Coproduct of 2-crossed modules. Applications to a definition of a tensor product for 2-crossed complexes

Timothy Porter

Emeritus Professor, University of Wales, Bangor

November 30, 2016

Some history

Algebraic models and homotopy types

Some relevant examples (non-exhaustive list).

- Crossed modules (Reidemeister, 1930s; Whitehead, 1940s - '50s)
- n-types - Fox, Whitehead 1950; Whitehead and Mac Lane, 1940s-50s; Loday, 1980s)
- $\mathcal{S}$-group(oid)s (Kan, 1958; Dwyer-Kan, 1984)
- Crossed complexes (Whitehead,1950; Brown-Higgins, 1970s-2010)
- 2-crossed modules (Conduché, 1984)
- 2-crossed complexes ...

Crossed modules

A crossed module consists of two groups, M and N, a morphism, $\partial : M \to N$, and an action of N on M, that we will take to be a left action. These are to satisfy two axioms: for all $m, m' \in M$ and $n \in N$,

- $\partial({}^{n}m) = n\partial(m)n^{-1}$;
- ${}^{\partial m}m' = mm'm^{-1}$, (Peiffer identity.)

If the first axiom is satisfied then we say we have a pre-crossed module. The crossed modules form a reflective subcategory within the category of pre-crossed modules.

If the bottom group, N is fixed throughout a discussion, then we may call $\partial : M \rightarrow N$ a crossed N-module, or, sometimes, a crossed module 'over' N.

$XMod$ = category of crossed modules.

$XMod/N$ = category of crossed N-crossed modules.

Examples of crossed modules

- Normal subgroups, and G-modules give examples of crossed modules;
- If (X, A) is a pair of pointed spaces, $\partial : \pi_2(X, A) \to \pi_1(A)$ is a crossed module.
- If G is a simplicial group, whose Moore complex, NG, has length 2, then that complex is a crossed module.

The Moore complex, (NG, ∂) of G has $NG_n = \bigcap_{k=1}^{n} Kerd_k$ with its boundary, ∂, being the restriction of d_0.

n-types

A continuous map $f : X \to Y$ is an *n-equivalence* if it induces a bijection on π_0 and for each choice of base point $x_0 \in X$, and each $k \leq n$, $\pi_k(f) : \pi_k(X, x_0) \to \pi_k(Y, f(x_0))$ is an isomorphism.

Two spaces that are linked by a zig-zag of *n*-equivalences are said *to have the same n-type*.

- Sets model 0-types.
- Group(oid)s model (connected) 1-types via π_1.
- (Mac Lane and Whitehead, 1950) Crossed modules model 2-types.
- 2-crossed modules (Conduché)[1], cat^2-groups (Loday), ... model 3-types
- cat^{n-1}-groups, and $(n-1)$-truncated $\mathcal{S}$-groupoids model n-types.
- Idea: Each type of model deserves to be studied in its own right both 'categorically' and 'algebraically' to assist calculation, e.g., via van Kampen theorems, Eilenberg-Zilber results, etc., but also simply out of algebraic curiosity.

[1] ... see later

We can do ‘better’.

- Fundamental groups, π_1, together with chain complexes of π_1-modules model 1-types together with a chain model of the universal cover.
- Crossed complexes model 2-types plus, again, chains on universal cover.
- and beyond ...?

Crossed complexes: some ideas

- Idea 1: a ‘crossed module with a tail’, that is, with a higher dimensional chain complex spliced to it. (This also allows greater freedom to construct homotopies, tensor products, etc.)
- Idea 2: positive chain complex of (possibly non-abelian) groups, abelian above level 2, but with a crossed module at its base.
- Idea 3: (as above) should model both the 2-type and the ‘chains on the universal cover’ of a CW-complex.

Crossed complexes: the definition

A (reduced) *crossed complex*, C, consists of a sequence of groups and morphisms

$$\mathrm{C} : \ldots \to C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \ldots \to C_3 \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1$$

satisfying the following:
CC1) $\delta_2 : C_2 \to C_1$ is a crossed module;
CC2) each C_n, for $(n > 2)$, is a left $C_1/\delta_2 C_2$-module and each δ_n, for $(n > 2)$, is a morphism of left $C_1/\delta_2 C_2$-modules, (for $n = 3$, this means that δ_3 commutes with the action of C_1 and that $\delta_3(C_3) \subseteq C_2$ must be a $C_1/\delta_2 C_2$-module);
CC3) $\delta\delta = 0$.
Morphisms are defined in the obvious way.
(There is also a non-reduced groupoid based version which we do not give in detail.)

Examples of crossed complexes I:

1) A *filtered space* is a space, X, together with a nested family $(X_n)_{n\in\mathbb{N}}$, e.g. this might be the filtration given by the skeleta of a CW-complex structure on X.
For any filtered space, $\underline{X} = (X_n)_{n\in\mathbb{N}}$, its *fundamental crossed complex*, $\underline{\pi}(\underline{X})$, is, in general, a non-reduced crossed complex with

$$\underline{\pi}(\underline{X})_n = (\pi_n(X_n, X_{n-1}, a))_{a\in X_0}$$

with $\underline{\pi}(\underline{X})_1$, the fundamental groupoid $\Pi_1 X_1 X_0$, and, for $n \geq 2$, $\underline{\pi}(\underline{X})_n$, the family, $(\pi_n(X_n, X_{n-1}, a))_{a\in X_0}$ having an action by $\Pi_1 X_1 X_0$ given by change of base point.
This crossed complex will only be reduced if X_0 consists just of one point.

Examples of crossed complexes II: from simplicial groups

2) Given any simplicial group, G, the formula,

$$\mathrm{C}(G)_{n+1} = \frac{NG_n}{(NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1})},$$

in higher dimensions with, at its 'bottom end', the crossed module,

$$\frac{NG_1}{d_0(NG_2 \cap D_2)} \to NG_0$$

gives a crossed complex with ∂ induced from the boundary in the Moore complex.

(Here D_n is the subgroup of G_n generated by the degenerate elements.)

Starting with an $\mathcal{S}$-groupoid, 'the same' formula would give a non-reduced crossed complex.

2-crossed complexes ... almost!

2-crossed complexes model 3-types plus ‘chains on universal cover’.

These, clearly, ‘must be’ 2-crossed modules plus a ‘tail’, so we first need:

2-crossed modules: definition

2-crossed modules, what are they?

- Simple answer: Moore complexes of length 2

$$\ldots \to 1 \to NG_2 \to NG_1 \to NG_0.$$

Detailed answer: A *2-crossed module* is a normal complex of groups, $L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$, together with an action of N on all three groups and a mapping $\{-,-\} : M \times M \to L$ such that

(i) the action of N on itself is by conjugation, and ∂_2 and ∂_1 are N-equivariant;

(ii) for all $m_0, m_1 \in M$, $\partial_2\{m_0, m_1\} = {}^{\partial_1 m_0}m_1.m_0 m_1^{-1} m_0^{-1}$;

(iii) if $\ell_0, \ell_1 \in L$, then

$$\{\partial_2\ell_0, \partial_2\ell_1\} = [\ell_0, \ell_1];$$

(iv) if $\ell \in L$ and $m \in M$, then

$$\{m, \partial\ell\}\{\partial\ell, m\} = {}^{\partial m}\ell.\ell^{-1};$$

(v) for all $m_0, m_1, m_2 \in M$,

(a) $\{m_0, m_1 m_2\} = \{m_0, m_1\}\{\partial\{m_0, m_2\}, (m_0 m_1 m_0^{-1})\}\{m_0, m_2\}$;

(b) $\{m_0 m_1, m_2\} = {}^{\partial m_0}\{m_1, m_2\}\{m_0, m_1 m_2 m_1^{-1}\}$;

(vi) if $n \in N$ and $m_0, m_1 \in M$, then

$${}^{n}\{m_0, m_1\} = \{{}^{n}m_0, {}^{n}m_1\}.$$

As stated before, 2-crossed modules model 3-types.

The $\{-,-\}$ lifts/covers the Peiffer identity so is called a Peiffer lifting, (cf. here back on slide 4 for the Peiffer identity.)

Any crossed complex gives a 2-crossed complex with trivial $\{-,-\}$, and conversely.

Crossed complexes form a reflexive subcategory within that of 2-crossed complexes.

Algebraic and categorical structures of these algebraic models

All these ‘models’ are algebras for many sorted algebraic theories so categories of such objects have nice limits, colimits, etc.

Problem: give explicit constructions of these categorical structures, which are amenable to calculation.

Products and limits are easy to give. Coproducts and colimits less so, but they are needed for example in calculations with homotopy types. They also linked to ideas on ‘free objects’, induction functors, etc.

Coproducts

Coproducts of

- 0-types (i.e. sets) - disjoint union.
- 1-types (groups or groupoids) - $G * H$ 'free product' (made of words in G and H)
- 2-types - two stage process:
  (i) induction along a morphism;
  (ii) coproduct in categories of the form, $XMod/N$.

(i) Induction along a morphism

Suppose we have $\varphi : G \to H$ is a morphism. Pullback induces a functor

$$\varphi^* : XMod/H \to XMod/G$$

and this has a left adjoint φ_* so, for C a crossed module over G, $\varphi_*(\mathsf{C})$ is one over H.

Given C over G and D over H, use $i_G : G \to G * H$ and $i_H : H \to G * H$ to induce and get $i_{G,*}(\mathsf{C})$ and $i_{H,*}(\mathsf{D})$ both over $G * H$.

(ii) Coproducts in $XMod/Q$

Form coproduct first as pre-crossed modules. (We will especially need the case $Q = G * H$ later.)
Take $\mathsf{M} = (M, P, \mu)$ and $\mathsf{N} = (N, P, \nu)$, two crossed modules, then

$$\mathsf{M} *_P \mathsf{N} = (M * N, P, \mu * \nu)$$

is a pre-crossed module.

Now form the associated crossed module by dividing out by the Peiffer subgroup to give $(\mathsf{M} * \mathsf{N})^{cr}$, which is the required coproduct of M and N.

Can extend to get coproducts of crossed complexes.

Finally: **Coproducts of 2-crossed modules and complexes**
This is joint work with Pilar Carrasco.
Plan:
(i) Induction along a (base) morphism of pre-crossed modules;
(ii) quasi-2-crossed complexes (plays the role of pre-crossed modules in previous discussion);
(iii) coproduct of quasi-2-crossed complexes, then
(iv) reflect back into $2 - XMod$.

(i) In a 2-crossed module, $\mathfrak{L} = (L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N)$, the bottom stage: $\mathsf{L} := M \xrightarrow{\partial_1} N$, is a pre-crossed module. There is a category, $2 - XMod/\mathsf{L}$, of 2-crossed modules 'over' L.

Given a morphism of precrossed modules, $\varphi : \mathsf{C} \to \mathsf{L}$, the pullback of $\mathfrak{L}$ along φ gives a 2-crossed module, $\varphi^*(\mathfrak{L})$, over C.

We get a functor $\varphi^* : 2 - XMod/\mathsf{L} \to 2 - XMod/\mathsf{C}$, which has a left adjoint φ_* with an explicit algebraic construction.

Given two 2-crossed modules, $\mathfrak{L}_1$ and $\mathfrak{L}_2$, form the coproduct $\mathsf{L}_1 * \mathsf{L}_2$ of their 'base' precrossed modules, and induce along the canonical morphisms to move everything to $2 - XMod/\mathsf{L}_1 * \mathsf{L}_2$.

Form the coproduct in $2 - XMod/\mathsf{L}_1 * \mathsf{L}_2$, using an intermediate form of structure:'quasi-2-crossed' module, then reflect back into $2 - XMod/\mathsf{L}_1 * \mathsf{L}_2$, to get the coproduct of $\mathfrak{L}_1$ and $\mathfrak{L}_2$.

Extend to $2 - Crs$.

Where to next?

Crossed complexes have a tensor product with nice properties.
It is constructed using coproducts of crossed complexes, free constructions etc.
There is an Eilenberg-Zilber theorem linking $\underline{\pi}(\underline{X \times Y})$ with $\underline{\pi}(\underline{X}) \otimes \underline{\pi}(\underline{Y})$. (Brown-Higgins, Tonks, etc.)
One can use the coproduct of 2-crossed complexes to define and construct a tensor product for 2-crossed complexes. (See Carrasco and Porter.)

Things to do:

To explore: Is there an Eilenberg-Zilber theorem for a 2-crossed complex analogue of $\underline{\pi}(\underline{X})$? We don't know, but it looks likely.

Even more distant: Is there a tensor product for $\mathcal{S}$-groupoids? Again it looks likely and should have an Eilenberg-Zilber theorem, but technically this seems very hard and we are missing some tool to make it work.
A lot of other things should generalise, but ...

The End ... for the moment!