Complete intersections in affine monomial curves. A computational approach via Graph Theory

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Based on the paper

I. Bermejo, I. García-Marco, J.J. Salazar-González, An algorithm for checking whether the toric ideal of an affine monomial curve is a complete intersection, *J. Symbolic Comput.* **42** (2007), 971–991.

Implementation

I. Bermejo, I. García-Marco, J.J. Salazar-González, cimonom.lib, Singular 3.0.3, 2007.

Main references

I. Bermejo, Ph. Gimenez, E. Reyes, R. H. Villarreal, Complete intersections in affine monomial curves, *Bol. Soc. Mat. Mexicana (3a) Serie* **11 (2)** (2005), 191–204.

M. Clausen, A. Fortenbacher, Efficient Solution of Linear Diophantine Equations, *J. Symbolic Comput.* **8** (1989), 201–216.

Ch. Delorme, Sous-monoïdes d'intersection complète de $\mathbb{N}$, *Ann. Sci. École Norm. Sup.* **9** (1976), 145–154.

J. Herzog, Generators and relations of abelian semigroups and semigroup rings, *Manuscripta Math.* **3** (1970), 175–193.

Definitions and basic results I

Let $R = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables over an arbitrary field K.

Denote by x^a the monomial $x_1^{a_1} \cdots x_n^{a_n}$, where $a = (a_1, \ldots, a_n) \in \mathbb{N}^n$.

A **binomial** f in R is a difference of two monomials, i.e., $f = x^a - x^b$ for some $a, b \in \mathbb{N}^n$. An ideal of R generated by binomials is called a **binomial ideal**.

Let $\{d_1, \ldots, d_n\}$ be a set of all-different positive integers and consider the affine monomial curve:

$$\Gamma = \{(t^{d_1}, \ldots, t^{d_n}) \in \mathbb{A}_K^n \,|\, t \in K\}.$$

The kernel of the homomorphism of K-algebras $\phi : R \longrightarrow K[t]$ induced by $x_i \longmapsto t^{d_i}$ is called the **toric ideal of** Γ and will be denoted by

$$\boxed{I(d_1, \ldots, d_n)}$$

Definitions and basic results II

- $I(d_1, \ldots, d_n)$ is a 1-dimensional binomial ideal.
- $I(d_1, \ldots, d_n)$ is generated by **quasi-homogeneous** binomials, i.e., homogeneous binomials when one gives degree d_i to variable x_i for all $i \in \{1, \ldots, n\}$.
- If either $\gcd\{d_1, \ldots, d_n\} = 1$ or $K = \overline{K}$, we get $\Gamma = V(I(d_1, \ldots, d_n))$, i.e., Γ is a toric variety.
- If K is an infinite field, $I(\Gamma) = I(d_1, \ldots, d_n)$.

$I(d_1, \ldots, d_n)$ is a **complete intersection** if there exists a system of quasi-homogeneous binomials $g_1, \ldots, g_{n-1}$ such that

$$I(d_1, \ldots, d_n) = (g_1, \ldots, g_{n-1}).$$

The definition **coincides with the usual one**.

Aim of the work

To obtain an efficient algorithm for checking whether or not $I(d_1, \ldots, d_n)$ is a complete intersection.

For all $i \in \{1, \ldots, n\}$ let us define

$$c_i := \min \left(\mathbb{Z}^+ d_i \cap \sum_{j \in \{1,\ldots,n\} \setminus \{i\}} \mathbb{N}\, d_j\right).$$

Herzog gives the following result when $n = 3$:

$I(d_1, d_2, d_3)$ is a complete intersection $\Leftrightarrow$ $\exists\, r\,, s : 1 \leq r < s \leq 3\,,$ such that $c_r = c_s$.

This result does not hold for $n > 3$.

The aim of this work is to design an efficient algorithm for checking whether $I(d_1, \ldots, d_n)$ is a complete intersection which is mainly **based on the computation of some** c_i.

First attempt I

Proposition. Let $I(d_1, \ldots, d_n)$ be a **complete intersection**. Then the following two conditions hold:

1. $\exists\, r, s : 1 \leq r < s \leq n$ such that $c_r = c_s$;

2. whenever $c_r = c_s$ for $r, s : 1 \leq r < s \leq n$, one has

   (a) $I(d_1, \ldots, \widehat{d_r}, \ldots, \widehat{d_s}, \ldots, d_{n+1})$ is a **complete intersection**, where $d_{n+1} := \gcd\{d_r, d_s\}$;

   (b) if one sets

   $$c_{n+1} := \min\left(\mathbb{Z}^+ d_{n+1} \cap \sum_{j \in \{1,\ldots,n\} \setminus \{r,s\}} \mathbb{N}\, d_j\right)$$

   then $c_{n+1} \in \mathbb{N}\, d_r + \mathbb{N}\, d_s$;

   (c) if for all $i \in \{1, \ldots, n\} \setminus \{r, s\}$ one sets

   $$c_i' := \min\left(\mathbb{Z}^+ d_i \cap \sum_{j \in \{1,\ldots,n+1\} \setminus \{i,r,s\}} \mathbb{N}\, d_j\right)$$

   then $c_i' = c_i$.

First attempt II

The two necessary conditions for $I(d_1, \ldots, d_n)$ to be a complete intersection in **Proposition** also turn out to be sufficient when $n = 4$.

Therefore, they provide an algorithm for determining whether or not $I(d_1, d_2, d_3, d_4)$ is a complete intersection that requires the design of procedures to solve the following problems:

- **To compute the smallest positive multiple of an integer that belongs to a semigroup.**

- **To check whether or not a positive integer belongs to a semigroup.**

First attempt III

This characterization does not hold for $n \geq 5$

Example. For $d_1 = 45\,, d_2 = 70\,, d_3 = 75\,, d_4 = 98$ and $d_5 = 147$, the toric ideal $I(d_1, d_2, d_3, d_4, d_5)$ **is not a complete intersection**.

Nevertheless, $c_1 = c_3 = 225$ and setting $d_6 := \gcd\{d_1, d_3\} = 15$, one has that

- $I(d_2, d_4, d_5, d_6)$ is a complete intersection
- $c_6 = \min(\mathbb{Z}^+ d_6 \cap \langle d_2, d_4, d_5 \rangle) = 210 \in \langle d_1, d_3 \rangle$
- $c_2' = c_2 = 210\,,\ c_4' = c_4 = c_5' = c_5 = 294$

In spite of this, **Proposition** is **essential to describing our algorithm**.

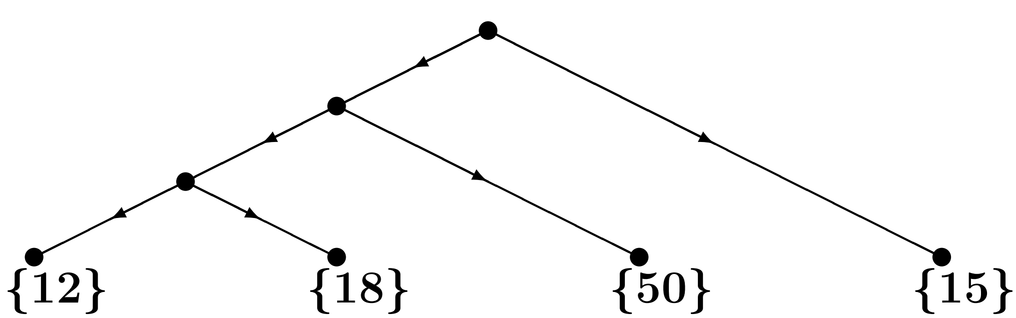
Binary trees labeled by $\{d_1, \ldots, d_n\}$

A **binary tree** is a connected directed rooted tree in which every node has either two children or zero.

Nodes with no children are called **terminal nodes** and the only node with no parent is called the **root** of the tree.

A binary tree with n terminal nodes is said to be **labeled by** $\{d_1, \ldots, d_n\}$ if its terminal nodes are labeled by $\{d_1\}, \ldots, \{d_n\}$.

The following picture is a **binary tree labeled by $\{12, 15, 18, 50\}$**:



Binary trees labeled by $\{d_1, \ldots, d_n\}$

Let $\mathcal{T}$ be a binary tree labeled by $\{d_1, \ldots, d_n\}$ and v a node of $\mathcal{T}$. Denote by Δ_v the subset of $\{d_1, \ldots, d_n\}$ such that the subtree of $\mathcal{T}$ whose root node is v is labeled by Δ_v.

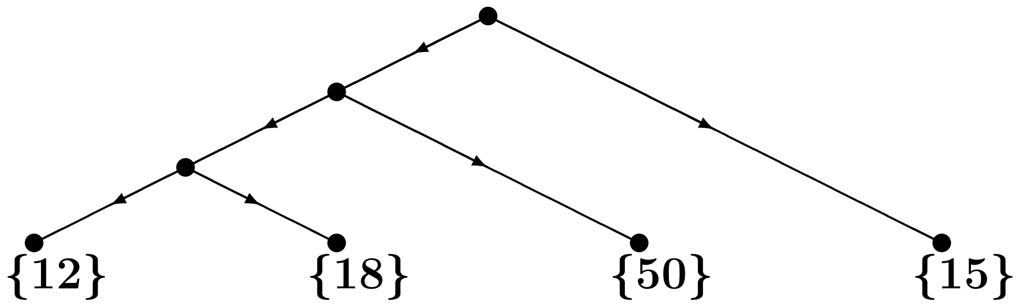
Theorem (BGRV). $I(d_1, \ldots, d_n)$ is a **complete intersection** $\Longleftrightarrow$ there is a binary tree $\mathcal{T}$ labeled by $\{d_1, \ldots, d_n\}$ such that, for each non-terminal node v of $\mathcal{T}$ with children v_1 and v_2, one has that

$$\mathrm{lcm}\left\{\gcd\{d_r \mid d_r \in \Delta_{v_1}\}, \gcd\{d_s \mid d_s \in \Delta_{v_2}\}\right\}$$

$$\in \textstyle\sum_{d_r \in \Delta_{v_1}} \mathbb{N}\, d_r \cap \sum_{d_s \in \Delta_{v_2}} \mathbb{N}\, d_s \,.$$

Binary trees labeled by $\{d_1, \ldots, d_n\}$

The binary tree labeled by $\mathbf{\{12, 15, 18, 50\}}$:



satisfies the arithmetical conditions stated in **Theorem (BGRV)**:

- $\mathbf{lcm}\left\{\mathbf{12, 18}\right\} = \mathbf{36} \in \langle \mathbf{12} \rangle \cap \langle \mathbf{18} \rangle$
- $\mathbf{lcm}\left\{\mathbf{gcd}\,\{\mathbf{12, 18}\}, \mathbf{50}\right\} = \mathbf{150} \in \langle \mathbf{12, 18} \rangle \cap \langle \mathbf{50} \rangle$
- $\mathbf{lcm}\left\{\mathbf{gcd}\{\mathbf{12, 18, 50}\}, \mathbf{15}\right\} = \mathbf{30} \in \langle \mathbf{15} \rangle \cap \langle \mathbf{12, 18, 50} \rangle$

$$\Downarrow$$

$\boldsymbol{I}(\mathbf{12, 15, 18, 50})$ is a **complete intersection**.

Binary trees labeled by $\{d_1, \ldots, d_n\}$

By **[BGRV, Remark 4.5]**, the binary tree encodes the following additional information:

- $\{x_1^3 - x_3^2, x_1^5 x_3^5 - x_4^3, x_1 x_3 - x_2^2\}$ is a system of quasi-homogeneous generators for $I(12, 15, 18, 50)$.
- The **Frobenius number** $\mathrm{g}(\mathcal{S})$ of the semigroup $\mathcal{S} := \langle 12, 15, 18, 50 \rangle$, i.e. , the largest integer not in $\mathcal{S}$, is 121.

$$x_1^3 - x_3^2 \quad \Longleftarrow \quad 36 = \underline{3} \cdot 12 = \underline{2} \cdot 18$$

$$x_1^5 x_3^5 - x_4^3 \quad \Longleftarrow \quad 150 = \underline{5} \cdot 12 + \underline{5} \cdot 18 = \underline{3} \cdot 50$$

$$x_1 x_3 - x_2^2 \quad \Longleftarrow \quad 30 = \underline{1} \cdot 12 + \underline{1} \cdot 18 = \underline{2} \cdot 15$$

$$\mathrm{g}(\mathcal{S}) = 121 \quad \Longleftarrow \quad 121 = 36 + 150 + 30 - 12 - 15 - 18 - 50$$

Binary trees labeled by $\{d_1, \ldots, d_n\}$

Theorem (BGRV) **does not provide an efficient algorithm** for checking whether or not $I(d_1, \ldots, d_n)$ is a complete intersection.

This is because one would need to verify the arithmetical conditions stated in the theorem in **every binary tree labeled by** $\{d_1, \ldots, d_n\}$.

Nevertheless, it is **extremely useful for obtaining our algorithm**.

Main result I

Let $\mathcal{T}$ be a binary tree labeled by $\{d_1, \ldots, d_n\}$ and v a node of $\mathcal{T}$ **different from the root node**.

Define

$$c_v := \min\left(\mathbb{Z}^+ \gcd\{d_r \mid d_r \in \Delta_v\} \cap \right.$$
$$\left. (\mathbb{N} \gcd\{d_s \mid d_s \in \Delta_w\} + \sum_{d_i \notin \Delta_v \cup \Delta_w} \mathbb{N}\, d_i)\right),$$

where v **and** w **are children of the same parent**.

Theorem. $I(d_1, \ldots, d_n)$ is a **complete intersection** $\Longleftrightarrow$ there exists a binary tree $\mathcal{T}$ labeled by $\{d_1, \ldots, d_n\}$ such that, for each non-terminal node v of $\mathcal{T}$ different from the root node with children v_1 and v_2, one has $c_{v_1} = c_{v_2}$ and $c_v \in \sum_{d_r \in \Delta_v} \mathbb{N}\, d_r$.

Main result II

$\Rightarrow$) This is a non-trivial consequence of **Proposition**. The result is proved by induction on n.

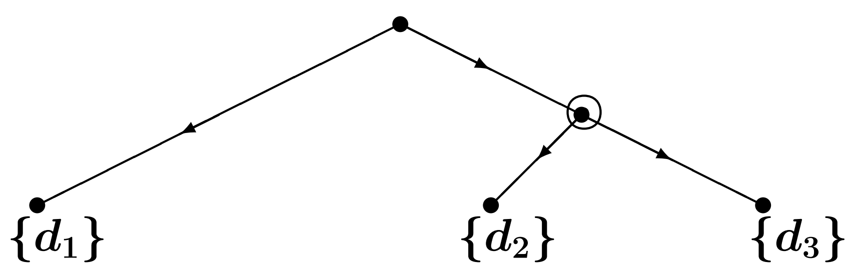
For $n = 3$, if $I(d_1, d_2, d_3)$ is a complete intersection, then there exist $r, s : 1 \leq r < s \leq 3$ such that $c_r = c_s$. Suppose that $c_2 = c_3$.

Setting $d_4 := \gcd\{d_2, d_3\}$ and

$$c_4 := \min\ (\mathbb{Z}^+ d_4 \cap \mathbb{N}\, d_1) = \operatorname{lcm}\{d_1, d_4\},$$

one has that $c_4 \in \langle d_2, d_3 \rangle$.

Thus, the **binary tree labeled by** $\{d_1, d_2, d_3\}$:

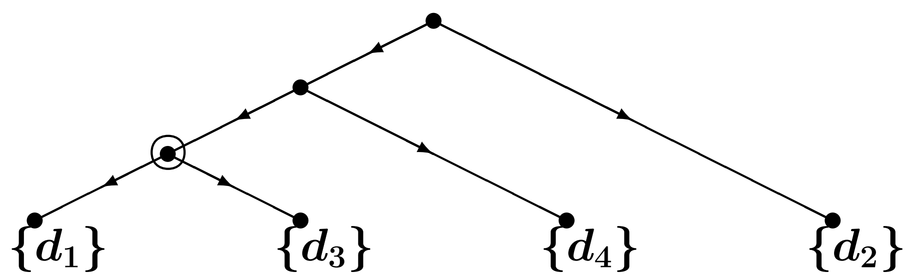


satisfies the arithmetical conditions stated in Theorem.

Main result III

⇐) A binary tree labeled by $\{d_1, \ldots, d_n\}$ that satisfies the arithmetical conditions stated in **Theorem** also satisfies the arithmetical conditions stated in **Theorem (BGRV)**.

The converse is not true. The binary tree labeled by $\{d_1, d_2, d_3, d_4\}$, where $d_1 = 12\,, d_2 = 15\,, d_3 = 18$ and $d_4 = 50$:



does not satisfy the arithmetical conditions in **Theorem**. Indeed, $c_1 = c_3 = 36$. Setting $d_5 := \gcd\{d_1, d_3\} = 6$ and

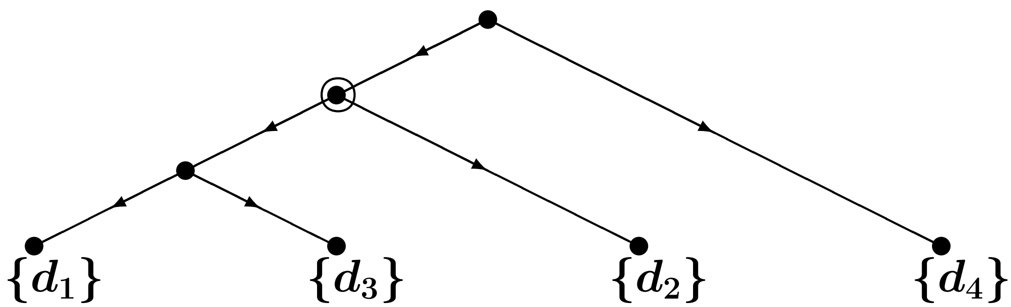
$$c_5 := \min\left(\mathbb{Z}^+ d_5 \cap \langle d_2, d_4\rangle\right) = 30\,,$$

one finds that $c_5 \in \langle d_1, d_3\rangle \longleftrightarrow c_5 = d_1 + d_3$

But $c_4 = \min\left(\mathbb{Z}^+ d_4 \cap \langle d_2, d_5\rangle\right) = 150$ **is different from** c_5.

Main result IV

The binary tree labeled by $\{d_1, d_2, d_3, d_4\}$, with $d_1 = 12\,, d_2 = 15\,, d_3 = 18$ and $d_4 = 50$:



satisfies the arithmetical conditions stated in **Theorem**.

Indeed, $c_2 = \min(\mathbb{Z}^+ d_2 \cap \langle d_4, d_5\rangle) = 30$ **is equal to** c_5 and setting $d_6 := \gcd\{d_2, d_5\} = 3$ and $c_6 := \mathrm{lcm}\{d_4, d_6\} = 150$ one finds that $c_6 \in \langle d_1, d_2, d_3\rangle \quad \longleftrightarrow \quad c_6 = 5d_1 + 5d_3$

Algorithm CI

Require: $\{d_1, \ldots, d_n\}$

Ensure: TRUE or FALSE

$G_1 := \{d_1, \ldots, d_n\}$
for $i = 1$ to n **do**
  $V_i := \{d_i\}$, $c_i := \min\left(\mathbb{Z}^+ d_i \cap \sum_{j \in \{1,\ldots,n\}\setminus\{i\}} \mathbb{N}\, d_j\right)$
end for

for $i = 1$ to $n-2$ **do**
  if $c_j \neq c_k$ for all $j, k : j \neq k$ and $d_j, d_k \in G_i$ **then**
    return FALSE
  end if
  Let $j, k : j \neq k$ such that $d_j, d_k \in G_i$ and $c_j = c_k$
  $d_{n+i} := \gcd\{d_j, d_k\}$, $V_{n+i} := V_j \cup V_k$
  $G_{i+1} := G_i \setminus \{d_j, d_k\} \cup \{d_{n+i}\}$
  $c_{n+i} := \min\left(\mathbb{Z}^+ d_{n+i} \cap \sum_{d_s \in G_{i+1}\setminus\{d_{n+i}\}} \mathbb{N}\, d_s\right)$
  if $c_{n+i} \notin \sum_{d_j \in V_{n+i}} \mathbb{N}\, d_j$ **then**
    return FALSE
  end if
end for

return TRUE

Algorithm CI

Given $\{d_1, \ldots, d_n\}$ such that $I(d_1, \ldots, d_n)$ is a complete intersection, **Algorithm CI** returns, with no additional effort, **a system of $n-1$ quasi-homogeneous generators for the toric ideal $I(d_1, \ldots, d_n)$**.

Moreover, when $\gcd\{d_1, \ldots, d_n\} = 1$, **Algorithm CI** also returns the **Frobenius number $\mathrm{g}(\mathcal{S})$ of the semigroup $\mathcal{S} := \sum_{i=1}^{n} \mathbb{N}\, d_i$** using the formula:

$$\mathrm{g}(\mathcal{S}) = \sum_{i \in \{1,\ldots,2n-2\}} c_i/2 - \sum_{i \in \{1,\ldots,n\}} d_i\,.$$

This is because a binary tree that satisfies the arithmetical conditions stated in **Theorem** also satisfies the arithmetical conditions stated in **Theorem (BGRV)**. Therefore, the first statement and the formula for $\mathrm{g}(\mathcal{S})$ are consequences of **[BGRV, Remark 4.5]**.

Computational aspects I

A direct implementation of **Algorithm CI** requires an **efficient procedure** to compute the values c_i.

Given $\{d_1, \ldots, d_n\}$, the optimization problem of computing

$$c_1 = \min \left(\mathbb{Z}^+ d_1 \cap \sum_{j \in \{2,\ldots,n\}} \mathbb{N}\, d_j\right)$$

can be formulated by the following Integer Linear Programming (ILP) model:

$$\begin{aligned}
x_1^* := \min \quad & x_1 && (1)\\
& d_1 + d_1 x_1 = d_2 x_2 + \cdots + d_n x_n && (2)\\
& x_1, x_2, \ldots, x_n \geq 0 && (3)\\
& x_1, x_2, \ldots, x_n \in \mathbb{Z} && (4)
\end{aligned}$$

Then $c_1 = d_1 + d_1 x_1^*$.

The computation of c_1 is a $\mathcal{NP}$**-hard** problem.

Computational aspects II

To compute x_1^*, we use a **Graph Theory** representation of the problem.

The approach is similar in spirit to that of **Clausen & Fortenbacher** to solve linear diophantine equations.

The idea is to represent each solution

$$(x_1, x_2, \ldots, x_n)$$

of (2)–(4) as a **walk** connecting two nodes in a **graph**, where the **weight** of the walk is x_1.

Then the combinatorial problem modeled in (1)–(4) is equivalent to **finding a shortest path** connecting these two nodes **in the graph**.

Computational aspects III

Consider the following directed weighted graph $\mathcal{G} = (V, A)$, where the **node set** is

$$V := \{0, 1, \ldots, d_1\},$$

the **arc set** is

$$A := \bigcup_{i=2}^{n} \{(v, (v - d_i) \bmod d_1) \mid v \in V\},$$

and for all $v \in V$ and $i \in \{2, \ldots, n\}$, the **weight of the arc** $(v, (v - d_i) \bmod d_1)$ is defined equal to

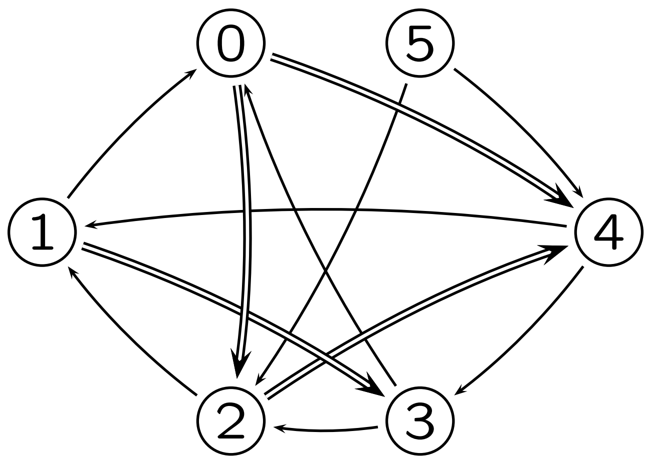
$$w_{(v,i)} := \left| \left\lfloor \frac{v - d_i}{d_1} \right\rfloor \right|.$$

Lemma. There is an onto map of the set of walks in $\mathcal{G}$ from d_1 to 0 with weight x_1 into the set of solutions $(x_1, x_2, \ldots, x_n)$ of (2)–(4).

Proof. $(v - d_i) \bmod d_1 = v - d_i + w_{(v,i)} d_1$

Computational aspects IV

The graph $\mathcal{G}$ for the instance $d_1 = 5\,, d_2 = 6\,, d_3 = 8$ is the following:



where arcs represented by **single lines** have a **weight** equal to **one**, and arcs represented by **double lines** have a **weight** equal to **two**.

The path $(5) \to (2) \to (1) \to (0)$

corresponds to the solution $(3, 2, 1)$ of (2)–(4):

$$\left.\begin{array}{r} d_1 - d_3 + 1 \cdot d_1 = 2 \\ 2 - d_2 - 1 \cdot d_1 = 1 \\ 1 - d_2 - 1 \cdot d_1 = 0 \end{array}\right| \Rightarrow d_1 + 3d_1 = 2d_2 + d_3$$

Computational aspects V

To compute x_1^* one can apply **Dijkstra's algorithm** to find a shortest path from d_1 to 0 in $\mathcal{G}$.

The **complexity** of the Dijkstra algorithm depends on the number of nodes and arcs in the graph. Since the number of nodes in $\mathcal{G}$ is d_1+1 and the number of arcs is $(n-1)(d_1+1)$, the optimization problem (1)–(4) can be solved in $\mathcal{O}(n \cdot d_1 + d_1 \cdot \log(d_1))$.

Corollary. Each c_i in **Algorithm CI** can be computed in **pseudo-polynomial time**.

Computational aspects VI

Our computational experiments show that **Algorithm CI** is able to solve **large-size instances**.

For instance, our implementation takes **less that one second** on a personal computer with Intel Pentium IV 3Ghz. to prove that the toric ideal $I(d_1, \ldots, d_{13})$ is a **complete intersection**, where

$$d_1 = 304920 \quad d_2 = 381150 \quad d_3 = 457380$$

$$d_4 = 571725 \quad d_5 = 97911 \quad d_6 = 223146$$

$$d_7 = 239085 \quad d_8 = 159390 \quad d_9 = 334719$$

$$d_{10} = 224112 \quad d_{11} = 238119 \quad d_{12} = 252126$$

$$d_{13} = 334949$$

Computational aspects VII

Additionally, the implementation also gives a **minimal set of quasi-homogeneous generators of the toric ideal**:

$$x_3^2 - x_1^3 \qquad x_8^3 - x_7^2$$

$$x_9^2 - x_6^3 \qquad x_{12}^8 - x_{10}^9$$

$$x_1\, x_3 - x_2^2 \qquad x_6\, x_9 - x_7\, x_8^2$$

$$x_{10}\, x_{12} - x_{11}^2 \qquad x_2^3 - x_4^2$$

$$x_{10}\, x_{11} - x_6\, x_7 \qquad x_6\, x_{10}\, x_{11} - x_5^7$$

$$x_7^3\, x_8 - x_1\, x_4 \qquad x_5^2\, x_8^2\, x_{11}\, x_{12} - x_{13}^3$$

and shows that the **Frobenius number** of the semigroup $\sum_{i=1}^{13} \mathbb{N}\, d_i$ is $\mathbf{6229597}$.