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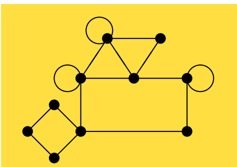
APPLICABLE ALGEBRA AND COMPUTER SCIENCE



On the first nonlinear syzygies of an edge ideal

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Joint work with **Oscar Fernández-Ramos** based on the preprint

O. F.-R. & P. G. [2008], "Nonlinear syzygies of smallest degree of an ideal associated to a graph".

Let $R := K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over an **arbitrary** field K.

Consider an ideal $I \subset R$ generated by a finite set $\mathbf{f} = \{f_1, \ldots, f_m\}$ of distinct **monomials of degree two**.

The **graph associated to I**, $G(I)$, has vertex and edge sets

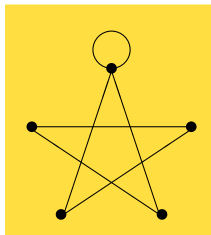
- $V_{G(I)} := \{(1), \ldots, (n)\}$ and
- $E_{G(I)} := \{(i, j);\ 1 \leq i \leq j \leq n \,/\, x_i x_j \in I\}$.

The **(simple) complement** $G(I)^c$ of $G(I)$ has the same vertex set and its edge set is formed by the edges in the complete simple graph that are not edges of $G(I)$, i.e.,

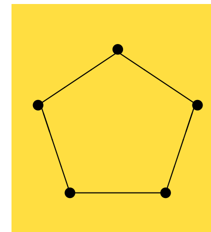
- $V_{G(I)^c} := \{(1), \ldots, (n)\}$ and
- $E_{G(I)^c} := \{(i, j);\ 1 \leq i < j \leq n \,/\, (i, j) \notin E_{G(I)}\}$.

Remark. I is square free $\Leftrightarrow$ $G(I)$ is simple (= has no loops). When this occurs, we say that I is an **edge ideal**.

$$(x_1^2, x_1x_3, x_3x_5, x_5x_2, x_2x_4, x_4x_1) \subset R := K[x_1, \ldots, x_5]$$



Graph $G(I)$ associated to I



Complement $G(I)^c$ of $G(I)$

DICTIONARY

ALGEBRA		**COMBINATORICS**
Properties of • the ideal $I = (\mathbf{f}) \subset R$, • the ring R/I, • the subalgebra $K[\mathbf{f}] \subset R$	$\longleftrightarrow$	Graph theoretical data of • the graph $G(I)$, • the graph $G(I)^c$, • any other graph

AN ILLUSTRATION OF THIS CORRESPONDENCE

The **normality** and the **polarizability** of the subalgebra $K[\mathbf{f}] \subset R$ are characterized in terms of $G(I)$ by the nonexistence of some configurations (induced subgraphs) in the two papers:

A. Simis, W. V. Vasconcelos & R. Villarreal [1998], "The integral closure of subrings associated to graphs", J. Algebra **199**, 281–289.

I. Bermejo, P. G. & A. Simis [2007], "Polar syzygies in characteristic zero: the monomial case", preprint available at `arXiv:math/0702265v1` (Feb. 07).

These combinatorial characterizations immediately imply that

POLARIZABILITY $\Rightarrow$ NORMALITY

GRADED BETTI NUMBERS

Consider a minimal graded free resolution of the ideal I:

$$0 \to \bigoplus_j R(-j)^{\beta_{p,j}} \longrightarrow \cdots \longrightarrow \bigoplus_j R(-j)^{\beta_{0,j}} \longrightarrow I \to 0$$

- $\beta_{i,j}$ is the number of generators of degree j in the ith syzygy module. The $\beta_{i,j}$'s are the **graded Betti numbers** of I.

- $\forall i = 0, \ldots, p$, $\beta_i := \sum_j \beta_{i,j}$ is the ith **Betti number** of I .

- This numerical information (degrees of the syzygies, graded Betti numbers, length of the resolution) can be displayed on a table columns $\leftrightarrow$ steps (labeled by $0, 1, \ldots, p$)
  rows $\leftrightarrow$ degrees
  where the entry in the jth row of the ith column is $\beta_{i,i+j}$. It is called the **Betti diagram** of I.

$$I = (x_1^2, x_1x_3, x_3x_5, x_5x_2, x_2x_4, x_4x_1) \subset R := K[x_1, \ldots, x_5]$$

Minimal graded free resolution of I:

$$0 \to R(-7) \longrightarrow \begin{matrix} R(-4) \\ \bigoplus \\ R(-5)^3 \end{matrix} \longrightarrow \begin{matrix} R(-3)^7 \\ \bigoplus \\ R(-4) \end{matrix} \longrightarrow R(-2)^6 \longrightarrow I \to 0$$

Betti diagram of I:

	0	1	2	3
2	6	7	1	-
3	-	1	3	1

DICTIONARY

ALGEBRA		**COMBINATORICS**
Graded Betti numbers, size of the Betti diagram, linearity of the resolution, . . .	$\longleftrightarrow$	Graph theoretical data of $G(I)$, $G(I)^c$, . . .

Remark. Since the graded Betti numbers may depend on the **characteristic of the field** K, one can not describe the whole Betti diagram in terms of the graphs (that only depend on the generating monomials) in general.

Indeed, even the size of the Betti diagram (number of rows and columns) depend on the characteristic of K:
\# rows = Castelnuovo-Mumford regularity of I (by definition).
\# columns $= p + 1$
$= n - \text{depth}\,(R/I)$ (by the Auslander-Buchsbaum formula).

But some Betti numbers and some properties of the minimal graded free resolution can be obtained from the graphs (and hence do not depend on the characteristic of K).

FIRST COLUMN

$$\beta_{0,2} = \#\text{ edges of } G(I) \quad \& \quad \beta_{0,j} = 0,\ \forall j \geq 3.$$

LINEAR SYZYGIES

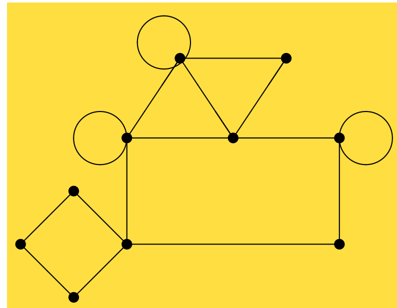
Linear syzygies are the syzygies of degree $j \geq 3$ for which $\beta_{j-2,j} \neq 0$. In terms of the Betti diagram, these syzygies are the ones that contribute to the first row.

Definition. I has **linear resolution** if all its syzygies are linear, i.e., if the Betti diagram has only one row.

Example. $G(I)$=complete simple graph $\Rightarrow I$ has linear resolution

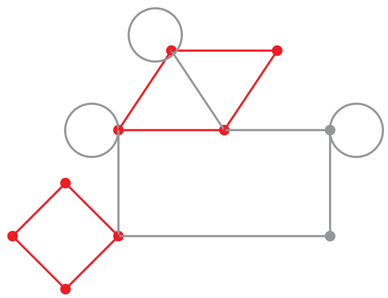
DEFINITIONS.

Let G be a graph.

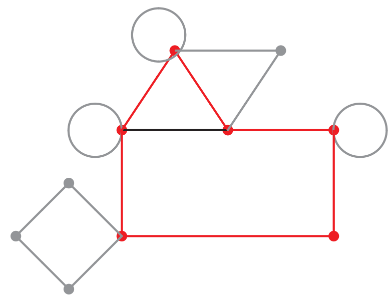


We introduce the concepts of

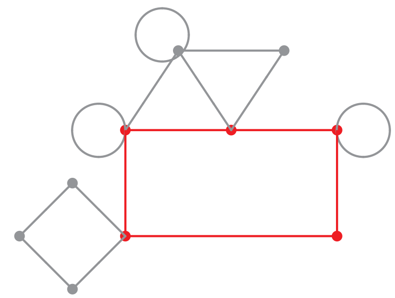
- **t-cycle** in G
- **chord** in a t-cycle
- **induced** t-cycle



4-cycles in G



6-cycle in G
with a **chord**



induced 5-cycle
(= no chord)

The graph G is **chordal** if it has no induced t-cycle with $t \geq 4$.

More generally, a subgraph H of G is **induced** if any edge in G joining two distinct vertices of H is an edge of H. When this occurs, we denote by $H \triangleleft G$.

THEOREM. If I is an **edge ideal**,

$$I \text{ has a linear resolution} \Leftrightarrow G(I)^c \text{ is chordal.}$$

R. Fröberg [1990], "On Stanley-Reisner rings". In: *Topics in Algebra*, Part 2 (Warsaw, 1988), Banach Center Publ. **26**, 57–70.

D. Eisenbud, M. Green, K. Hulek & S. Popescu [2005], "Restricting linear syzygies: algebra and geometry", Compos. Math. **141**, 1460–1478.

SOME BETTI NUMBERS GIVEN BY $G(I)$

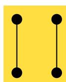
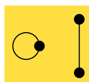
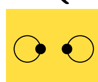
When I is an **edge ideal**, the first row and the second diagonal of the Betti diagram are described in the following papers:

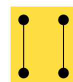
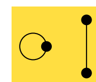
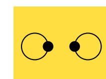
M. Roth & A. Van Tuyl [2007], "On the linear strand of an edge ideal", Comm. Algebra **35**, 821–832.

M. Katzman [2006], "Characteristic-independence of Betti numbers of graph ideals", J. Combin. Theory Ser. A **113**, 435–454.

NONLINEAR SYZYGIES OF SMALLEST DEGREE

THEOREM. Assume that I has a nonlinear resolution and let r be the smallest integer (≥ 4) such that $\beta_{i,r} \neq 0$ for some $i \leq r-3$.

1. If there exists $H \triangleleft G(I)$ consisting of 2 disjoint edges of $G(I)$, i.e., $H =$ , , or  then $r = 4$ and

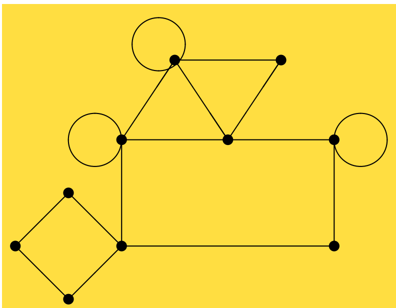
$\beta_{1,4}$ $= \#\{H \triangleleft G(I)$ such that $H =$ , , or  $\}$.

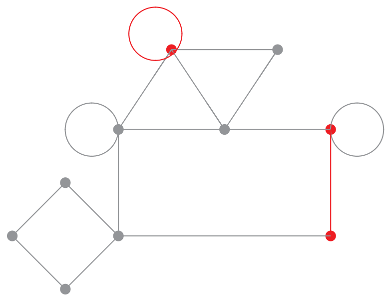
2. Otherwise, r is the smallest integer such that there exists an r-cycle $C \triangleleft G(I)^c$ and

$$\beta_{r-3,r} = \#\{\ r\text{-cycles } C \triangleleft G(I)^c\}$$

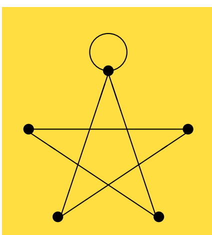
(and $\beta_{i,r} = 0$ for all $i < r-3$).

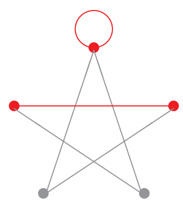
EXAMPLES. We compute the number of nonlinear syzygies of smallest degree of I when $G(I)$ is one of the following graphs:



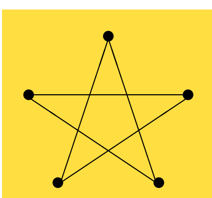


	0	1	2	3
2	16	30	18	...
3	-	45	147	...
4	-	-	20	...

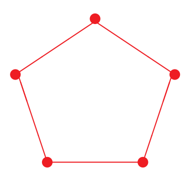




	0	1	2	3
2	6	7	1	-
3	-	1	3	1



$\Rightarrow G(I)^c =$



	0	1	2
2	5	5	-
3	-	-	1

PROOF (sketch).

- Generalization of Katzman's result when $G(I)$ has loops, i.e., removing the hypothesis I squarefree (edge ideal).

- Case study: when $G(I)^c$ is an n-cycle. In this case, the whole resolution is described:

$$0 \longrightarrow R(-n) \longrightarrow R(-n+2)^{\beta_{n-4}} \longrightarrow \cdots \longrightarrow R(-2)^{\beta_0} \longrightarrow I \longrightarrow 0$$

i.e.,

	0	1	...	$n-4$	$n-3$
2	β_0	β_1	...	β_{n-4}	-
3	-	-	...	-	1

where for all i, $0 \leq i \leq n-4$, $\beta_i = n\frac{i+1}{n-i-2}\binom{n-2}{i+2}$.

- Whenever $G(I)^c$ has an induced r-cycle,

$$\beta_{r-3,r} \geq \#\{\text{ induced } r\text{-cycles in } G(I)^c\}.$$

One uses the **multigraded resolution** for this.