Intersection homotopy type of complex varieties with isolated singularities

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Objective: Use techniques of homotopical algebra to study topological invariants for singular spaces (and in particular, for complex algebraic varieties).

We will consider the combination of three "enhancements" of cohomology:

- Intersection cohomology (Goresky-MacPherson)
  - Restores Poincaré duality for singular spaces.
  - Defined for any topological pseudomanifold $X \mapsto IH^*_{\bullet}(X;\mathbb{Q})$.
  - Topological invariant (not homotopy invariant!).
- Rational homotopy (Quillen, Sullivan)
  - $X \in \mathrm{Top} \mapsto \mathcal{A}^*_{pl}(X) \in \mathsf{CDGA}_{\mathbb{Q}}$ such that $H^*(\mathcal{A}^*_{pl}(X)) \cong H^*(X;\mathbb{Q})$.
  - Contains more information than $H^*(X;\mathbb{Q})$ (e.g. Massey products).
- Mixed Hodge theory (Deligne)
  - If X is a complex algebraic variety, then $H^k(X;\mathbb{Q})$ has a filtration W such that W_m/W_{m-1} "looks like" the cohomology of a compact Kähler manifold.
  - The weight filtration W is an algebraic invariant (not a topological invariant!)

The rational cohomology $H^*(X;\mathbb{Q})$ of a compact Kähler manifold X (or a smooth projective variety) satisfies the Kähler package:

- Poincaré duality.
- Weak and Hard Lefschetz Theorems.
- Hodge decomposition.
- Hodge signature Theorem.

All the previous "enhancements" of $H^*(X;\mathbb{Q})$ are trivial in this case:

- $IH^*_{\bullet}(X;\mathbb{Q}) \cong H^*(X;\mathbb{Q})$.
- (Deligne-Griffiths-Morgan-Sullivan) Every Kähler manifold is formal:

$$\mathcal{A}^*_{pl}(X) \xleftarrow{\sim} \cdots \xrightarrow{\sim} H^*(X;\mathbb{Q}).$$

  In particular, higher order Massey products vanish.
- The weight filtration W is pure: $0 = W_{k-1} \subset W_k = H^k(X;\mathbb{Q})$.

A question of Goresky

Theorem (Goresky-MacPherson, Beilinson-Bernstein-Deligne, Saito, ...)

If X = complex projective variety $\Rightarrow IH^*_{\overline{m}}(X;\mathbb{Q})$ satisfies the Kähler package.

"*It remains as open question whether there is an intersection homology analogue to the rational homotopy theory of Sullivan. For example, one would like to know when Massey triple products are defined in intersection homology and whether they always vanish on a projective algebraic variety*" (Goresky 84').

- Intersection homotopy theory (Chataur-Saralegi-Tanré '14)
  - X pseudomanifold $\mapsto I\mathcal{A}^*_{\bullet}(X;\mathbb{Q})$ perverse cdga such that $H^*(I\mathcal{A}^*_{\bullet}(X)) \cong IH^*_{\bullet}(X;\mathbb{Q})$.
  - Contains Massey products in $IH^*_{\bullet}(X;\mathbb{Q})$.
  - It is a topological invariant (not homotopy invariant!).
- (Chataur, C. '16) Complex projective varieties with isolated singularities.
  - mixed Hodge theory for $I\mathcal{A}^*_{\bullet}(X;\mathbb{Q})$.
  - Partial results of intersection-formality.

- X = complex projective variety of dimension n. $\Sigma := Sing(X)$.
- Assume $\dim \Sigma = 0$ (isolated singularities). $X_{reg} := X - \Sigma$.
- In this case: perversity = non-negative integer.
  $m := n-1$ (middle perversity), $t := 2n-2$ (top perversity).

$$IH^k_{\bar{p}}(X;R) \cong \begin{cases} H^k(X;R) & \text{if } k > p+1 \\ \mathrm{Im}\left(H^k(X;R) \longrightarrow H^k(X_{reg};R)\right) & \text{if } k = p+1 \\ H^k(X_{reg};R) & \text{if } k \leq p \end{cases}.$$

Properties

- $IH^*_{\bar{0}}(X;R) \cong H^*(\overline{X};R)$, where $\overline{X} \longrightarrow X$ is a normalization.
- $IH_{\bar{\infty}}(X;R) \cong H^*(X_{reg};R)$.
- $IH^k_{\bar{p}}(X;R) \cong IH^{2n-k}_{\overline{t-p}}(X;R)^{\vee}$, for all $p \leq t$ and all $k \leq 2n$.

- The modules $IH_{\overline{p}}^{k}$ with the products $IH_{\overline{p}}^{k} \otimes IH_{\overline{q}}^{l} \longrightarrow IH_{\overline{p+q}}^{k+l}$ and the maps $IH_{\overline{p}} \longrightarrow IH_{\overline{q}}$ for all $p \leq q$, define a perverse commutative graded algebra: commutative monoid in the category of functors from

$$\mathcal{P} := (\mathbb{Z}_{\geq 0}, \geq) \longrightarrow G^{*}(R\mathrm{-mod}).$$

- $\mathcal{V}_{\mathbb{C}} =$ complex projective varieties with isolated singularities and stratified morphisms ($f(X_{reg}) \subset Y_{reg}$). We have a functor

$$IH_{\bullet}^{*}(-;R) : \mathcal{V}_{\mathbb{C}} \longrightarrow \mathcal{P}\mathsf{CGA}_{R}.$$

- A morphism $f : X \longrightarrow Y$ is called intersection rational homotopy equivalence if and only if $f^{*} : IH_{\bullet}^{*}(Y;\mathbb{Q}) \xrightarrow{\cong} IH_{\bullet}^{*}(X;R)$.
- This is stronger than rational homotopy equivalence.

- A perverse commutative differential graded algebra (over $\mathbf{k}$) is a commutative monoid in the category of functors from $\mathcal{P} = (\mathbb{Z}_{\geq 0}, \leq)$ to $C^+_{\mathbf{k}}$:
  - Bigraded $\mathbf{k}$-vector spaces $A^*_{\overline{\bullet}} = \{A^i_{\overline{p}}\}$ with $i, p \geq 0$.
  - linear differential $d : A^i_{\overline{p}} \to A^{i+1}_{\overline{p}}$,
  - associative product $\mu : A^i_{\overline{p}} \otimes A^j_{\overline{q}} \to A^{i+j}_{\overline{p+q}}$ with unit $\eta : \mathbf{k} \to A^0_{\overline{0}}$.
  - a poset map $A^i_{\overline{q}} \to A^i_{\overline{p}}$ for every $q \leq p$.

  $+$ Leibnitz, commut. and compatibility of d and μ with poset maps:

$$\begin{tikzcd} A_{\overline{p}} \otimes A_{\overline{q}} \arrow[r, "\mu"] \arrow[d] & A_{\overline{p+q}} \arrow[d] \\ A_{\overline{p'}} \otimes A_{\overline{q'}} \arrow[r, "\mu"] & A_{\overline{p'+q'}} \end{tikzcd} \quad ; \quad \begin{tikzcd} A_{\overline{p}} \arrow[r, "d"] \arrow[d] & A_{\overline{p}} \arrow[d] \\ A_{\overline{p'}} \arrow[r, "d"] & A_{\overline{p'}} \end{tikzcd} .$$

- Quasi-isomorphisms: morphisms $f : A^*_{\overline{\bullet}} \longrightarrow B^*_{\overline{\bullet}}$ such that $f^*_{\overline{p}} : H^*(A_{\overline{p}}) \xrightarrow{\cong} H^*(B_{\overline{p}})$ for all p.

Theorem (Hovey '09)

The category of perverse cdga's admits a Quillen model structure with $\mathcal{W} =$ quasi-isomorphisms and $\mathcal{F}ib =$ surjections.

- $X =$ complex projective variety of dimension n, $\Sigma = Sing(X)$. $\dim \Sigma = 0$.
- $L = L(\Sigma, X)$ link of Σ in X ($\simeq$ compact real manifold of dimension $2n-1$).
- We have an inclusion $\iota : L \hookrightarrow X_{reg}$.

$$\begin{tikzcd} I\mathcal{A}^*_{\overline{p}}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \tau_{\leq p}\mathcal{A}^*_{pl}(L) \arrow[d] \\ \mathcal{A}^*_{pl}(X_{reg}) \arrow[r, "\iota^*"] & \mathcal{A}^*_{pl}(L) \end{tikzcd}\ .$$

- This gives a functor $I\mathcal{A}_{\overline{\bullet}} : \mathcal{V}_{\mathbb{C}} \longrightarrow \mathrm{Ho}(\mathcal{P}\mathrm{CDGA}_{\mathbb{Q}})$ such that

$$H^*(I\mathcal{A}_{\overline{\bullet}}(X)) \cong IH^*_{\overline{\bullet}}(X;\mathbb{Q}).$$

Properties

- $I\mathcal{A}_{\overline{0}}(X) \simeq \mathcal{A}_{pl}(\overline{X})$, where $\overline{X} \to X$ is a normalization of X.
- $I\mathcal{A}_{\overline{\infty}}(X) \simeq \mathcal{A}_{pl}(X_{reg})$.
- $IA^*_{\overline{\bullet}}(X)$ is a topological invariant of X (not homotopy invariant!).

- Let $\mathbb{Q} \subseteq \mathbf{K}$. Then X is intersection-formal over $\mathbf{K}$ if $I\mathcal{A}^*_{\bullet}(X) \otimes \mathbf{K}$ and $IH^*_{\bullet}(X; \mathbf{K})$ are isomorphic in $\mathrm{Ho}(\mathcal{P}\mathsf{CDGA}_{\mathbf{K}})$.
- X Intersection-formal $\Rightarrow \overline{X}$ and X_{reg} formal.
- Let $\mathcal{P}_n := \{0, 1, \cdots, 2n-2\}$. We have a forgetful functor

$$U : \mathcal{P}\mathsf{CDGA}_{\mathbf{K}} \longrightarrow \mathcal{P}_n\mathsf{CDGA}_{\mathbf{K}}$$

  defined by forgetting all products $A_{\overline{p}} \times A_{\overline{q}} \to A_{\overline{p+q}}$ such that $p + q > t$.
- X is GM-intersection-formal if $U(I\mathcal{A}^*_{\bullet}(X) \otimes \mathbf{K})$ and $U(IH^*_{\bullet}(X; \mathbf{K}))$ are isomorphic in $\mathrm{Ho}(\mathcal{P}_n\mathsf{CDGA}_{\mathbf{K}})$.
- GM-intersection-formality detects the vanishing of Massey products in Goresky and MacPherson's intersection cohomology.
- GM-intersection-formal $\not\Rightarrow X_{reg}$ formal.

Mixed Hodge theory

- X = complex projective variety with isolated singularities. $\Sigma = Sing(X)$.
- (Hironaka) There is a cartesian diagram

$$\begin{tikzcd} D \arrow[r, "j"] \arrow[d, "g"'] & \widetilde{X} \arrow[d, "f"] \\ \Sigma \arrow[r, "i"] & X \end{tikzcd}$$

  where $\widetilde{X}$ is smooth projective, f iso outside Σ and $D := f^{-1}(X)$ SNCD.
- Simplification: assume D is smooth (e.g. ordinary isolated singularities).
- Restriction morphism $j^s : H^s(\widetilde{X};\mathbb{Q}) \to H^s(D;\mathbb{Q})$.
- Gysin map $\gamma^s : H^{s-2}(D;\mathbb{Q}) \to H^s(\widetilde{X};\mathbb{Q})$.

$$\begin{tikzcd} E_1^{r,s}(X_{reg}) := \arrow[d] & H^{s-2}(D;\mathbb{Q}) \arrow[r, "\gamma^s"] \arrow[d, "Id"] & H^s(\widetilde{X};\mathbb{Q}) \arrow[d, "j^s"] \\ E_1^{r,s}(L) := & H^{s-2}(D;\mathbb{Q}) \arrow[r, "j^s \circ \gamma^s"] & H^s(D;\mathbb{Q}) \\ & r=-1 & r=0 \end{tikzcd}$$

- (Deligne / Durfee) $E_2(X_{reg}) \cong H^*(X_{reg};\mathbb{Q})$ and $E_2(L) \cong H^*(L;\mathbb{Q})$ are independent of the chosen resolutions.
- The map $H^*(\widetilde{X}) \times H^*(D) \longrightarrow H^*(D)$ given by $(x,a) \mapsto j^*(x)\cdot a$ and the cup products of $H^*(\widetilde{X})$ and $H^*(D)$ make $E_1(X_{reg})$ into a cdga.
- Same for $E_1(L)$ with the cup product of $H^*(D)$.

Theorem (Chataur, C.)

The morphism $E_1(\iota) : E_1(X_{reg}) \otimes \mathbb{C} \longrightarrow E_1(L) \otimes \mathbb{C}$ is a model of the morphism $\iota : \mathcal{A}_{pl}(X_{reg}) \otimes \mathbb{C} \longrightarrow \mathcal{A}_{pl}(L) \otimes \mathbb{C}$.

The (complex) intersection homotopy type of X is encoded in the perverse weight spectral sequence:

$$\begin{tikzcd}
IE^*_{1,\overline{p}}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \tau_{\leq p}E_1^{*,*}(L) \arrow[d] \\
E_1^{*,*}(X_{reg}) \arrow[r, "\iota^*"] & E_1^{*,*}(L)
\end{tikzcd} .$$

- A mixed Hodge diagram $\mathbf{A}$ for a topological space X is a filtered cdga $(A_{\mathbb{Q}}, W)$, such that $A_{\mathbb{Q}} \simeq \mathcal{A}_{pl}(X)$, a bifiltered cdga $(A_{\mathbb{C}}, W, F)$ and a string of quasi-isomorphisms

$$(A_{\mathbb{Q}}, W) \otimes \mathbb{C} \xleftarrow{\sim} \cdot \xrightarrow{\sim} (A_{\mathbb{C}}, W).$$

  $+$ axioms making its cohomology into a graded mixed Hodge structure.

Idea of proof.

- (Morgan) $\exists$ a mixed Hodge diagram $\mathbf{A}(X_{reg})$ for X_{reg}.
- (Durfee-Hain) $\exists$ a mixed Hodge diagram $\mathbf{A}(L)$ for L.
- (C.-Guillén) If $\mathbf{A} \longrightarrow \mathbf{B}$ is a morphism of mixed Hodge diagrams then

$$\begin{tikzcd} A_{\mathbb{C}} \arrow[d] & M \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d] & E_1(A_{\mathbb{C}}, W) \arrow[d] \\ B_{\mathbb{C}} & M' \arrow[l, "\sim"'] \arrow[r, "\sim"] & E_1(B_{\mathbb{C}}, W) \end{tikzcd}$$

- Use Navarro-Aznar's simple $\mathbf{s}_{TW} : \Delta\mathrm{MHD} \longrightarrow \mathrm{MHD}$ to describe $E_1(A_{\mathbb{Q}}(X_{reg})) \simeq E_1(X_{reg})$ and $E_1(A_{\mathbb{Q}}(L)) \simeq E_1(L)$. $\square$

How to use this to study intersection-formality?

- Try to build a morphism of perverse cdga's

$$IE_{1,\bar{\bullet}}^{*,*}(X) \xleftarrow{\sim} \cdot \xrightarrow{\sim} IE_{2,\bar{\bullet}}^{*,*}(X) \cong IH_{\bar{\bullet}}^{*}(X;\mathbb{Q}).$$

- This will give $I\mathcal{A}_{\bar{\bullet}}(X) \otimes \mathbb{C} \xleftarrow{\sim} \cdot \xrightarrow{\sim} IH_{\bar{\bullet}}^{*}(X;\mathbb{C})$.

Corollary

If the weight filtration on $IH^k(X;\mathbb{Q})$ is trivial (pure of weight k), for all $k \geq 0$, then X is intersection-formal over $\mathbb{C}$.

Examples:

- Complex projective varieties whose cohomology satisfies Poincaré duality.
- Complex projective varieties that are $\mathbb{Q}$-homology manifolds.
  - Weighted projective spaces, V-manifolds, Cayley Cubic, Kummer surface.

- By purely topological reasons, every simply connected projective surface is GM-intersection-formal.
- (Simpson, Kapovich, Kollár) There exist non-formal projective surfaces.

Theorem (Chataur, C.)

Every complex projective surface with isolated singularities X is GM-intersection-formal over $\mathbb{C}$. If X as only one singular point, then X_{reg} is also formal and X is intersection-formal over $\mathbb{C}$.

Theorem (Chataur, C.)

Let X be a complex projective variety of dimension n with isolated singularities $\Sigma = Sing(X)$.

(1) *If the link of σ in X is $(n-2)$-connected for all $\sigma \in \Sigma$ then X is formal over $\mathbb{Q}$.*

(2) *If the singularities are ordinary, then X is GM-intersection-formal over $\mathbb{C}$.*

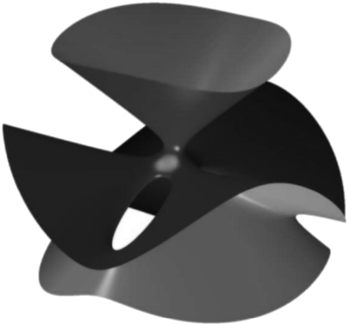
(3) *If $\Sigma = \{*\}$ then X_{reg} is formal over $\mathbb{Q}$ and X is intersection-formal over $\mathbb{C}$.*

Examples where (1) applies:

- Hypersurfaces with isolated singularities and complete intersections.
- $Y \hookrightarrow X$ closed immersion with $Sing(X) \subset Y$. Then X/Y is formal.

$$S : \left\{ x_0 + x_1 + x_2 + x_3 + x_4 + x_5 = 0,\ x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0 \right\}.$$

Projective threefold with 10 ordinary isolated singular points.



Real representation of the Segre cubic

Example: the Segre cubic

- Resolution of the Segre cubic:

$$\begin{tikzcd} D = \bigsqcup_{10} \mathbb{CP}^1 \times \mathbb{CP}^1 \arrow[r, "j"] \arrow[d, "g"'] & \overline{\mathcal{M}}_{0,6} \arrow[d, "f"] \\ \Sigma = \{10 \text{ pts }\} \arrow[r, "i"] & S \end{tikzcd}$$

- Non-trivial Betti numbers of $\overline{\mathcal{M}}_{0,6}$: $b_0 = b_6 = 1$ and $b_2 = b_4 = 16$.
- For $\sigma_i \in \Sigma$, $L_i := L(\sigma_i, X) \simeq S^2 \times S^3$.

$H^*(S;\mathbb{Q}) \cong$

$\mathbb{Q}$
0
$\mathrm{Ker}(j^4) \cong \mathbb{Q}^6$
$\mathrm{Coker}(j^2) \cong \mathbb{Q}^5$
$\mathrm{Ker}(j^2) \cong \mathbb{Q}$
0
$\mathbb{Q}$

$j^s : H^s(\overline{\mathcal{M}}_{0,6};\mathbb{Q}) \to H^s(D;\mathbb{Q})$ denotes the restriction map

$\pi_2 \cong \mathbb{Q}$, $\pi_3 \cong \pi_4 \cong \mathbb{Q}^5$,
$\pi_5 \cong \mathbb{Q}^{15}$, $\pi_6 \cong \mathbb{Q}^{50}$
$\pi_7 \cong \mathbb{Q}^{116}, \cdots$

Example: the Segre cubic

$$IH^*_{\overline{p}}(S;\mathbb{Q}) \cong \underset{0\le p\le 1}{\begin{array}{|c|}\hline \mathbb{Q} \\ \hline 0 \\ \hline H^2(S;\mathbb{Q})^\vee \oplus Exc^\vee \\ \hline Van \\ \hline H^2(S;\mathbb{Q}) \\ \hline 0 \\ \hline \mathbb{Q} \\ \hline \end{array}} \; ; \; \underset{p=2}{\begin{array}{|c|}\hline \mathbb{Q} \\ \hline 0 \\ \hline H^2(S;\mathbb{Q})^\vee \oplus Exc^\vee \\ \hline 0 \\ \hline H^2(S;\mathbb{Q}) \oplus Exc \\ \hline 0 \\ \hline \mathbb{Q} \\ \hline \end{array}} \; ; \; \underset{3\le p\le 4}{\begin{array}{|c|}\hline \mathbb{Q} \\ \hline 0 \\ \hline H^2(S;\mathbb{Q})^\vee \\ \hline Van^\vee \\ \hline H^2(S;\mathbb{Q}) \oplus Exc \\ \hline 0 \\ \hline \mathbb{Q} \\ \hline \end{array}}$$

- $Van := H^3(S;\mathbb{Q}) \cong \mathrm{Coker}(j^2) \cong \mathbb{Q}^5$.
- $\gamma^s : H^{s-2}(D;\mathbb{Q}) \to H^s(\overline{\mathcal{M}}_{0,6};\mathbb{Q})$ is the Gysin map.
- $Exc \cong \mathbb{Q}^5$ is defined via the decomposition

$$H^2(\overline{\mathcal{M}}_{0,6};\mathbb{Q}) \cong \mathrm{Ker}(j^2) \oplus \mathrm{Coker}(\gamma^2) \oplus Exc.$$

Thank you!