataur-Saralegi-Tanré ’14)
  - $X$ pseudomanifold $\mapsto IA^*_\bullet(X; \mathbb{Q})$ perverse cdga such that $H^*(IA^*_\bullet(X)) \cong IH^*_\bullet(X; \mathbb{Q})$.
  - Contains Massey products in $IH^*_\bullet(X; \mathbb{Q})$.
  - It is a topological invariant (not homotopy invariant!).

- (Chataur, C. ’16) Complex projective varieties with isolated singularities.
  - mixed Hodge theory for $IA^*_\bullet(X; \mathbb{Q})$.
  - Partial results of intersection-formality.
Intersection cohomology for varieties with isolated singularities

- $X =$ complex projective variety of dimension $n$. $\Sigma := Sing(X)$.
- Assume $\dim \Sigma = 0$ (isolated singularities). $X_{\text{reg}} := X - \Sigma$.
- In this case: perversity $= \text{non-negative integer}$.
  - $m := n - 1$ (middle perversity), $t := 2n - 2$ (top perversity).

$$IH^k_p(X; R) \cong \begin{cases} 
H^k(X; R) & \text{if } k > p + 1 \\
\text{Im} \left( H^k(X; R) \longrightarrow H^k(X_{\text{reg}}; R) \right) & \text{if } k = p + 1 \\
H^k(X_{\text{reg}}; R) & \text{if } k \leq p
\end{cases}$$

Properties

- $IH^*_0(X; R) \cong H^*(\overline{X}; R)$, where $\overline{X} \longrightarrow X$ is a normalization.
- $IH^*_\infty(X; R) \cong H^*(X_{\text{reg}}; R)$.
- $IH^k_p(X; R) \cong IH^{2n-k}_{t-p}(X; R)^\vee$, for all $p \leq t$ and all $k \leq 2n$. 
Intersection homotopy equivalence

- The modules $IH^k_p$ with the products $IH^k_p \otimes IH^l_q \to IH^{k+l}_{p+q}$ and the maps $IH_p \to IH_q$ for all $p \leq q$, define a perverse commutative graded algebra: commutative monoid in the category of functors from
  \[ \mathcal{P} := (\mathbb{Z}_{\geq 0}, \geq) \to G^* (R{-\text{mod}}). \]

- $\mathcal{V}_\mathbb{C} =$ complex projective varieties with isolated singularities and stratified morphisms $(f(X_{\text{reg}}) \subset Y_{\text{reg}})$. We have a functor
  \[ IH^*_\bullet (\cdot; R) : \mathcal{V}_\mathbb{C} \to \mathcal{P} \text{CGA}_R. \]

- A morphism $f : X \to Y$ is called intersection rational homotopy equivalence if and only if $f^* : IH^*_\bullet (Y; \mathbb{Q}) \xrightarrow{\sim} IH^*_\bullet (X; R)$.

- This is stronger than rational homotopy equivalence.
A perverse commutative differential graded algebra (over \(k\)) is a commutative monoid in the category of functors from \(\mathcal{P} = (\mathbb{Z}_{\geq 0}, \leq)\) to \(C^+_k\):

- Bigraded \(k\)-vector spaces \(A^*_\bullet = \{A^i_p\}\) with \(i, p \geq 0\).
- Linear differential \(d : A^i_p \rightarrow A^{i+1}_p\),
- Associative product \(\mu : A^i_p \otimes A^j_q \rightarrow A^{i+j}_{p+q}\) with unit \(\eta : k \rightarrow A^0_0\).
- A poset map \(A^i_q \rightarrow A^i_p\) for every \(q \leq p\).

+ Leibnitz, commut. and compatibility of \(d\) and \(\mu\) with poset maps:

\[
\begin{align*}
A^p_q \otimes A^q_q & \xrightarrow{\mu} A^p_{p+q} ; \\
A^p_q \xrightarrow{d} A^p_p.
\end{align*}
\]

- Quasi-isomorphisms: morphisms \(f : A^*_\bullet \longrightarrow B^*_\bullet\) such that \(f^*_p : H^*(A^p) \xrightarrow{\cong} H^*(B^p)\) for all \(p\).

**Theorem (Hovey ’09)**

The category of perverse cdga’s admits a Quillen model structure with \(\mathcal{W} = \text{quasi-isomorphisms}\) and \(\mathcal{Fib} = \text{surjections}\).
Perverse algebraic model

- $X$ = complex projective variety of dimension $n$, $\Sigma = Sing(X)$. $\dim \Sigma = 0$.
- $L = L(\Sigma, X)$ link of $\Sigma$ in $X$ ($\cong$ compact real manifold of dimension $2n - 1$).
- We have an inclusion $\iota : L \hookrightarrow X_{\text{reg}}$.

\[
\begin{align*}
IA^*_p(X) & \longrightarrow \tau_{\leq p}A^*_{pl}(L) . \\
\downarrow & \quad \downarrow \\
A^*_{pl}(X_{\text{reg}}) & \overset{\iota^*}{\longrightarrow} A^*_{pl}(L)
\end{align*}
\]

- This gives a functor $IA^- : \mathcal{V}_\mathbb{C} \longrightarrow \text{Ho}(\mathcal{PCDGA}_\mathbb{Q})$ such that

\[
H^*(IA^-(X)) \cong IH^*(X; \mathbb{Q}).
\]

Properties

- $IA^0_0(X) \cong A_{pl}(\overline{X})$, where $\overline{X} \to X$ is a normalization of $X$.
- $IA^\infty(X) \cong A_{pl}(X_{\text{reg}})$.
- $IA^*_\bullet(X)$ is a topological invariant of $X$ (not homotopy invariant!).
Let $Q \subseteq K$. Then $X$ is intersection-formal over $K$ if $IA^*_\bullet(X) \otimes K$ and $IH^*_\bullet(X; K)$ are isomorphic in $Ho(\mathcal{P}_{\text{CDGA}}_K)$.

$X$ Intersection-formal $\Rightarrow \overline{X}$ and $X_{\text{reg}}$ formal.

Let $\mathcal{P}_n := \{0, 1, \cdots, 2n - 2\}$. We have a forgetful functor

$$U: \mathcal{P}_{\text{CDGA}}_K \longrightarrow \mathcal{P}_n \text{CDGA}_K$$

defined by forgetting all products $A_p \times A_q \rightarrow A_{p+q}$ such that $p + q > t$.

$X$ is GM-intersection-formal if $U(IA^*_\bullet(X) \otimes K)$ and $U(IH^*_\bullet(X; K))$ are isomorphic in $Ho(\mathcal{P}_n \text{CDGA}_K)$.

GM-intersection-formality detects the vanishing of Massey products in Goresky and MacPherson’s intersection cohomology.

GM-intersection-formal $\not\Rightarrow X_{\text{reg}}$ formal.
**Mixed Hodge theory**

- $X =$ complex projective variety with isolated singularities. $\Sigma = Sing(X)$.
- (Hironaka) There is a cartesian diagram

\[
\begin{array}{ccc}
D & \xrightarrow{j} & \tilde{X} \\
g \downarrow & & \downarrow f \\
\Sigma & \xrightarrow{i} & X
\end{array}
\]

where $\tilde{X}$ is smooth projective, $f$ iso outside $\Sigma$ and $D := f^{-1}(X)$ SNCD.

- Simplification: assume $D$ is smooth (e.g. ordinary isolated singularities).
- Restriction morphism $j^s : H^s(\tilde{X}; \mathbb{Q}) \to H^s(D; \mathbb{Q})$.
- Gysin map $\gamma^s : H^{s-2}(D; \mathbb{Q}) \to H^s(\tilde{X}; \mathbb{Q})$.

\[
\begin{array}{ccc}
E^{r,s}_{1}(X_{\text{reg}}) := & \quad & H^{s-2}(D; \mathbb{Q}) \xrightarrow{\gamma^s} H^s(\tilde{X}; \mathbb{Q}) \\
\downarrow Id & & \downarrow j^s \\
E^{r,s}_{1}(L) := & \quad & H^{s-2}(D; \mathbb{Q}) \xrightarrow{j^s \circ \gamma^s} H^s(D; \mathbb{Q})
\end{array}
\]

- $r = -1$ 
- $r = 0$
Mixed Hodge theory

- (Deligne / Durfee) $E_2(X_{\text{reg}}) \cong H^*(X_{\text{reg}}; \mathbb{Q})$ and $E_2(L) \cong H^*(L; \mathbb{Q})$ are independent of the chosen resolutions.
- The map $H^*(\tilde{X}) \times H^*(D) \to H^*(D)$ given by $(x, a) \mapsto j^*(x) \cdot a$ and the cup products of $H^*(\tilde{X})$ and $H^*(D)$ make $E_1(X_{\text{reg}})$ into a cdga.
- Same for $E_1(L)$ with the cup product of $H^*(D)$.

**Theorem (Chataur, C.)**

The morphism $E_1(\iota) : E_1(X_{\text{reg}}) \otimes \mathbb{C} \to E_1(L) \otimes \mathbb{C}$ is a model of the morphism $\iota : A_{pl}(X_{\text{reg}}) \otimes \mathbb{C} \to A_{pl}(L) \otimes \mathbb{C}$.

The (complex) intersection homotopy type of $X$ is encoded in the perverse weight spectral sequence:

$$
\begin{array}{ccc}
IE_{1, \mathbb{C}}^*(X) & \xrightarrow{\tau_{\leq p}} & \tau_{\leq p} E_{1,*}^*(L) \\
\downarrow & & \downarrow \\
E_{1,*,*}(X_{\text{reg}}) & \xrightarrow{\iota^*} & E_{1,*,*}(L)
\end{array}
$$
A mixed Hodge diagram $\mathbf{A}$ for a topological space $X$ is a filtered cdga $(A_Q, W)$, such that $A_Q \cong A_{pl}(X)$, a bifiltered cdga $(A_C, W, F)$ and a string of quasi-isomorphisms

$$(A_Q, W) \otimes \mathbb{C} \xleftarrow{\sim} \cdots \xrightarrow{\sim} (A_C, W).$$

+ axioms making its cohomology into a graded mixed Hodge structure.

**Idea of proof.**

(Morgan) $\exists$ a mixed Hodge diagram $\mathbf{A}(X_{reg})$ for $X_{reg}$.

(Durfee-Hain) $\exists$ a mixed Hodge diagram $\mathbf{A}(L)$ for $L$.

(C.-Guillén) If $\mathbf{A} \to \mathbf{B}$ is a morphism of mixed Hodge diagrams then

$$
\begin{array}{ccc}
A_C & \xleftarrow{\sim} & M & \xrightarrow{\sim} & E_1(A_C, W) \\
\downarrow & & \downarrow & & \downarrow \\
B_C & \xleftarrow{\sim} & M' & \xrightarrow{\sim} & E_1(B_C, W)
\end{array}
$$

Use Navarro-Aznar’s simple $s_{TW} : \Delta \text{MHD} \to \text{MHD}$ to describe $E_1(A_Q(X_{reg})) \cong E_1(X_{reg})$ and $E_1(A_Q(L)) \cong E_1(L)$. □
How to use this to study intersection-formality?

- Try to build a morphism of perverse cdga’s

$$IE_{1,\bullet}^*(X) \xymatrix{ & \ar[l] \sim & \ar[r] \sim & IE_{2,\bullet}^*(X) \cong IH_{\bullet}^*(X; \mathbb{Q}).}$$

- This will give $$IA_{\bullet}(X) \otimes \mathbb{C} \xymatrix{ & \ar[l] \sim & \ar[r] \sim & IH_{\bullet}^*(X; \mathbb{C}).}$$

**Corollary**

*If the weight filtration on $IH^k(X; \mathbb{Q})$ is trivial (pure of weight $k$), for all $k \geq 0$, then $X$ is intersection-formal over $\mathbb{C}$.*

**Examples:**

- Complex projective varieties whose cohomology satisfies Poincaré duality.
- Complex projective varieties that are $\mathbb{Q}$-homology manifolds.
  - Weighted projective spaces, $V$-manifolds, Cayley Cubic, Kummer surface.
• By purely topological reasons, every simply connected projective surface is GM-intersection-formal.

• (Simpson, Kapovich, Kollár) There exist non-formal projective surfaces.

Theorem (Chataur, C.)

Every complex projective surface with isolated singularities $X$ is GM-intersection-formal over $\mathbb{C}$. If $X$ has only one singular point, then $X_{\text{reg}}$ is also formal and $X$ is intersection-formal over $\mathbb{C}$. 
Theorem (Chataur, C.)

Let $X$ be a complex projective variety of dimension $n$ with isolated singularities $\Sigma = \text{Sing}(X)$.

1. If the link of $\sigma$ in $X$ is $(n-2)$-connected for all $\sigma \in \Sigma$ then $X$ is formal over $\mathbb{Q}$.

2. If the singularities are ordinary, then $X$ is GM-intersection-formal over $\mathbb{C}$.

3. If $\Sigma = \{\ast\}$ then $X_{\text{reg}}$ is formal over $\mathbb{Q}$ and $X$ is intersection-formal over $\mathbb{C}$.

Examples where (1) applies:

- Hypersurfaces with isolated singularities and complete intersections.
- $Y \hookrightarrow X$ closed immersion with $\text{Sing}(X) \subset Y$. Then $X/Y$ is formal.
Example: the Segre cubic

\[ S : \{ x_0 + x_1 + x_2 + x_3 + x_4 + x_5 = 0, x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0 \} . \]

Projective threefold with 10 ordinary isolated singular points.

Real representation of the Segre cubic
Example: the Segre cubic

Resolution of the Segre cubic:

\[ D = \bigsqcup_{i=1}^{10} \mathbb{CP}^1 \times \mathbb{CP}^1 \xrightarrow{j} \overline{\mathcal{M}}_{0,6} \]
\[ \Sigma = \{10 \text{ pts} \} \xrightarrow{i} S \]

Non-trivial Betti numbers of \( \overline{\mathcal{M}}_{0,6} \): \( b_0 = b_6 = 1 \) and \( b_2 = b_4 = 16 \).

For \( \sigma_i \in \Sigma \), \( L_i := L(\sigma_i, X) \cong S^2 \times S^3 \).

\[
\begin{array}{c|c|c|c}
 & \mathbb{Q} & \mathbb{Q}^0 & \mathbb{Q}^6 \\
\hline
0 & & & \\
\hline
\text{Ker}(j^4) & \cong & & \\
\hline
\text{Coker}(j^2) & \cong & \mathbb{Q}^5 & \\
\hline
\text{Ker}(j^2) & \cong & \mathbb{Q} & \\
\hline
0 & & & \\
\hline
\mathbb{Q} & & & \\
\end{array}
\]

\[ j^s : H^s(\overline{\mathcal{M}}_{0,6}; \mathbb{Q}) \to H^s(D; \mathbb{Q}) \]
denotes the restriction map

\[ \pi_2 \cong \mathbb{Q}, \pi_3 \cong \pi_4 \cong \mathbb{Q}^5, \]
\[ \pi_5 \cong \mathbb{Q}^{15}, \pi_6 \cong \mathbb{Q}^{50} \]
\[ \pi_7 \cong \mathbb{Q}^{116}, \ldots \]
Example: the Segre cubic

\[ IH_p^*(S; \mathbb{Q}) \xrightleftharpoons{\sim} \]

\[
\begin{array}{|c|c|c|}
\hline
 & 0 \leq p \leq 1 & p = 2 & 3 \leq p \leq 4 \\
\hline
\mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\
0 & 0 & 0 \\
H^2(S; \mathbb{Q})^\vee \oplus \text{Exc}^\vee & H^2(S; \mathbb{Q})^\vee \oplus \text{Exc}^\vee & H^2(S; \mathbb{Q})^\vee \\
\text{Van} & 0 & \text{Van}^\vee \\
H^2(S; \mathbb{Q}) & H^2(S; \mathbb{Q}) \oplus \text{Exc} & H^2(S; \mathbb{Q}) \oplus \text{Exc} \\
0 & 0 & 0 \\
\mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\
\hline
\end{array}
\]

- \( \text{Van} := H^3(S; \mathbb{Q}) \cong \text{Coker}(j^2) \cong \mathbb{Q}^5 \).
- \( \gamma^s : H^{s-2}(D; \mathbb{Q}) \to H^s(\mathcal{M}_{0,6}; \mathbb{Q}) \) is the Gysin map.
- \( \text{Exc} \cong \mathbb{Q}^5 \) is defined via the decomposition

\[ H^2(\mathcal{M}_{0,6}; \mathbb{Q}) \cong \text{Ker}(j^2) \oplus \text{Coker}(\gamma^2) \oplus \text{Exc}. \]
Thank you!