Computing cohomology of groups and spaces

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Problem

Compute the (co)homology

$$H^*(G,A) = H^*(BG,A) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z},A)$$

of a discrete group G.

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of a discrete group G.

More generally, G could be a simplicial group.

EXAMPLE 1

(Number Crunching)

Theorem

The Mathieu group M_{23} *has trivial integral homology* $H_n(M_{23}, \mathbb{Z}) = 0$ *in dimensions* $n = 1, 2, 3$.

Proof.

R.J. Milgram, "The cohomology of the Mathieu group M_{23}", *J. Group Theory* 3 (2000), no. 1, 7–26.

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□

Computer Proof.

```
gap> GroupHomology(MathieuGroup(23),1);
[  ]
gap> GroupHomology(MathieuGroup(23),2);
[  ]
gap> GroupHomology(MathieuGroup(23),3);
[  ]
```

Analysis of computer proof

- $|M_{23}| = 10200960 = 2^7.3^2.5.7.23$

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- Each Sylow p-subgroup P is small so, by brute force, construct low dimensional skeleta of a contractible CW-space $X_{(p)}$ with free P-action.

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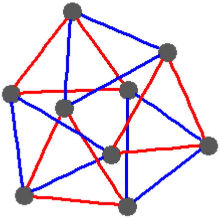
- $X^1_{(3)} =$ 

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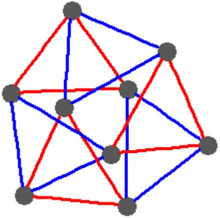
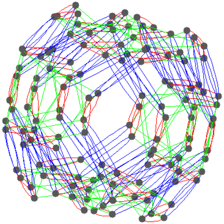
- $X^1_{(3)} =$  $X^1_{(2)} =$ 

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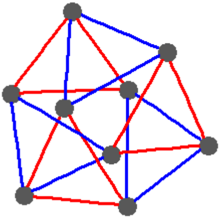
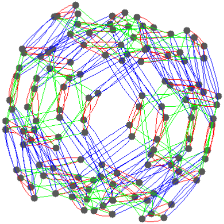
- $X^1_{(3)} =$  $X^1_{(2)} =$ 
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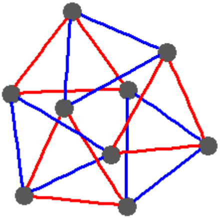
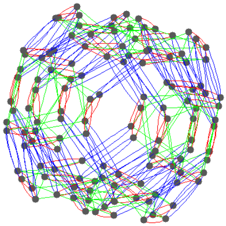
- $X^1_{(3)} =$  $X^1_{(2)} =$ 
- $C_*(X_{(p)})$ is a free $\mathbb{Z}P$-resolution of $\mathbb{Z}$.
- During the construction of $X_{(p)}$ record an explicit contracting homotopy $h_* \colon C_*(X_{(p)}) \to C_{*+1}(X_{(p)})$.

- There is a surjection $H_n(P,\mathbb{Z}) \to H_n(G,\mathbb{Z})_{(p)}$ whose kernel is described (Cartan-Eilenberg) in terms of induced homomorphisms

$$\iota_x \colon H_n(P,\mathbb{Z}) \to H_n(xPx^{-1},\mathbb{Z})$$

where x ranges over double coset representatives.

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- ι_x constructed using h_*.

EXAMPLE 2

(Twisted Tensor Product)

Theorem

For an odd prime p the group $K_p = \ker(\mathrm{SL}_2(\mathbb{Z}_{p^3}) \to \mathrm{SL}_2(\mathbb{Z}_p))$ has third integral homology group of exponent p^3.

Proof.

W. Browder and J. Pakianathan, "Cohomology of uniformly powerful p-groups", *Trans. Amer. Math. Soc.* 352 (2000), no. 6, 2659–2688. □

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Computer Proof.

```
gap> K5:=MaximalSubgroups(SylowSubgroup(
                              SL(2,Integers mod 5^3),5))[2];;
gap> GroupHomology(K5,3);
[ 5, 5, 5, 5, 5, 5, 125 ]
```

Analysis of computer proof

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$$1 \longrightarrow N \longrightarrow G \longrightarrow Q \longrightarrow 1$$

  and

  - a free $\mathbb{Z}N$-resolution $R_*^N \longrightarrow \mathbb{Z}$
  - a free $\mathbb{Z}Q$-resolution $R_*^Q \longrightarrow \mathbb{Z}$

  then the differential on the tensor product of chain complexes $R^N \otimes_{\mathbb{Z}} R^Q$ can be perturbed to produce a free $\mathbb{Z}G$-resolution

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- There are several explanations of this perturbation. We use a Lemma of CTC Wall .

Let A be a ring. (e.g. $A = \mathbb{Z}G$.) Let

$$C_* \colon \to C_n \to C_{n-1} \to \cdots \to C_0$$

be an A-resolution of some A-module M, where the A-modules C_n are **not** assumed to be free.

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Suppose that, for each p, we have a **free** A-resolution of C_p

$$D_{p*}\colon \to D_{p,q} \to D_{p,q-1} \to \cdots \to D_{p,0} \to C_p$$

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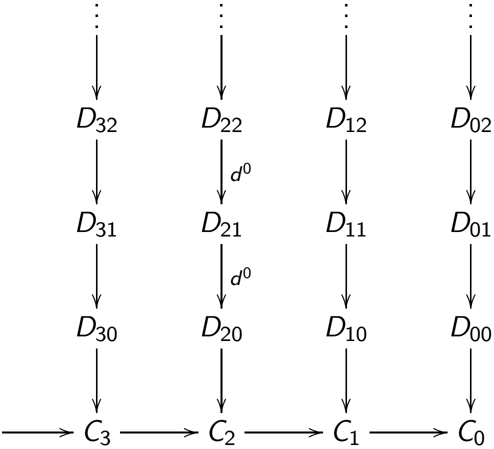
Lemma (C.T.C. Wall)

There exists a free A-resolution $R_* \to M$ with

$$R_n = \bigoplus_{p+q=n} D_{p,q}.$$



$$d^0 d^0 = 0$$

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & D_{32} & \longrightarrow & D_{22} & \longrightarrow & D_{12} & \longrightarrow & D_{02} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & D_{31} & \longrightarrow & D_{21} & \longrightarrow & D_{11} & \longrightarrow & D_{01} \\
 & & \downarrow & & \downarrow{\scriptstyle d^0} & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & D_{30} & \xrightarrow{d^1} & D_{20} & \longrightarrow & D_{10} & \longrightarrow & D_{00} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
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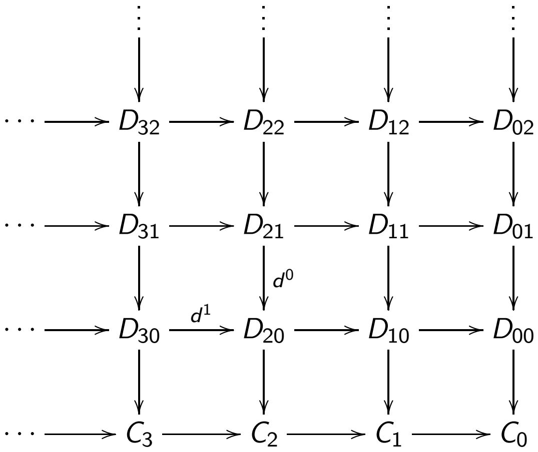
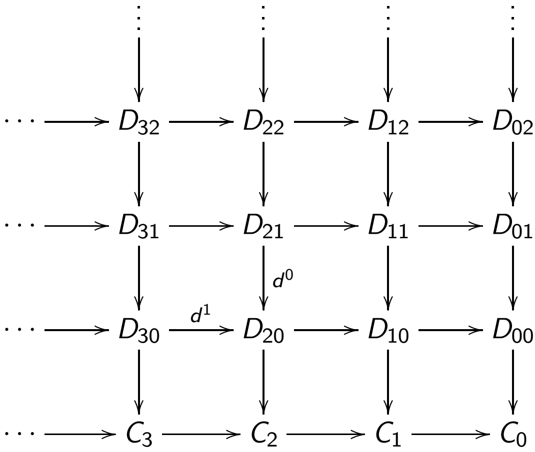
\end{array}
$$



$$\partial = d^0 + d^1$$



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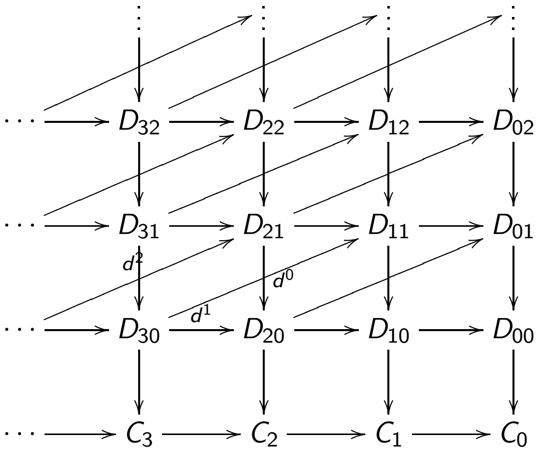
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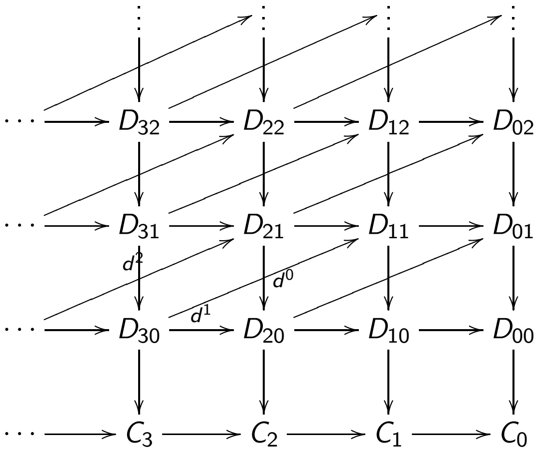
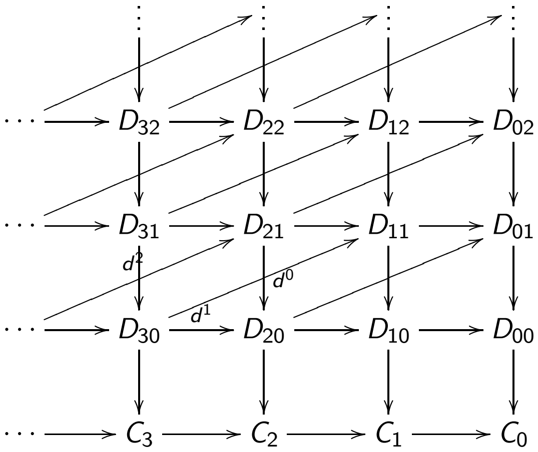


$$\partial = d^0 + d^1 + d^2$$



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but for $d^2 d^2 \neq 0$ etc

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There is a free A-resolution $R_* \to M$ with

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A contracting homotopy on R_* can be constructed using homotopies on D_{p*} and C_*

E X A M P L E 3

(Linear Algebra & Gröbner Bases)

Theorem

The mod 2 cohomology $H^n(M_{11}, \mathbb{Z}_2)$ of the Mathieu group M_{11} is a vector space of dimension equal to the coefficients of x^n in the Poincaré series

$$(x^4 - x^3 + x^2 - x + 1)/(x^6 - x^5 + x^4 - 2x^3 + x^2 - x + 1)$$

for all n.

Proof.

P.J. Webb, "A local method in group cohomology" *Comment. Math. Helv.* 62 (1987), no. 1, 135–167. □

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Computer proof for $n \leq 20$.

```
gap> PoincareSeriesPrimePart(MathieuGroup(11),2,20);
(x^4-x^3+x^2-x+1)/(x^6-x^5+x^4-2*x^3+x^2-x+1)
```

Analysis of computer proof

- For the field $\mathbb{F}$ of p elements any free $\mathbb{F}G$-module $(\mathbb{F}G)^n$ can be treated as a vector space of dimension $n \times |G|$. Linear algebra can be used to determine minimal generators for kernels of $\mathbb{F}G$-homomorphisms.

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But how do we compute a Poincaré series which is known to be correct in all degrees?

Theorem (Well-Known)

The quaternion group G of order 8 has cohomology ring

$$H^*(G, \mathbb{F}) = \mathbb{F}[x, y, e] / < x^2 + xy + y^2,\ y^3 >$$

where $\mathbb{F}$ is the field of two elements, x, y have degree 1 and e has degree 2.

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Computer Proof. (Paul Smith)

- The central extension $1 \to C_2 \to G \to C_2 \times C_2 \to 1$ yields the LHS spectral sequence

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- Our CTC Wall resolution defines the derivation $d_2\colon E_2^* \to E_2^*$ by $d_2(x) = d_2(y) = 0,\ d_2(z) = x^2+xy+y^2$.

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- Using the CTC Wall resolution to obtain the differential on E_3^*, repeat to find

$$E_4^* = E_\infty^* = \mathbb{F}[x, y, z^2]/ < x^2 + xy + y^2,\ y^3 >$$

E X A M P L E S   5

(Convex Hulls & Perturbations)

Theorem (Dutour & E)

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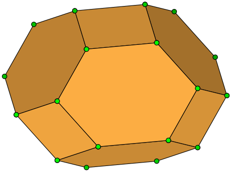
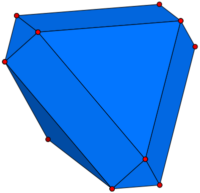
- $|M_{24}| = 244823040 = 2^{10}.3^3.5.7.11.23$ but the Sylow subgroup approach doesn't work on my laptop.
- $M_{24} < S_{24}$ acts on $\mathbb{R}^{24}$ by permuting the standard basis. Let $v = (1, 2, 3, 4, 5, 0, \ldots, 0) \in \mathbb{R}^{24}$ and compute the polytope

$$P(M_{24}) = \mathrm{ConvexHull}(v^{M_{24}}).$$

- Low dimensional illustrations using POLYMAKE: $P(S_4)$ and $P(A_4)$ with $V = (1, 2, 3, 4)$





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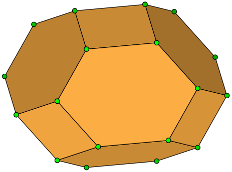

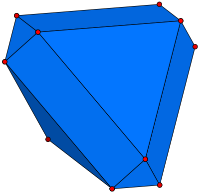

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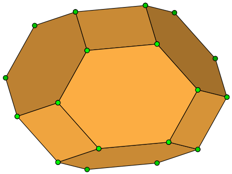

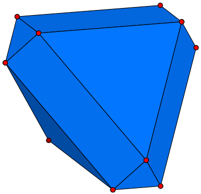

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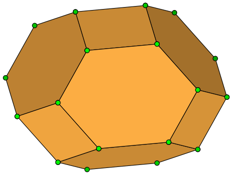
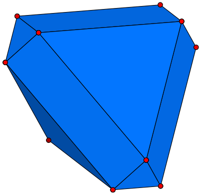
- $C_*(P)$ is a $\mathbb{Z}M_{24}$-resolution of $\mathbb{Z}$ but is not free.

- Low dimensional illustrations using POLYMAKE: $P(S_4)$ and $P(A_4)$ with $V = (1, 2, 3, 4)$





- The computation of $P = P(M_{24})$ is helped by using the 5-transitivity of M_{24} to first prove that P is simple.
- $C_*(P)$ is a $\mathbb{Z}M_{24}$-resolution of $\mathbb{Z}$ but is not free.
- We use CTC Wall's lemma to enlarge $C_*(P)$ to a free resolution.

Theorem

$H^*(M, \mathbb{Z})$ has now been computed for each of the 62 7-dimensional Hantzsche-Wendt manifolds M.

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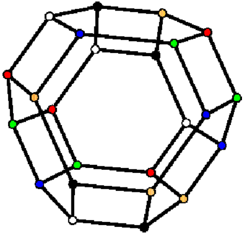
- The resulting $C_*(\mathbb{R}^n)$ is a free $\mathbb{Z}G$-resolution.

Low dimensional illustrations.

G=SpaceGroup(3,9) has fundamental domain

$$D(G, v) =$$ 

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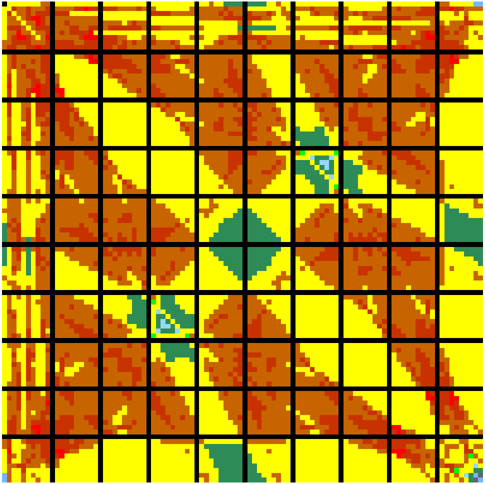
The domains depend only on the x and y coordinates of v.



E X A M P L E S 6

(Polytopal Combinatorics)

Theorem

The Artin group $G =< x, y, z : xyx = yxy, xz = zx,$ $yzyzy = zyzyz >$ *has*

$$H^n(G, \mathbb{Z}) = \mathbb{Z}(0 \leq n \leq 3), H^n(G, \mathbb{Z}) = 0 \ (n \geq 4).$$

Proof.

C. Landi, "Cohomology rings of Artin groups", Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., 11 no. 1 (2000), 41-65. □

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Computer proof

```
gap> D:=[[1,[2,3]],[2,[3,5]]];;
gap> GroupCohomology(D,1);
[ 0 ]
gap> GroupCohomology(D,2);
[ 0 ]
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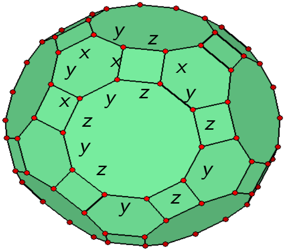
[ 0 ]
```

Analysis of computer proof

Let X be the "canonical" quotient of the 3-dimensional polytope



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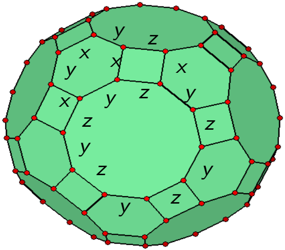

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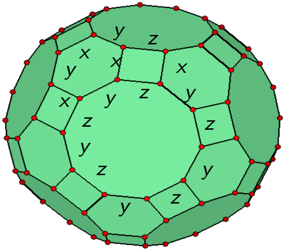

This polytope is $P(W_G)$ where $W_G = G/ < x^2 = y^2 = z^2 = 1 >$.

It was shown independently by C. Squier and M. Salvetti that such a space X is aspherical. Hence $H^*(G, \mathbb{Z}) = H^*(X, \mathbb{Z})$.

Conjecture

The Artin group $G' =< w, x, y, z : wxw = xwx, wy = yw, wzw = zwz, xyx = yxy, xz = zx,\ yzyzy = zyzyz >$ has

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Computer evidence

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[ 0 ]
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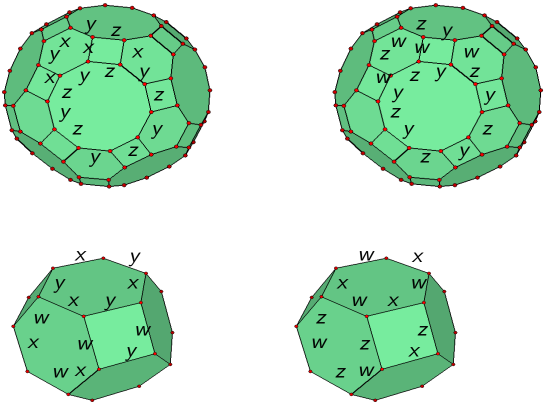
[ 0 ]
gap> GroupCohomology(D,3);
[ 2, 2, 0, 0 ]
```

Analysis of computer evidence

Let X' be the "canonical" path-connected quotient of the four polytopes:



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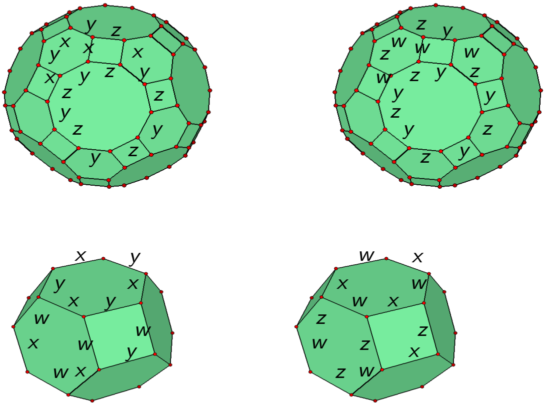

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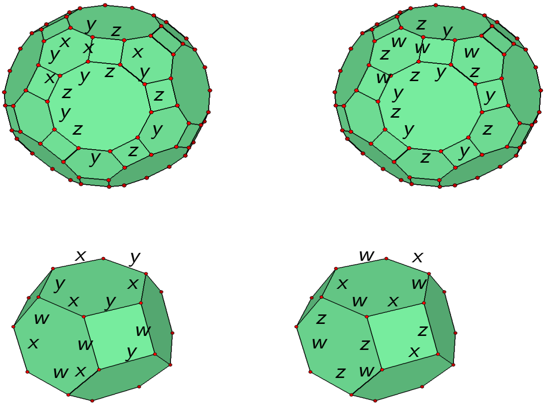

It is not known if X' is aspherical. Remark: $W_{G'}$ is infinite whereas W_G was finite.

HOMOTOPY 2-TYPES

(Ideas, No Examples Yet!)

A *homotopy n-type* is represented by a connected CW-space X with $\pi_i X = 0$ for $i \geq n+1$.

$$B\colon (\text{groups}) \xrightarrow{\simeq} (\text{homotopy } 1-\text{types})$$

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Problem. (with Ana Romero)

Compute $H^*(G, A) = H^*(BG, A)$.

$$\begin{array}{ccccc} B\colon (\mathrm{cat}^1-\mathrm{groups}) & \xrightarrow{\mathcal{N}} & (\text{simplicial groups}) & & \\ & & \Big\downarrow{\scriptstyle \mathcal{N}} & & \\ & & (\text{bisimplicial sets}) & \xrightarrow{\Delta} & (\text{simplicial sets}) \end{array}$$

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$H_*(G,\mathbb{Z})$ is the homology of the total complex of the bicomplex:

$$\begin{array}{ccccccc}
 & & \downarrow & & \downarrow & & \downarrow \\
\longrightarrow & & F\mathcal{N}_2\mathcal{N}_2(G) & \longrightarrow & F\mathcal{N}_2\mathcal{N}_1(G) & \longrightarrow & F\mathcal{N}_2\mathcal{N}_0(G) \\
 & & \downarrow & & \downarrow & & \downarrow \\
\longrightarrow & & F\mathcal{N}_1\mathcal{N}_2(G) & \longrightarrow & F\mathcal{N}_1\mathcal{N}_1(G) & \longrightarrow & F\mathcal{N}_1\mathcal{N}_0(G) \\
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We could replace each column by

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Idea for future work: Use CTC Wall's lemma to obtain a suitable total complex. And/OR use the KENZO approach.

THE END

THANK YOU!