Algunos aspectos computacionales en geometría algebraica

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Basado en un proyecto común con

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Part I:

Basic computations in algebraic geometry

- dimension and Hilbert function
- syzygies
- projective dimension and depth, regularity

Dimension and Hilbert function (polynomial and series)

Let K be an arbitrary field and $R = K[x_0, \ldots, x_n]$ the polynomial ring in $n+1$ variables. Consider $I \subset R$, a **homogeneous** ideal.

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KEY IDEA: 'Reduce questions on arbitrary ideals to questions on monomial ideals which are much easier (in fact, often combinatorial in nature)'. [Decker-Lossen, 2006]

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KEY INGREDIENTS:

- Gröbner basis (and Buchberger’s algorithm) in order to determine $\mathrm{in}\,(I)$, the initial ideal of I w.r.t. some monomial ordering.

- Macaulay’s theorem (1927): R/I and $R/\mathrm{in}\,(I)$ have the same Hilbert function (polynomial and series).

In particular, $\dim R/I = \dim R/\mathrm{in}\,(I)$ which is easy to compute (see [Cox-Little-O’Shea, 92]).

Finer numerical information: syzygies

Consider a minimal graded free resolution (m.g.f.r.) of I,

$$0 \to \bigoplus_j R(-j)^{\beta_{p,j}} \xrightarrow{\phi_p} \cdots \xrightarrow{\phi_2} \bigoplus_j R(-j)^{\beta_{1,j}} \xrightarrow{\phi_1} \bigoplus_j R(-j)^{\beta_{0,j}} \xrightarrow{\phi_0} I \to 0\,.$$

The $\beta_{i,j} \neq 0$ are the **graded Betti numbers** of I (and setting $\beta_i := \sum_j \beta_{i,j}$, one has that $\beta_0, \ldots, \beta_p$ are its **Betti numbers**).

This is a finer numerical information because one gets the Hilbert function of R/I from $\{\beta_{i,j}(I)\}$, the set of Betti numbers of I.

A g.f.r. of in$\,(I)$ –in particular a minimal one– can be ‘lifted’ to a g.f.r. of I –not minimal in general–

$$\Rightarrow \quad \beta_{i,j}(I) \leq \beta_{i,j}(\mathrm{in}\,(I)),\ \forall i, j\ .$$

Equality does not hold in general.

Construction of a minimal graded free resolution

Buchberger's algorithm also gives the first syzygies of a module and hence provides a minimal graded free resolution.

Example. Consider the ideal $I \subset \mathbb{Q}[x_0, \ldots, x_3]$ generated by

$$x_0^2 - 3x_0x_1 + 5x_0x_3,\ x_0x_1 - 3x_1^2 + 5x_1x_3,\ x_0x_2 - 3x_1x_2,$$

$$2x_0x_3 - x_1x_3,\ x_1^2 - x_1x_2 - 2x_1x_3\,.$$

For the reverse lexicographic order, $\text{in}\,(I) = (x_0^2,\ x_0x_1,\ x_0x_2,\ x_0x_3,\ x_1^2,\ x_1x_2x_3)$, and the m.g.f.r. of I and $\text{in}\,(I)$ are

$$0 \longrightarrow R(-4)^2 \longrightarrow R(-3)^6 \longrightarrow R(-2)^5 \longrightarrow I \longrightarrow 0$$

$$0 \longrightarrow R(-5) \longrightarrow \begin{array}{c} R(-4)^4 \\ \oplus \\ R(-5) \end{array} \longrightarrow \begin{array}{c} R(-3)^7 \\ \oplus \\ R(-4)^2 \end{array} \longrightarrow \begin{array}{c} R(-2)^5 \\ \oplus \\ R(-3) \end{array} \longrightarrow \text{in}\,(I) \longrightarrow 0$$

Projective dimension and depth, regularity

$$0 \to \bigoplus_j R(-j)^{\beta_{p,j}} \xrightarrow{\phi_p} \cdots \xrightarrow{\phi_1} \bigoplus_j R(-j)^{\beta_{0,j}} \xrightarrow{\phi_0} I \to 0\,.$$

DEFINITION. The size of a the minimal resolution is given by

- the projective dimension of I, $\mathrm{pd}\,(I) := p$, and
- the Castelnuovo-Mumford regularity of I,

$$\mathrm{reg}\,(I) := \max\{j - i \text{ such that } \beta_{i,j} \neq 0\}\,.$$

By the Auslander-Buchsbaum formula, $\mathrm{depth}\,(R/I) = n - \mathrm{pd}\,(I)$ and hence

$$\begin{aligned} \mathrm{reg}\,(I) &\leq \mathrm{reg}\,(\mathrm{in}\,(I)) \\ \mathrm{pd}\,(I) &\leq \mathrm{pd}\,(\mathrm{in}\,(I)) \\ \mathrm{depth}\,(R/I) &\geq \mathrm{depth}\,(R/\mathrm{in}\,(I)). \end{aligned}$$

Part II:

Computation of the regularity

- Bayer and Stillman
- monomial ideals of nested type
- projective changes of coordinates
- implementation of the algorithms

Computational issues: Bayer-Stillman

One would like to find some conditions **(C)** such that:

- When I satisfies **(C)**, $\operatorname{reg}(I) = \operatorname{reg}(\operatorname{in}(I))$ and $\operatorname{depth}(R/I) = \operatorname{depth}(R/\operatorname{in}(I))$.

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- When I satisfies **(C)**, $\operatorname{in}(I)$ should have 'good' properties that make easy the computation of $\operatorname{reg}(\operatorname{in}(I))$ and $\operatorname{depth}(R/\operatorname{in}(I))$.

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- One should easily **check** when conditions **(C)** are satisfied.

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- One should easily **check** when conditions **(C)** are satisfied.

- Given a homogeneous ideal I that does not satisfy **(C)**, one should get an ideal satisfying **(C)** making some **projective change of coordinates** (which?).

From now on, the monomial order is the **reverse lexicographic order**. Denote by $d := \dim R/I$ and by $I^{\mathrm{sat}} := I : (x_0, \ldots, x_n)^\infty$.

[BS87] D. Bayer and M. Stillman.
A criterion for detecting m-regularity.
Invent. Math. **87**:1–11, 1987.

The **BS-Conditions:** x_n is a nzd (nonzero divisor) on R/I^{sat},
x_{n-1} is a nzd on $R/(I, x_n)^{\mathrm{sat}}$, ...

$$\forall i,\ n \geq i \geq n-d+1,\ x_i \text{ is a nzd on } R/(I, x_n, \ldots, x_{i+1})^{\mathrm{sat}}.$$

Theorem ([BS87]). If I satisfies the BS-Conditions then

$$\operatorname{reg}(I) = \operatorname{reg}(\operatorname{in}(I))$$
$$\operatorname{depth}(R/I) = \operatorname{depth}(R/\operatorname{in}(I)).$$

THEOREM ([BS87]).
Let $I \subset R := K[x_0, \ldots, x_n]$ be a homogeneous ideal.

- I satisfies the BS-Conditions $\Leftrightarrow$ in (I) does.

- There is a dense Zariski open subset $\mathcal{U} \subset \mathrm{GL}(n+1, K)$ such that in $(\varphi(I))$ is constant and Borel fixed for $\varphi \in \mathcal{U}$ (Galligo, 74) (generic initial ideal, Gin (I))

  and Borel fixed ideals satisfy the BS-Conditions.

- If the **characteristic of** K **is** 0 and I is Borel fixed, then

  * pd $(R/I) = p$ where p is the least integer such that none of the minimal generators of I involve $x_{p+1}, \ldots, x_n$, and hence depth $(R/I) = n - p$.

  * reg $(I) =$ maxdeg (I).

ALGORITHM. Assume that K is infinite.

Algorithm. Assume that K is infinite.

- Choose randomly an element φ in $\mathrm{GL}(n+1, K)$ and compute $\mathrm{in}\,(\varphi(I))$, which is (hopefully!) $\mathrm{Gin}\,(I)$.

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- $\mathrm{reg}\,(I) = \mathrm{maxdeg}\,(\mathrm{Gin}\,(I))$ and $\mathrm{depth}\,(R/I) = n - p$ where p is the least integer such that none of the m.g. of $\mathrm{Gin}\,(I)$ involve $x_{p+1}, \ldots, x_n$ **if Char** $(K) = 0$.

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PROBLEMS.

- Probabilistic algorithm : no test for $\mathrm{Gin}\,(I)$.

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PROBLEMS.

- Probabilistic algorithm : no test for $\mathrm{Gin}\,(I)$.

- $\mathrm{Char}\,(K) > 0$?

ALGORITHM. Assume that K is infinite.

- Choose randomly an element φ in $\mathsf{GL}(n+1, K)$ and compute in$(\varphi(I))$, which is (hopefully!) Gin(I).

- reg(I) = maxdeg(Gin(I)) and depth$(R/I) = n - p$ where p is the least integer such that none of the m.g. of Gin(I) involve $x_{p+1}, \ldots, x_n$ **if Char**$(K) = 0$.

PROBLEMS.

- Probabilistic algorithm : no test for Gin(I).

- Char$(K) > 0$?

- in$(\varphi(I))$ can not be obtained in general.

Monomial ideals of nested type

One would like to

- compute depth and regularity of monomial ideals satisfying the BS-conditions,
- check easily when a monomial ideal satisfies the BS-conditions,
- in general, find 'smaller' changes of coordinates (not the generic ones) that put variables in the correct position.

[BG06] I. Bermejo and P.G.
Saturation and Castelnuovo-Mumford regularity.
J. Algebra **303**: 592–617, 2006.

Let $I \subset R = K[x_0, \ldots, x_n]$ be a monomial ideal with $d := \dim R/I$ (K **arbitrary** field).

DEFINITION/PROPOSITION ([BG06]). I is **of nested type** if one of the following equivalent properties holds:

1. I satisfies the BS-Conditions.

2. For any prime ideal $\mathfrak{p} \subset R$ associated to I, there exists $i \in \{0, \ldots, n\}$ such that $\mathfrak{p} = (x_0, \ldots, x_i)$.

3.
   - $\forall i \in \{0, \ldots, n-d\}$, $\exists k_i \geq 1$ such that $x_i^{k_i} \in I$ &
   - $I : (x_n)^\infty \subseteq I : (x_{n-1})^\infty \subseteq \cdots \subseteq I : (x_{n-d+1})^\infty$.

4. $\forall i \in \{0, \ldots, n\}, \; I : (x_i)^\infty = I : (x_0, \ldots, x_i)^\infty$.

EXAMPLE. Any Borel fixed ideal is of nested type (K arbitrary).

Let $I \subset R$ be a **monomial ideal of nested type** and let p be the least integer such that no min. gen. of I involves $x_{p+1}, \ldots, x_n$.

PROPOSITION. $\mathrm{pd}\,(R/I) = p$, and hence $\mathrm{depth}\,(R/I) = n - p$.

THEOREM ([BG06]). Set $d := \dim R/I$.

$$\bullet\ \mathrm{reg}\,(I) = \max\{ \begin{array}{l} \mathrm{sat}\,(I \cap K[x_0, \ldots, x_p])\ , \\ \mathrm{sat}\,(I|_{x_p=1} \cap K[x_0, \ldots, x_{p-1}])\ , \\ \mathrm{sat}\,(I|_{x_{p-1}=1} \cap K[x_0, \ldots, x_{p-2}])\ , \\ \quad\vdots \\ \mathrm{sat}\,(I|_{x_{n-d+1}=1} \cap K[x_0, \ldots, x_{n-d}])\ \} \end{array}$$

where the **satiety** $\mathrm{sat}\,(J)$ of a homogeneous ideal $J \subset R$ is the least integer m such that, for all $s \geq m$, $J_s = (J^{\mathrm{sat}})_s$. Moreover, $\mathrm{sat}\,(J) = \max_{1 \leq i \leq r} \{\deg(h_i);\ h_i \notin J\} + 1$ where $h_1, \ldots, h_r$ are homogeneous and $J : (x_0, \ldots, x_n) = (h_1, \ldots, h_r)$.

• $\mathrm{reg}\,(I) = \max\{\mathrm{reg}\,(\mathfrak{q}_i);\ 1 \leq i \leq r\}$ where $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ is the (unique) irredundant irreducible decomposition of I.

Computation of $\operatorname{reg}(I)$ when $\operatorname{in}(I)$ is of nested type

If K is arbitrary and $I \subset R = K[x_0, \ldots, x_n]$ is a homogeneous ideal such that $\operatorname{in}(I)$ **is of nested type**, $\operatorname{depth}(R/I)$ and $\operatorname{reg}(I)$ are obtained using the results in [BS87] and the previous formulae from [BG06]. This occurs for example when

- $\dim R/I = 0$, or
- R/I Cohen-Macaulay & $K[x_{n-d+1}, \ldots, x_n]$ is a N.N. of R/I, or
- $\dim R/I = 1$ & $K[x_n]$ is a Noether Normalization of R/I, or
- $\dim R/I = 2$, $K[x_{n-1}, x_n]$ is a N.N. of R/I & I prime
(e.g., if I is the defining ideal of a projective toric curve).

Recall that if $J \subset R$ is an ideal with $\dim R/J = d$, one has that $K[x_{n-d+1}, \ldots, x_n]$ is a **Noether Normalization** of R/J if $K[x_{n-d+1}, \ldots, x_n] \hookrightarrow R/J$ is an integral ring extension.

EXAMPLE. Let $I \subset \mathbb{C}[x_0, \ldots, x_4]$ be the defining ideal of the projective monomial curve parametrically defined by

$$x_0 = s^5t^{15},\ x_1 = s^9t^{11},\ x_2 = s^{11}t^9,\ x_3 = s^{20},\ x_4 = t^{20}\,.$$

One knows before hand that in(I) is of NT. One has that

$$\begin{aligned} \text{in}\,(I) &= (x_0^4, x_1^3, x_2^5, x_1x_2, x_0x_2^3, x_0^3x_1^2, x_0^3x_1x_3) \\ &= \quad (x_0, x_1^3, x_2^3) \\ &\quad \bigcap\ (x_0^4, x_1^3, x_2) \\ &\quad \bigcap\ (x_0^2, x_1, x_2^5) \\ &\quad \bigcap\ (x_0, x_1^2, x_2^5, x_3) \end{aligned}$$

and hence

$$\begin{aligned} \text{reg}\,(I) &= \max\{0+2+2, 3+2+0, 1+0+4, 0+1+4+0\}+1 \\ &= 6\,. \end{aligned}$$

Remark. If $\dim R/I = 2$ & $K[x_{n-1}, x_n]$ is a N.N. of R/I, $\mathrm{in}\,(I)$ may not be of nested type and one can have $\mathrm{reg}\,(I) \neq \mathrm{reg}\,(\mathrm{in}\,(I))$.

[BG00] I. Bermejo and P.G.
On Castelnuovo-Mumford regularity of projective curves.
Proc. Amer. Math. Soc. **128**: 1293–1299, 2000.

Example ([BG00, Rmk 2.10]). $I \subset R = \mathbb{Q}[x_0, \ldots, x_3]$

$$I = (x_0^2 - 3x_0x_1 + 5x_0x_3,\ x_0x_1 - 3x_1^2 + 5x_1x_3,\ x_0x_2 - 3x_1x_2,$$

$$2x_0x_3 - x_1x_3,\ x_1^2 - x_1x_2 - 2x_1x_3).$$

Then $\mathrm{in}\,(I) = (x_0^2,\ x_0x_1,\ x_0x_2,\ x_1^2,\ x_1x_2x_3)$ and

- $\mathrm{reg}\,(I) = 2 < \mathrm{reg}\,(\mathrm{in}\,(I)) = 3$
- $\mathrm{depth}\,(R/I) = 1 > \mathrm{depth}\,(R/\mathrm{in}\,(I)) = 0.$

Computation of reg (I) when $K[x_{n-d+1},\ldots,x_n]$ is a N.N.

Set $N := \frac{d(d-1)}{2}$. For $\gamma = (\gamma_1,\ldots,\gamma_N) \in \mathbb{A}^N_K$, consider the homogeneous linear transformation $\Psi(\gamma) : R \to R$ defined by:

$$\begin{array}{lclcl} x_n & \mapsto & x_n + \gamma_1 x_{n-1} + \gamma_2 x_{n-2} + \cdots\cdots + \gamma_{d-1} x_{n-d+1} \\ x_{n-1} & \mapsto & x_{n-1} + \gamma_d x_{n-2} + \cdots\cdots + \gamma_{2d-3} x_{n-d+1} \\ & \vdots & \\ x_{n-d+2} & \mapsto & x_{n-d+2} + \gamma_{\frac{d(d-1)}{2}} x_{n-d+1} \end{array}$$

Theorem ([BG06]). There is a dense Zariski open subset $\mathcal{U}$ of $\mathbb{A}^N_K$ such that in$(\Psi(\gamma)(I))$ is constant and of nested type for $\gamma \in \mathcal{U}$. We call it the **monomial ideal of nested type associated to** I and denote it by $N(I)$.

Example ([BG00, Rmk 2.10]). In the previous example, in$(I) = (x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^2, x_1x_2x_3)$ is not of NT. Applying a change of coordinate of the form $\Psi : R \to R$, $x_3 \mapsto x_3 + \gamma x_2$, one gets that $N(I) :=$ in$(\Psi(I)) = (x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2) = (x_0, x_1) \cap (x_0, x_1^2, x_2) \cap (x_0^2, x_1, x_2)$, and hence reg$(I) = 2$.

How do we know that we have a N.N. in this example?

[BG01] I. Bermejo and P.G.
Computing the Castelnuovo-Mumford regularity of some subschemes of $\mathbb{P}^n_K$ using quotients of monomial ideals.
J. Pure Appl. Algebra **164**: 23–33, 2001.

Lemma (Noether normalization test, [BG01]). If

- in (I) contains powers of the variables $x_0, \ldots, x_{n-d}$ and
- any monomial in in (I) involves at least one of these variables,

then $d = \dim R/I$ and $K[x_{n-d+1}, \ldots, x_n]$ is a N.N. of R/I.

Example. In the previous example, $I \subset R = \mathbb{Q}[x_0, \ldots, x_3]$ and

$$\text{in}\,(I) = (x_0^2, x_0x_1, x_0x_2, x_0x_3, x_1^2, x_1x_2x_3)$$

and hence $\mathbb{Q}[x_2, x_3]$ is a N.N. of R/I.

In next example, we know before hand that we have a N.N.:

Big Example. Let $I \subset R = \mathbb{C}[x_0, \dots, x_{10}]$ be the defining ideal of the 3-dimensional projective toric variety $\mathcal{V} \subset \mathbb{P}^{10}_{\mathbb{C}}$ defined by

$$x_0 = st^6u^4v^4,\ x_1 = st^4u^3v^7,\ x_2 = u^{11}v^4,\ x_3 = s^6t^3u^4v^2,$$

$$x_4 = st^7uv^6,\ x_5 = tu^{10}v^4,\ x_6 = s^3t^3u^3v^6,$$

$$x_7 = s^{15},\ x_8 = t^{15},\ x_9 = u^{15},\ x_{10} = v^{15}\,.$$

The ideal I is generated by 389 binomials of degree ≤ 17.

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$$x_7 = s^{15},\ x_8 = t^{15},\ x_9 = u^{15},\ x_{10} = v^{15}\,.$$

The ideal I is generated by 389 binomials of degree ≤ 17. Using the implementation of our results in the Singular library `mregular.lib` we get that depth $(R/I) = 1$ and reg $(I) = 29$.

I. Bermejo, P.G. and G.-M. Greuel.
mregular.lib, a **Singular** library for computing the regularity of a homogeneous ideal. Available at **http://www.singular.uni-kl.de**, 2007.

D. Grayson and M. Stillman.
Macaulay2 a software system for research in Algebraic Geometry. Available at **http://www.math.uiuc.edu/Macaulay2**, 2008.

Computation of $\mathrm{reg}\,(I)$ in general.

Set $N := dn - \frac{d(d-1)}{2}$. For $\gamma = (\gamma_1, \ldots, \gamma_N) \in \mathbb{A}_K^N$, consider the homogeneous linear transformation $\Psi(\gamma) : R \to R$ defined by:

$$\begin{array}{rcccccccc} x_n & \mapsto & x_n & + & \gamma_1 x_{n-1} & + & \gamma_2 x_{n-2} & + \cdots\cdots + & \gamma_n x_0 \\ x_{n-1} & \mapsto & & & x_{n-1} & + & \gamma_{n+1} x_{n-2} & + \cdots\cdots + & \gamma_{2n-1} x_0 \\ & \vdots & & & & & & & \\ x_{n-d+1} & \mapsto & & & & & x_{n-d+1} & + \cdots\cdots + & \gamma_N x_0 \end{array}$$

Theorem ([BG06]). There is a dense Zariski open subset $\mathcal{U}$ of $\mathbb{A}_K^N$ such that $\mathrm{in}\,(\Psi(\gamma)(I))$ is constant and of nested type for $\gamma \in \mathcal{U}$. We call it the **monomial ideal of nested type associated to** I and denote it by $N(I)$.

Example. Let $I \subset R = \mathbb{Q}[x_0, \ldots, x_8]$ be the monomial ideal generated by all the squarefree monomials of degree 2. Then $d = \dim R/I = 1$ and applying a change of coordinate of the form $\Psi : R \to R$, $x_8 \mapsto x_8 + \gamma_1 x_7 + \ldots + \gamma_8 x_0$, one gets that $N(I) := \mathrm{in}\,(\Psi(I)) = (x_0, \ldots, x_7)^2$ and hence $\mathrm{reg}\,(I) = 2$.

Consequence: a new algorithm for Noether normalization.

If $d = \dim R/I$, one has that

$$\boxed{\begin{array}{c}\text{in}\,(I)\ \text{is}\\ \text{of N.T.}\end{array}} \quad \Leftrightarrow \quad \begin{cases} x_0^{k_0}, \ldots, x_{n-d}^{k_{n-d}} \in \text{in}\,(I) \text{ for } k_0, \ldots, k_{n-d} \geq 1 \\ \text{and} \\ \text{in}\,(I) : (x_n)^\infty \subseteq \cdots \subseteq \text{in}\,(I) : (x_{n-d+1})^\infty \end{cases}$$

$$\Downarrow$$

$$\boxed{\begin{array}{c}K[x_{n-d+1}, \ldots, x_n]\\ \text{is a N.N. of } R/I\end{array}} \quad \Leftrightarrow \quad x_0^{k_0}, \ldots, x_{n-d}^{k_{n-d}} \in \text{in}\,(I) \text{ for } k_0, \ldots, k_{n-d} \geq 1$$

Thus the previous linear transformation put variables in **Noether position**. This improves the usual triangular changes of coordinates (see previous example).

REMARK. If works also in the nonhomogeneous context.

Part III:

A theoretical application

- **monomial varieties of codimension 2**
- **a closed formula for the regularity**
- **upper bounds (Eisenbud-Goto conjecture)**
- **classification of the extremal case**

[BGM06] I. Bermejo, P.G. and M. Morales.
Castelnuovo-Mumford regularity of projective monomial varieties of codimension two.
J. Symbolic Comput. **41**: 1105–1124, 2006.

Monomial varieties of codimension two.

DEFINITION. Let K be an algebraically closed field. A **projective monomial variety $\mathcal{V}$ of codimension 2** is defined parametrically

$$x_1 = u_1^a\,, \quad \ldots\ldots\,, \quad x_n = u_n^a\,,$$

$$z = u_1^{b_1} \cdots u_n^{b_n}\,, \quad y = u_1^{c_1} \cdots u_n^{c_n}\,,$$

where $a > 0$ is an integer and $\mathbf{b} = (b_1, \ldots, b_n), \mathbf{c} = (c_1, \ldots, c_n)$ are two vectors in $\mathbb{N}^n$ such that $\mathbf{b} \neq \mathbf{c}$, $\sum_{j=1}^n b_j = \sum_{j=1}^n c_j = a$, and $(b_j, c_j) \neq (0, 0)$ with $b_j, c_j < a$ for all $j \in \{1, \ldots, n\}$.

Let $I(\mathcal{V}) \subset R := K[z, y, x_1, \ldots, x_n]$ be the defining ideal of $\mathcal{V}$ and $\mathcal{G}$ its reduced Gröbner basis w.r.t. the reverse lexicographic order.

PROPOSITION([M95]). 1. $\mathcal{G}$ is a min. syst. of gen. of $I(\mathcal{V})$.

2. $n - 1 \leq \operatorname{depth}(R/I(\mathcal{V})) \leq n$.

3. $\operatorname{depth}(R/I(\mathcal{V})) = n$ if and only if $\#\mathcal{G} \leq 3$.

A closed formula for $\mathrm{reg}\,(\mathcal{V}) := \mathrm{reg}\,(I(\mathcal{V}))$, the regularity of $\mathcal{V}$.

THEOREM([BGM06]). 1. If $\mathcal{V}$ is a **complete intersection** then $\mathrm{reg}\,(\mathcal{V}) = \deg h_0 + \deg h_1 - 1$ where $\mathcal{G} = \{h_0, h_1\}$.

2. If $\mathcal{V}$ is **arithmetically Cohen-Macaulay** and is **not a complete intersection**, denote by h_0, h_1, h_2 the elements of $\mathcal{G}$ where the leading terms of h_0 and h_1 are pure powers of z and y resp. Then, $\mathrm{reg}\,(\mathcal{V}) = \max\{\deg h_0 + \deg_y h_2,\ \deg h_1 + \deg_z h_2\} - 1$.

3. If $\mathcal{V}$ is **not arithmetically Cohen-Macaulay**, consider the partition $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$ where $\mathcal{G}_2$ is formed by the elements of $\mathcal{G}$ whose leading term involves the variable y.
If $\mathcal{G}_1 = \{g_0, \ldots, g_r\}$ and $\mathcal{G}_2 = \{f_0, \ldots, f_s\}$ where the elements in both sets have been ordered by decreasing degree in the variable z of their leading term, denote by $\ell(i)$ the least integer $j \leq s$ such that $\deg_z(f_j) \leq \deg_z(g_i)$ for all $i \in \{0, \ldots, r\}$. Then,

$$\mathrm{reg}\,(\mathcal{V}) = \max_{1 \leq i \leq r} \{\deg g_i + \max\{\deg f_{\ell(i)}, \deg f_{\ell(i-1)}\}\} - 2.$$

Sketch of the Proof.

• $\mathcal{G}$ is explicitly described in

[M95] M. Morales.
Équations des variétés monomiales en codimension deux.
J. Algebra **175**:1082–1095, 1995.

• When $\mathcal{V}$ is a.C.-M., in $(I(\mathcal{V}))$ is of nested type.

• When $\mathcal{V}$ is not a.C.-M., in $(I(\mathcal{V}))$ is not of nested type, but one can describe $N(I(\mathcal{V}) + x_n)$, the monomial ideal of nested type associated to $I(\mathcal{V}) + x_n$.

• It turned out to be a monomial ideal involving only the 3 variables z, y, x_1 and the result follows from the explicit description of its irredundant irreducible decomposition.

Bounds for the regularity.

The m.g.f.r. of codimension 2 lattice ideals is described in:

[PS98] I. Peeva and B. Sturmfels.
Syzygies of codimension 2 lattice ideals.
Math. Z. **229**:163–194, 1998.

Some results are deduced for the regularity of this class of ideals that include our $I(\mathcal{V})$. In particular:

- $\operatorname{reg}(\mathcal{V}) \leq 2 \operatorname{maxdeg} I(\mathcal{V}) - 2$ whenever $\mathcal{V}$ is not a complete intersection.

- $\operatorname{reg}(\mathcal{V}) \leq \deg \mathcal{V} - 1$ (Eisenbud-Goto bound).

These results are recovered from our formula.

Classification of varieties where equality holds.

THEOREM([BGM06]). The projective monomial varieties $\mathcal{V} \subset \mathbb{P}_K^{n+1}$ of codimension two satisfying that $\operatorname{reg}(\mathcal{V}) = \deg \mathcal{V} - 1$ are:

- For all $d \geq 3$, the smooth monomial curve in $\mathbb{P}_K^3$ of degree d, i.e., the monomial curves in $\mathbb{P}_K^3$ defined by

$$z = u_1 u_2^{d-1},\ y = u_1^{d-1} u_2,\ x_1 = u_1^d,\ x_2 = u_2^d.$$

- The C.I. monomial curve in $\mathbb{P}_K^3$ of degree 4 defined by

$$z = u_1^2 u_2^2,\ y = u_1^3 u_2,\ x_1 = u_1^4,\ x_2 = u_2^4.$$

- The C.I. monomial surface in $\mathbb{P}_K^4$ of degree 4 defined by

$$z = u_2^2 u_3^2,\ y = u_1^2 u_2 u_3,\ x_1 = u_1^4,\ x_2 = u_2^4,\ x_3 = u_3^4.$$

- The C.I. monomial surface in $\mathbb{P}_K^4$ of degree 4 defined by

$$z = u_2 u_3,\ y = u_1 u_2,\ x_1 = u_1^2,\ x_2 = u_2^2,\ x_3 = u_3^2.$$

- The C.I. monomial variety in $\mathbb{P}_K^5$ of degree 4 defined by

$$z = u_3 u_4,\ y = u_1 u_2,\ x_1 = u_1^2,\ x_2 = u_2^2,\ x_3 = u_3^2,\ x_4 = u_4^2.$$