Pommaret bases and Koszul homology for quasi-stable ideals

Mehdi Sahbi
mehdi_sahbi@yahoo.de

29. Februar 2008

Motivation

- Many invariants of homological nature can be read off from a Pommaret basis:
  - Projective dimension
  - Depth
  - Castelnuovo-Mumford regularity
- A Pommaret basis is to a large extent determined by the structure of the ideal.
- Goal: Construct the homology from the Pommaret basis (here in the monomial case)

Pommaret bases

- An example of involutive bases
- Special kind of Gröbner bases (in general not reduced) with additional combinatorial properties
- closely related to the involution analysis of symbols in the formal theory of differential equation (see works of Seiler)

Pommaret bases

- Consider the multiindex $\mu = (0, \dots, 0, \mu_k, \dots, \mu_n)$, $\mu_k \neq 0$
  - cls $\mu := k$, the class of μ
  - Multiplicative variables (allowed prolongations) for x^μ: $x_1, \dots, x_k$
  - Pommaret cone of x^μ: $\mathcal{C}_P(x^\mu) = x^\mu \cdot \{x^l \mid l_i = 0 \text{ for } i > k\}$
- A set of monomials M is a Pommaret basis, if $< M >_P := \bigcup_{m \in M} \mathcal{C}_P(m) =< M >$

But:
Pommaret bases do not always exist!
Monomial ideals having a Pommaret basis are called quasi-stable
$\longrightarrow$ In this work we only consider this class of ideals!

Example

- Consider the ideal generated by $< x^3y, y^3 > \subseteq \mathcal{K}[x, y]$
- cls$(x^3y) = 1 \longrightarrow$ Prolongations in direction of x_1
- cls$(y^3) = 2 \longrightarrow$ Prolongations in direction of x_1 and x_2
- Pommaret basis of the ideal: $\{x^3y, x^3y^2, y^3\}$
  $\longrightarrow$ Completion algorithm (see works of Seiler)

Koszul homology I

- $\mathcal{V}$ an n-dimensional linear space over $\mathbb{K}$ $(char\mathbb{K} = 0)$
- $S\mathcal{V}$ the symmetric algebra over $\mathcal{V}$
- $\Lambda\mathcal{V}$ the exterior algebra

Definition. *The Koszul complex $K_r(S\mathcal{V})$ at degree r over $\mathcal{V}$ is given by*

$$0 \to S_{r-n}\mathcal{V} \otimes \Lambda^n\mathcal{V} \xrightarrow{\partial} S_{r-n+1}\mathcal{V} \otimes \Lambda^{n-1}\mathcal{V} \xrightarrow{\partial} \ldots \xrightarrow{\partial} S_r\mathcal{V} \to 0$$

where

$$\partial(w_1 \ldots w_q \otimes v_1 \ldots v_p) = \sum_{i=1}^{p} w_1 \ldots w_q v_i \otimes v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_p.$$

We set $S_j\mathcal{V} = 0$ for $j < 0$.

Koszul homology II

- Let $\{x_1, \dots, x_n\}$ be a basis of $\mathcal{V}$
- We identify $S\mathcal{V} \cong \mathbb{K}[x_1, \dots, x_n] = \mathcal{P}$
- Basis of $S_q\mathcal{V}$: all terms x^μ with μ a multiindex of length q.
- Basis of $\Lambda^p\mathcal{V}$:
  We consider a sorted repeated index I of length p
  i.e. $I = (i_1, \dots, i_p)$ with $1 \leq i_1 < \dots < i_p \leq n$.
  We define $x^I := x_{i_1} \wedge \dots \wedge x_{i_p}$.
  The set of all x^I for alle possible sorted repeated indices I of length p provides a basis of $\Lambda^p\mathcal{V}$.

$\longrightarrow$ ∂ w.r.t. the above bases:

$$\partial(x^\mu \otimes x^I) = \sum_{j=1}^{p} (-1)^{j+1} x^{\mu + 1_{i_j}} \otimes x^{I \setminus \{i_j\}}.$$

Koszul homology III

Definition. *Let* $\mathcal{M}$ *be a graded module over the symmetric algebra* $\mathcal{P} = S\mathcal{V}$*. Its Koszul complex* $(K(\mathcal{M}), \partial)$ *is the tensor product complex* $\mathcal{M} \otimes_{\mathcal{P}} K(S\mathcal{V})$*. The Koszul homology of* $\mathcal{M}$ *is the corresponding bigraded homology; the homology group at* $\mathcal{M}_q \otimes \Lambda^p\mathcal{V}$ *is denoted by* $H_{q,p}(\mathcal{M})$*.*

Remark: The Koszul homology corresponds to a minimal free resolution

Koszul homology IV

- **Proposition.**

  $$H_q(\mathcal{I}) \cong H_{q+1}(\mathcal{P}/\mathcal{I}),$$

  where the second isomorphism is induced by the Koszul differential ∂.
- Strategy: Compute the homology of $\mathcal{P}/\mathcal{I}$ (simpler!).

The P-graph

- Vertices: The Pommaret generators of the ideal
- Edges: Let h be a Pommaret generator and x_i a non-multiplicative variable for h. Then there exists a unique generator $\bar{h}$ such that $x_i \cdot h$ is contained in the Pommaret cone of $\bar{h}$. In this case we have an edge from h to $\bar{h}$.

- x_i is a K-variable for h, if $x_i \cdot h \neq \bar{h}$
- x_i is an R-variable for h, if $x_i \cdot h = \bar{h}$

Example

- $\mathcal{I} = \langle x^2y, y^2z, y^3, z^2 \rangle$
- $\mathcal{H} = \{x^2y, x^2yz, x^2y^2, y^3, y^2z, z^2\}$
- The P-graph

$$\begin{tikzcd}
x^2y \arrow[r, "y"] \arrow[d, "z"] & x^2y^2 \arrow[r, "y"] \arrow[d, "z"] & y^3 \arrow[dl, "z"] \\
x^2yz \arrow[r, "y"] \arrow[d, "z"] & y^2z \arrow[dl, "z"] & \\
z^2 & &
\end{tikzcd}$$

Cycles from the P-graph I

- A-cycles:
  - $x^\mu \in \mathcal{H}$
  - cls $x^\mu = k$
  - $K = \{x_i \mid x_i$ non-multiplicative and x_i is a K-variable for $x^\mu\}$

  For all $L \subseteq K$ it holds:

$$[x^{\mu - 1_k} \otimes x_k \wedge L] \in H_{|L|+1}(\mathcal{P}/\mathcal{I})$$

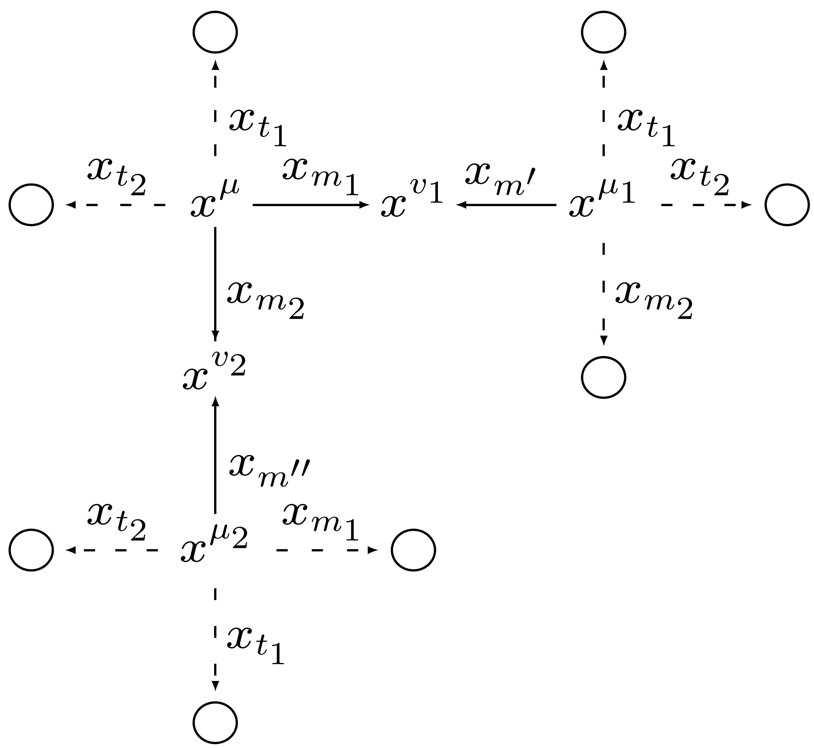
Example

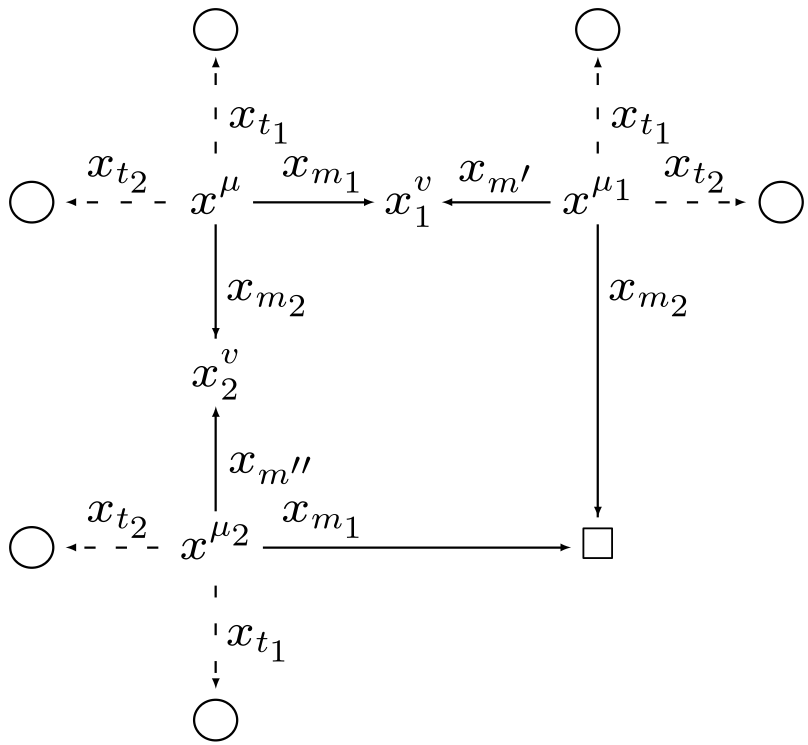
$$
\begin{tikzcd}
x^2y \arrow[r, "y"] \arrow[d, "z"] & x^2y^2 \arrow[r, "y"] \arrow[d, "z"] & y^3 \arrow[dl, "z"] \\
x^2yz \arrow[r, "y"] \arrow[d, "z"] & y^2z \arrow[dl, "z"] & \\
z^2 & &
\end{tikzcd}
$$

$[xy^2 \otimes x] \in H_1(\mathcal{P}/\mathcal{I})$
$[xy^2 \otimes x \wedge y] \in H_2(\mathcal{P}/\mathcal{I})$
$[xy^2 \otimes x \wedge z] \in H_2(\mathcal{P}/\mathcal{I})$
$[xy^2 \otimes x \wedge y \wedge z] \in H_3(\mathcal{P}/\mathcal{I})$

Cycles from the P-graph II

- B-cycles:




x_{t_1}
x_{t_1}
x_{t_2}
x^{μ}
x_{m_1}
x_1^{v}
$x_{m'}$
x^{μ_1}
x_{t_2}
x_{m_2}
x_{m_2}
x_2^{v}
$x_{m''}$
x_{t_2}
x^{μ_2}
x_{m_1}
x_{t_1}

The cycle is:

$$\begin{aligned}\omega_B = & x^{\mu-1_k} \otimes x_k \wedge x_{t_1} \wedge x_{t_2} \wedge x_{m_1} \wedge x_{m_2} \\ & + x^{\mu+1_{m_1}-1_{m'}-1_k} \otimes x_k \wedge x_{t_1} \wedge x_{t_2} \wedge x_{m_2} \wedge x_{m'} \\ & - x^{\mu+1_{m_2}-1_{m''}-1_k} \otimes x_k \wedge x_{t_1} \wedge x_{t_2} \wedge x_{m_1} \wedge x_{m''}.\end{aligned}$$

$\longrightarrow [\omega_B] \in H_5(\mathcal{P}/\mathcal{I})$

There is a rule for the signs!

Example

$$
\begin{tikzcd}
 & xz \arrow[r, "z"] \arrow[d, "y"] & z^2 \\
xy \arrow[r, "z"] \arrow[d, "y"] & xyz \arrow[ur, "z"] \arrow[d, "y"] & \\
y^2 \arrow[r, "z"] & y^2z \arrow[uur, "z"] &
\end{tikzcd}
$$

$$[y \otimes x \wedge z - z \otimes x \wedge y] \in H_2(\mathcal{P}/\mathcal{I})$$

Relations

- The obtained set of cycles still contains
  - Boundaries
  - Equivalent cycles

- **Proposition:** The relations to consider are of the form

$$\partial(x^{\mu-1_k} \otimes x_k \wedge x^I),$$

where $k = \text{cls}\ \mu$, $x^\mu \in \mathcal{H}$ and there exists $i \in I$, such that x_i is an R-variable for x^μ.

Computation of the homology

- Given the set of cycles and all relations we are now able to compute the basis
- We have designed a (canonical) term rewriting system to do this job
- Result: A (apriori) subbasis of the homology
- Claim: The algorithm computes a basis of the Koszul homology of $\mathcal{P}/\mathcal{I}$.

Proof (strategy)

- Consider the Syzygy resolution
- Simplification delivers the Betti numbers
- The Koszul homology corresponds to a minimal resolution.
- Equality of the dimensions (obtained from the algorihm) and the Betti numbers proves the statement
  - Since the linear spaces are finite!

Pommaret-Schreyer Theorem

- Given a Pommaret basis of the module $\mathcal{M}$, we have a **Pommaret basis** of the first syzygy module (w.r.t. some term order).
- **Iteration constructs a syzygy resolution**

Syzygy resolution

Let $\mathcal{H}$ be a Pommaret basis of the polynomial module $\mathcal{M} \subseteq P^m$. If we denote by $b_0^{(k)}$ the number of generators $h \in \mathcal{H}$ such that cls $\text{le}_{\prec} h = k$ and set $d = \min\{k \mid b_0^{(k)} > 0\}$, then $\mathcal{M}$ possesses a finite free resolution

$$0 \longrightarrow P^{r_{n-d}} \longrightarrow \ldots \longrightarrow P^{r_1} \longrightarrow P^{r_0} \longrightarrow \mathcal{M} \longrightarrow 0$$

of length $n - d$ where the ranks of the free modules are given by

$$r_i = \sum_{k=1}^{n-i} \binom{n-k}{i} b_0^{(k)}.$$

$\longrightarrow$ Stronger version of Hilbert´s Syzygy Theorem

Extended Schreyer theorem

Corollary: Let $\mathcal{I} \subseteq \mathcal{P}$ be a quasi-stable ideal. Then we have explicit bases for the whole Koszul homology.

Final remarks

- Our algorithm deals only with the direction basis$\longrightarrow$homology
- Seiler has investigated the other direction
- The generalisation of this work to the polynomial case is still an open question
- The homological nature of Pommaret bases should be better understood
- Pommaret bases encode also some geometric aspects such as
  - Noether normalisation
  - primary decomposition (in the monomial case)

- A further investigation direction