Algebraic Analysis of Physical Field Theories

Werner M. Seiler

AG Computational Mathematics

UNI KASSEL
VERSITÄT

I consider that I understand an equation when I can predict the properties of its solutions, without actually solving it.

P.A.M. Dirac

Field Theories

Point mechanics: finite-dimensional systems consisting of points or rigid bodies; (gen.) coordinates function of *time*

$\rightsquigarrow$ equations of motion ODEs

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Point mechanics: finite-dimensional systems consisting of points or rigid bodies; (gen.) coordinates function of *time*

$\rightsquigarrow$ equations of motion ODEs

Field theory: observables (“fields”) depend on *time* and *space*

$\rightsquigarrow$ field equations PDEs

Field Theories

Maxwell's equations (in vacuum)

$$\frac{\partial \mathbf{E}}{\partial t} = \boldsymbol{\nabla} \times \mathbf{B} \qquad \boldsymbol{\nabla} \cdot \mathbf{E} = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla} \times \mathbf{E} \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0$$

$\mathbf{E}$ electric field $\qquad$ $\mathbf{B}$ magnetic field

$\leadsto$ 8 equations for 6 unknowns

$\leadsto$ *over-determined system* ?

Field Theories

Incompressible **Navier-Stokes equations**

$$\mathbf{u}_t + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = \nu \Delta \mathbf{u} - \boldsymbol{\nabla} p$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

$\mathbf{u}$ velocity field $\qquad p$ pressure

$\leadsto$ 4 equations for 4 unknowns

$\leadsto$ *well-determined system* ???

Field Theories

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$\leadsto$ 4 equations for 4 unknowns
$\leadsto$ *well-determined system* ???

cross-derivative yields integrability condition

$$\Delta p = -\boldsymbol{\nabla} \cdot \big((\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u}\big)$$

Geometric Modelling

$\mathcal{X}$ space-time (independent variables)
$\mathcal{U}$ field values (often vector space)

"field" $\rightsquigarrow$ smooth function $u : \mathcal{X} \to \mathcal{U}$

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Def: *jet bundle* of order q $\rightsquigarrow$

$$J_q(\mathcal{X},\mathcal{U}) = \left\{ [u]^{(q)}_{\bar{x}} \mid u \in \mathcal{C}^\infty(\mathcal{X},\mathcal{U}),\ \bar{x} \in \mathcal{X} \right\}$$

Geometric Modelling

- manifold of dimension $n + m\binom{n+q}{q}$
- coord.: $\bar{\mathbf{x}}, \left(u^\alpha_\mu\right)^{1\leq\alpha\leq m}_{0\leq|\mu|\leq q}$ with $u^\alpha_\mu = \frac{\partial^{|\mu|}u^\alpha}{\partial x^\mu}(\bar{x})$
- $J_0(\mathcal{X},\mathcal{U}) = \mathcal{X} \times \mathcal{U}$
- rich geometric structure
- for $r > q$ natural projections

$$\pi^r_q : J_r(\mathcal{X},\mathcal{U}) \longrightarrow J_q(\mathcal{X},\mathcal{U})$$

Geometric Modelling

Def: *differential equation* of order q $\rightsquigarrow$ (regular) submanifold $\mathcal{R}_q \subseteq J_q(\mathcal{X}, \mathcal{U})$

- zero set of functions $F : J_q(\mathcal{X}, \mathcal{U}) \to \mathbb{R}$
- similar to algebraic geometry
- no distinction: *scalar* $\Leftrightarrow$ *system*

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function $u \in \mathcal{C}^\infty(\mathcal{X}, \mathcal{U})$ $\rightsquigarrow$ *prolongation*

$$j_q u \in \mathcal{C}^\infty\big(\mathcal{X}, J_q(\mathcal{X}, \mathcal{U})\big) \text{ with } j_q u(x) = [u]_x^{(q)}$$

Def: u *solution* $\rightsquigarrow$ $\operatorname{im}(j_q u) \subseteq \mathcal{R}_q$

Field Equations

- **phenomenological models:** *idealisation* and *modelling* yield together with *"first principles"* field equations

  example: *Navier-Stokes*

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- **phenomenological models:** *idealisation* and *modelling* yield together with *"first principles"* field equations

  example: *Navier-Stokes*

- **variational models:** field equations follow from *variational principle* ("nature is lazy and stingy")

  example: *Yang-Mills*

Field Equations

variational systems are described by *Lagrangian density* $\mathcal{L} : J_1(\mathcal{X}, \mathcal{U}) \to \mathbb{R}$

fields $\mathbf{u}$ evolve such that *action functional*

$$S[\mathbf{u}] = \int_{\mathcal{X}} \mathcal{L}\bigl(j_1\mathbf{u}(\mathbf{x})\bigr)\,d\mathbf{x}$$

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$\rightsquigarrow$ *Euler-Lagange equations*

$$\sum_i \frac{d}{dx^i}\left(\frac{\partial \mathcal{L}}{\partial u_i^\alpha}\right) - \frac{\partial \mathcal{L}}{\partial u^\alpha} = 0$$

Field Equations

Example: *massive scalar field*

$$\mathcal{L} = \frac{1}{2}(u_t^2 - u_x^2) - \frac{1}{2}m^2u^2$$

Euler-Lagrange $\rightsquigarrow$ *Klein-Gordon equation*

$$u_{tt} - u_{xx} + m^2u = 0$$

Gauge Symmetries

- assume *Lie group* $\mathcal{G}$ operates on $\mathcal{U}$ ($\mathcal{E} = \mathcal{X} \times \mathcal{U}$ *principal fibre bundle*)
- *local* gauge transformation: map $\mathcal{X} \longrightarrow \mathcal{G}$ (at every space-time point a *different* group element may be used)
- *gauge symmetry* $\rightsquigarrow$ action remains invariant
- physical interpretation: solutions related by gauge transformation describe *same* physical state; may be identified

Gauge Symmetries

Example: *$U(1)$-Yang-Mills in $1+1$ dimensions*

Lagrangian: $\mathcal{L} = \frac{1}{2}(u_x - v_t)^2$

Euler-Lagrange (2 equations for 2 unknowns)

$$u_{xx} - v_{xt} = 0 \qquad u_{xt} - v_{tt} = 0$$

invariant under transformations

$$u \mapsto u + \Lambda_t \quad v \mapsto v + \Lambda_x$$

for arbitrary function $\Lambda(x,t)$

$\rightsquigarrow$ *under-determined system* ???

General Differential Equations

$$u_t = f(t, \mathbf{x}, u, u_{\mathbf{x}})$$

General Differential Equations

$$\mathbf{u}_t = \mathbf{f}(t, \mathbf{x}, \mathbf{u}, \mathbf{u}_\mathbf{x})$$

normal (Cauchy-Kovalevskaya) system

- same number of equations and unknowns
- further *structural* assumption: existence of distinguished variable t ("time")
- existence and uniqueness theorem for *analytic* solutions and equations $\rightsquigarrow$ more general results only for special classes of equations
- gauge theories are *never* in this form

General Differential Equations

$$
\begin{aligned}
\mathbf{v}_t &= \mathbf{g}(t, \mathbf{x}, \mathbf{v}, \mathbf{w}, \mathbf{v}_{\mathbf{x}}, \mathbf{w}_{\mathbf{x}}, \mathbf{w}_t) \\
0 &= \mathbf{h}(t, \mathbf{x}, \mathbf{v}, \mathbf{w}, \mathbf{v}_{\mathbf{x}}, \mathbf{w}_{\mathbf{x}})
\end{aligned}
$$

general system (“PDAE”)

- generally not yet in normal form!
  (but usually for gauge theories)
- generally same *order*, but different *class*

General Differential Equations

Problems:

- existence of solutions
  - formal consistency
  - analytic theory

General Differential Equations

Problems:

- existence of solutions
- uniqueness of solutions
  - size of formal solution space
  - "degrees of freedom", "mobility"
  - *how many and which initial or boundary conditions?*

General Differential Equations

Problems:

- existence of solutions
- uniqueness of solutions
- classifications
  - under-, well- or over-determined system?
  - *elliptic or hyperbolic system?*

General Differential Equations

Problems:

- existence of solutions
- uniqueness of solutions
- classifications
- *detection of singular behaviour*
  - *impasse or funnel points*
  - *singular integrals*
  - *shocks*
  - *local normal forms*

General Differential Equations

Problems:

- existence of solutions
- uniqueness of solutions
- classifications
- *detection of singular behaviour*
- *existence of symmetries*
- *quantisation*
- *numerical integration*
- ...

Formal Integrability

Two natural operations with differential equations

Projection: $\mathcal{R}^{(s)}_{q-s} = \pi^q_{q-s}(\mathcal{R}_q)$
(eliminate all equations of order $> q - s$)

Prolongation: $\mathcal{R}_{q+r} = J_r(\mathcal{X}, \mathcal{R}_q) \cap J_{q+r}(\mathcal{X}, \mathcal{U})$
(differentiate every equation in system r times with respect to all variables)

Formal Integrability

In general: $\mathcal{R}_q^{(s)} = \pi_q^{q+s}(\mathcal{R}_{q+s}) \subsetneq \mathcal{R}_q$

$\rightsquigarrow$ **integrability conditions**

- computationally two mechanisms:
  - differentiation of lower-order equations
  - (gen.) cross-derivatives
- for ODEs only first one possible
- second one requires algebraic treatment

Formal Integrability

Def: $\mathcal{R}_q$ **formally integrable**, if

$$\forall r > 0 : \mathcal{R}_{q+r}^{(1)} = \mathcal{R}_{q+r}$$

- *infinite* definition!
- integrability conditions obstruct *order by order* construction of formal power series solutions
- truncated series *valid* approximations only for formally integrable equations
- construction of *some* integrability conditions easy; when do we know *all*?

Formal Integrability

Ordinary differential equations:

Prop: $\mathcal{R}_q$ either inconsistent or $\mathcal{R}_q^{(s)}$ formally integrable for some $0 \leq s \leq \dim \mathcal{R}_q$

Proof: simple dimensional argument

Formal Integrability

Ordinary differential equations:

Prop: $\mathcal{R}_q$ either inconsistent or $\mathcal{R}_q^{(s)}$ formally integrable for some $0 \leq s \leq \dim \mathcal{R}_q$

Proof: simple dimensional argument

Partial differential equations:

Thm: $\mathcal{R}_q$ either inconsistent or $\mathcal{R}_{q+r}^{(s)}$ formally integrable for some $r, s \in \mathbb{N}_0$

Proof: non-trivial *Noetherian* arguments

Geometric Symbol

$\pi^q_{q-1} : J_q(\mathcal{X},\mathcal{U}) \to J_{q-1}(\mathcal{X},\mathcal{U})$ *affine* bundle modelled on vector bundle $S_q(T^*\mathcal{X}) \otimes T\mathcal{U}$

Fundamental identification:

$$T\bigl(J_q(\mathcal{X},\mathcal{U})\bigr) \supset V\pi^q_{q-1} \cong S_q(T^*\mathcal{X}) \otimes T\mathcal{U}$$

Evaluation in local basis $(\omega_1, \dots, \omega_n)$ of $T^*\mathcal{X}$

$$S(T^*\mathcal{X}) = \bigoplus_{q\geq 0} S_q(T^*\mathcal{X}) \cong \mathbb{R}[x^1, \dots, x^n]$$

$\leadsto$ *natural polynomial structure in jet hierarchy*

Geometric Symbol

Def: *(geometric) symbol* of $\mathcal{R}_q$ $\rightsquigarrow$

$$\mathcal{N}_q = T\mathcal{R}_q \cap V\pi^q_{q-1}$$

- vertical part of tangent space $T\mathcal{R}_q$
- solution space of linearised principal part considered as LSE with matrix

$$\left(\frac{\partial F^\tau}{\partial u^\alpha_\mu}\right)^{\tau=1,\dots,p}_{\alpha=1,\dots,m;\ \ |\mu|=q}$$

in variables $\dot{u}^\alpha_\mu$ with $\alpha = 1, \dots, m,\ \ |\mu| = q$

Geometric Symbol

- same construction for prolonged equation $\mathcal{R}_{q+1} \subseteq J_{q+1}(\mathcal{X},\mathcal{U})$ described by $D_{\mathbf{x}}\mathbf{F} = 0$ $\rightsquigarrow$ *prolonged symbol*

$$\mathcal{N}_{q+1} = T\mathcal{R}_{q+1} \cap V\pi^{q+1}_{q}$$

- iteration to arbitrary order $\rightsquigarrow$ *symbolic system:* $\mathcal{N}_q, \mathcal{N}_{q+1}, \mathcal{N}_{q+2}, \dots$

  (modern point of view: graded *co*module $\mathcal{N}$ over symmetric *co*algebra $\mathfrak{S}(T^*\mathcal{X})$)

Principal Symbol

- replace variable $\dot{u}^\alpha_\mu$ by product $\boldsymbol{\chi}^\mu \dot{u}^\alpha$ with vector $\boldsymbol{\chi} = (\chi_1, \dots, \chi_n)$ (one-form $\chi \in T^*\mathcal{X}$)
- yields $(p \times m)$ matrix

$$T_q[\boldsymbol{\chi}] = \left(\sum_{|\mu|=q} \frac{\partial F^\tau}{\partial u^\alpha_\mu} \boldsymbol{\chi}^\mu \right)^{\tau=1,\dots,p}_{\alpha=1,\dots,m}$$

of homogeneous polynomials of degree q in $\boldsymbol{\chi}$

Principal Symbol

- rows of $T_q[\chi]$ generate graded submodule of free module $\mathbb{R}[\chi]^m$ $\rightsquigarrow$ *symbol module* $\mathcal{M}$ (*annulator* of symbol *comodule* $\mathcal{N}$)
- component $\mathcal{M}_{q+r}$ generated by principal symbol $T_{q+r}[\chi]$ of $\mathcal{R}_{q+r}$
- *syzygies* of $\mathcal{M}$ induce (gen.) *cross-derivatives* for construction of integrability conditions
  $\rightsquigarrow$ finite criterion for formal integrability

Principal Symbol

$U(1)$-Yang-Mills equations

$$\mathcal{R}_2 : \begin{cases} v_{tt} - u_{tx} = 0 \\ v_{tx} - u_{xx} = 0 \end{cases}$$

symbol matrix: $\begin{pmatrix} 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}$

principal symbol: $T_2[\boldsymbol{\chi}] = \begin{pmatrix} -\chi_t\chi_x & \chi_t^2 \\ -\chi_x^2 & \chi_t\chi_x \end{pmatrix}$

Principal Symbol

Def: differential equation $\mathcal{R}_q$

- *under-determined* $\rightsquigarrow$ $\nexists\,\chi : T_q[\chi]$ injective
- *well-determined* $\rightsquigarrow$ $\exists\,\chi : T_q[\chi]$ bijective
- *over-determined* $\rightsquigarrow$ else

Caution: classification makes sense only for *formally integrable* equations!

Degrees of Freedom

Idea: determine (asymptotic) size of *formal* solution space of formally integrable differential equation $\mathcal{R}_q$

Degrees of Freedom

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Observation: Taylor coefficients of order $q+r$ satisfy inhomogeneous LSE with homogeneous solution space $\mathcal{N}_{q+r} \implies$ number of *parametric* coefficients of order $q+r$ given by

$$\dim_{\mathbb{R}} \mathcal{N}_{q+r} = \dim_{\mathbb{R}} \left(\mathbb{R}[\boldsymbol{\chi}]^m/\mathcal{M}\right)_{q+r}$$

Degrees of Freedom

Def: *Hilbert function* H of $\mathcal{R}_q$ $\rightsquigarrow$
Hilbert function of $\bar{\mathcal{M}} = \mathbb{R}[\boldsymbol{\chi}]^m/\mathcal{M}$
(d.h. $H(r) = \dim_{\mathbb{R}} \bar{\mathcal{M}}_r$ for $r \in \mathbb{N}$)

Prop: Hilbert function $H(r)$ becomes *Hilbert polynomial* $h(r) \in \mathbb{Q}[r]$ for $r \gg 0$

- *dimension:* $\dim \bar{\mathcal{M}} = \deg h$
- *multiplicity:* $\operatorname{mult} \bar{\mathcal{M}} = (\deg h)! \cdot \operatorname{lc} h \quad (\in \mathbb{N})$

Degrees of Freedom

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- *multiplicity:* $\operatorname{mult} \bar{\mathcal{M}} = (\deg h)! \cdot \operatorname{lc} h \quad (\in \mathbb{N})$
- *Cartan genus:* $d = \dim \bar{\mathcal{M}} + 1$
- *index of generality:* $e = \operatorname{mult} \bar{\mathcal{M}}$

Degrees of Freedom

Interpretation: formal solution space of $\mathcal{R}_q$ parametrised by e functions of d variables plus functions of fewer variables

Field theories:

$$d = n - 1 \quad \Longrightarrow \quad e \ \textit{degrees of freedom}$$

Degrees of Freedom

Interpretation: formal solution space of $\mathcal{R}_q$ parametrised by e functions of d variables plus functions of fewer variables

Proposition:

- $\mathcal{R}_q$ under-determined $\Longleftrightarrow$ $d = n$
- $\mathcal{R}_q$ well-determined $\Longleftrightarrow$
$$d = n - 1 \;\wedge\; e = mq$$
- $\mathcal{R}_q$ over-determined $\Longleftrightarrow$
$$d < n - 1 \;\vee\; e < mq$$

Degrees of Freedom

Interpretation: formal solution space of $\mathcal{R}_q$ parametrised by e functions of d variables plus functions of fewer variables

Examples:

- Maxwell: $d = 3 \quad e = 4$
- Navier-Stokes: $d = 3 \quad e = 3$
- Yang-Mills: $d = 2 \quad e = 1$

Gauge Symmetries Revisited

Observation: gauge symmetries lead always to *under-determined* field equations

Goal: find *"gauge correction"* for d and e by formally subtracting effect of gauge symmetries

Idea: model gauge symmetries mathematically as *Lie pseudogroup* (Lie groupoid), i.e. as solutions of differential equation

Gauge Symmetries Revisited

Set $\mathcal{E} = \mathcal{X} \times \mathcal{U}$ and introduce $I_r(\mathcal{E}) \subset J_r(\mathcal{E}, \mathcal{E})$ the space of r-jets of *invertible* maps $\gamma : \mathcal{E} \to \mathcal{E}$

Def: *(local) Lie pseudogroup* $\rightsquigarrow$
(solution space of) differential equation $\mathcal{G}_r \subseteq I_r(\mathcal{E})$ such that (wherever defined)

1. $\mathrm{id}_\mathcal{E}$ solution
2. γ_1, γ_2 solutions $\Longrightarrow$ $\gamma_1 \circ \gamma_2$ solution
3. γ solution $\Longrightarrow$ γ^{-1} solution

Gauge Symmetries Revisited

Set $\mathcal{E} = \mathcal{X} \times \mathcal{U}$ and introduce $I_r(\mathcal{E}) \subset J_r(\mathcal{E}, \mathcal{E})$ the space of r-jets of *invertible* maps $\gamma : \mathcal{E} \to \mathcal{E}$

γ *gauge transformation*, if fibration $\mathrm{pr}_1 : \mathcal{E} \to \mathcal{X}$ preserved, i.e. local coordinate form

$$\bar{x} = f(x) \qquad \bar{u} = g(x, u)$$

Gauge Symmetries Revisited

Def: $\mathcal{G}_r$ *gauge symmetry group* of $\mathcal{R}_q$ $\rightsquigarrow$
$\gamma \cdot u$ again solution of $\mathcal{R}_q$ for every solution γ of $\mathcal{G}_r$ and u of $\mathcal{R}_q$

G, H Hilbert functions of $\mathcal{G}_r$ and $\mathcal{R}_q$, resp. $\rightsquigarrow$
gauge corrected Hilbert function: $\bar{H} = H - G$
gauge corrected Cartan genus: $\bar{d} = \deg \bar{h} + 1$
gauge corrected index of generality: $\bar{e} = (\deg \bar{h})! \cdot \mathrm{lc}\, \bar{h}$

Gauge Symmetries Revisited

$U(1)$-Yang-Mills equations in $1+1$ dimensions

- gauge transformations:

$$\bar{u} = u + \Lambda_t \quad \bar{v} = v + \Lambda_x$$

  with arbitrary function $\Lambda(x,t)$

- representation as Lie pseudogroup:

$$\frac{\partial \bar{u}}{\partial u} = 1 \qquad \frac{\partial \bar{u}}{\partial v} = 0 \qquad \frac{\partial \bar{u}}{\partial x} = \frac{\partial \bar{v}}{\partial t}$$

$$\frac{\partial \bar{v}}{\partial u} = 0 \qquad \frac{\partial \bar{v}}{\partial v} = 1$$

Gauge Symmetries Revisited

$U(1)$-Yang-Mills equations in $1+1$ dimensions
Hilbert polynomials:

$$h(s) = s + 2 \qquad g(s) = s + 2$$
$$\bar{h}(s) = 0$$

Cartan genus and index of generality

$$\bar{d} = 0 \qquad \bar{e} = 0$$

under-determinacy solely due to gauge symmetry
$\rightsquigarrow$ modulo gauge *one-dimensional* solution space

Gauge Symmetries Revisited

$U(1)$-Yang-Mills equations in $1+3$ dimensions

Hilbert polynomials:

$$h(s) = \frac{1}{6}s^3 + \frac{7}{2}s^2 + \frac{25}{3}s + 4 \quad \Longrightarrow \quad d = 4$$

$$g(s) = \frac{1}{6}s^3 + \frac{3}{2}s^2 + \frac{13}{3}s + 4 \quad \Longrightarrow \quad \bar{h}(s) = 2s^2 + 4s$$

Cartan genus and index of generality

$$\bar{d} = 3 \qquad \bar{e} = 4$$

modulo gauge symmetry solution space of same size as Maxwell's equations

Involution

Let $\mathcal{R}_q$ be formally integrable; from certain order $\bar{q} \geq q$ on symbol module $\mathcal{M}$ "particularly nice"

- $\operatorname{Syz}(T_{\bar{q}}[\chi])$ generated in degree 1
- Hilbert function polynomial
- Hilbert polynomial directly readable off $T_{\bar{q}}[\chi]$
- Koszul homology $H_{r,p}(\bar{\mathcal{M}})$ vanishes for $r \geq \bar{q}$
- ...

($\bar{q} = \operatorname{reg} \bar{\mathcal{M}} \leadsto$ *Castelnuovo-Mumford regularity*)

Involution

Let $\mathcal{R}_q$ be formally integrable; from certain order $\bar{q} \geq q$ on symbol module $\mathcal{M}$ "particularly nice"

Def: $\mathcal{R}_{\bar{q}}$ *involutive*

- classical concept; goes back to Cartan
- known effective tests expensive or coordinate dependent *(Cartan test)*
- important for existence *and* uniqueness results

Involution

Theorem: (Cartan-Kähler)

analytic involutive differential equation $\mathcal{R}_q \Longrightarrow$ "appropriate" initial value problem possesses unique analytic solution

- proof by reduction to sequence of normal systems $\rightsquigarrow$ *Cauchy-Kovalevskaya*
- additional structural assumptions allow for statements in more general functionen spaces (e.g. uniqueness of continuous solutions of linear equations $\rightsquigarrow$ *Holmgren Theorem*)
- involution essential for proof

Message of the Day

Mathematics is being lazy. Mathematics is letting the principles do the work for you so that you do not have to do the work for yourself.

G. Pólya

Involution is the central principle for general systems of partial differential equations!