

Koszul Homology and Resolutions

revisited through

Effective Homology

```
;; Clock -> 2002-01-17, 19h 25m 36s.
Computing the boundary of the generator 19 (dimension 7) :
<TnPr <TnPr <TnPr S3 <<Abar[2 S1][2 S1]>>> <<Abar>>> <<Abar>>>
End of computing.
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End of computing.
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Homology in dimension 6 :

Component Z/12Z

---done---

```
;; Clock -> 2002-01-17, 19h 27m 15s.
```

Talk in honour of Jacques Calmet

*Francis Sergeraert, Institut Fourier, Grenoble, France
Logroño, February 28, 2008*

Plan

1. **Simplest** example for an application of **spectral sequence**.
2. Rewriting it as a **BPL application**.
 $\Rightarrow$ **effective** version of the same result.
3. Doing the same for the **Aramova-Herzog bicomplex**.
4. Computing **effective resolutions** in **Commutative Algebra**.

Easy and convenient application of spectral sequences:
the Bicomplex Spectral Sequence.

Particular case:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 0 \leftarrow & C_{02} & \xleftarrow{d'} & C_{12} & \xleftarrow{d'} & C_{22} & \leftarrow \cdots \\
 & \downarrow d'' & & \downarrow d'' & & \downarrow d'' & \\
 0 \leftarrow & C_{01} & \xleftarrow{d'} & C_{11} & \xleftarrow{d'} & C_{21} & \leftarrow \cdots \\
 & \downarrow d'' & & \downarrow d'' & & \downarrow d'' & \\
 0 \leftarrow & C_{00} & \xleftarrow{d'} & C_{10} & \xleftarrow{d'} & C_{20} & \leftarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
 \end{array}$$

with every row and column exact

except maybe at C_{*0} and C_{0*} .

Totalization of a **bicomplex**.

$$T_i := C_{i0} \oplus C_{i-1,1} \oplus \cdots \oplus C_{0i}$$

Double-Complex Property:

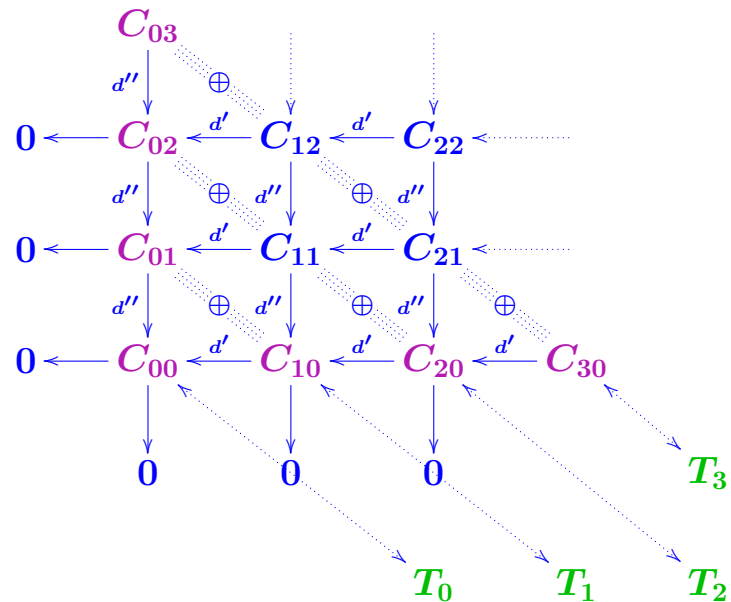
$$d'd'' + d''d' = 0$$

$$\Rightarrow d := d' \boxplus d'' = \text{differential}$$

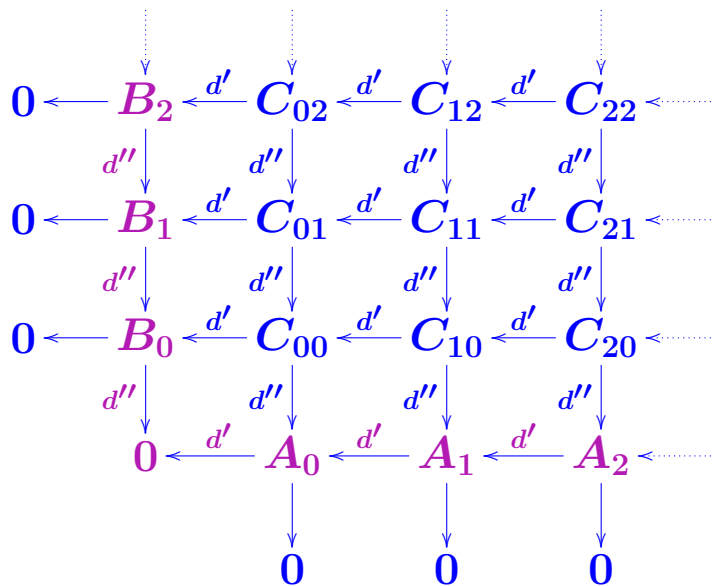
$$\Rightarrow (T_i, d) = \text{Chain complex.}$$

Definition :

$$H_i(\{C_{jk}, d'_{jk}, d''_{jk}\}) = H_i^T := H_i\{0 \leftarrow T_0 \xleftarrow{d} T_1 \xleftarrow{d} T_2 \xleftarrow{d} \cdots\}$$



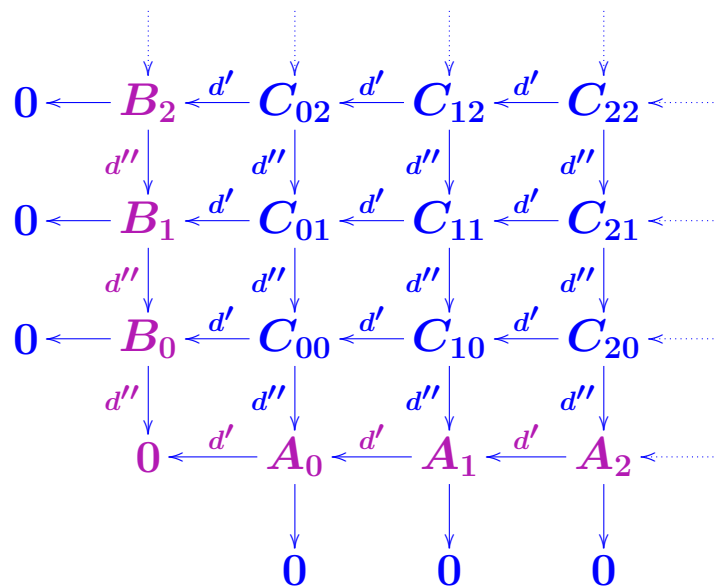
Other construction:



with: $B_i := C_{0i}/d'(C_{1i})$

$A_i := C_{i0}/d''(C_{i1})$

But new problem!

$\Rightarrow$


The **violet lines** are **chain complexes** in general **non-exact**.

$\Rightarrow$ Homology groups:

$$H_i^A := H_i\{0 \leftarrow A_0 \xleftarrow{d'} A_1 \xleftarrow{d'} A_2 \xleftarrow{d'} \cdots\}$$

$$H_i^B := H_i\{0 \leftarrow B_0 \xleftarrow{d''} B_1 \xleftarrow{d''} B_2 \xleftarrow{d''} \cdots\}$$

Theorem :

In the special case of a first quadrant bicomplex

with rows and columns exact

except maybe at C_{*0} and C_{0*} ,

then there are canonical isomorphisms:

$$H_i^B \cong H_i^T \cong H_i^A$$

Usual proof = **Filtering** the **totalization**

$\Rightarrow$ **Spectral sequence.**

Vertical **filtration**:

$F_i(T) :=$

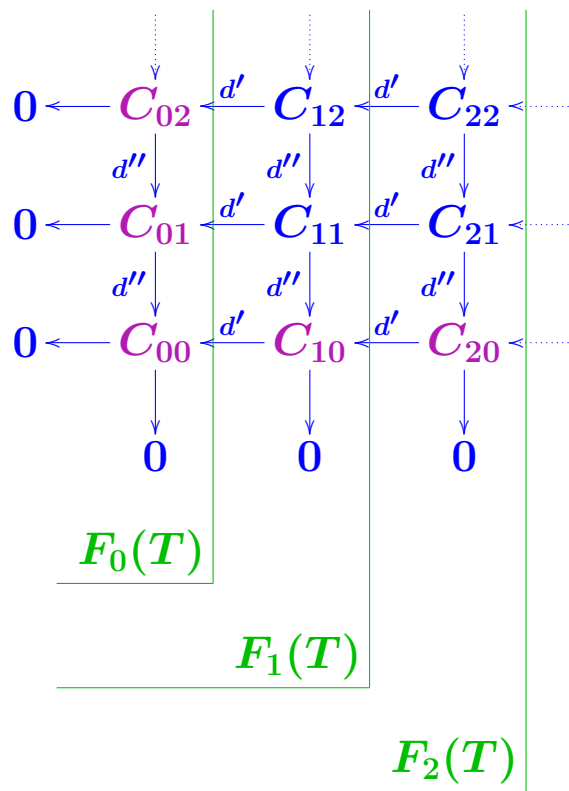
$\{C_{jk}\}$ **vanished** for $j > i$.

$\Rightarrow$

$$F_i(T)/F_{i-1}(T) = C_{i*}$$

= Column i

Acyclic except in C_{i0} .



$F_i(T)/F_{i-1}(T)$ **acyclic** except along the ***0** axis

$\Rightarrow$ the **spectral sequence** “**degenerates**”.

E^1 -page of the **spectral sequence**:

$$\leftarrow \cdots 0 \leftarrow 0 \leftarrow 0 \leftarrow 0 \leftarrow \cdots$$

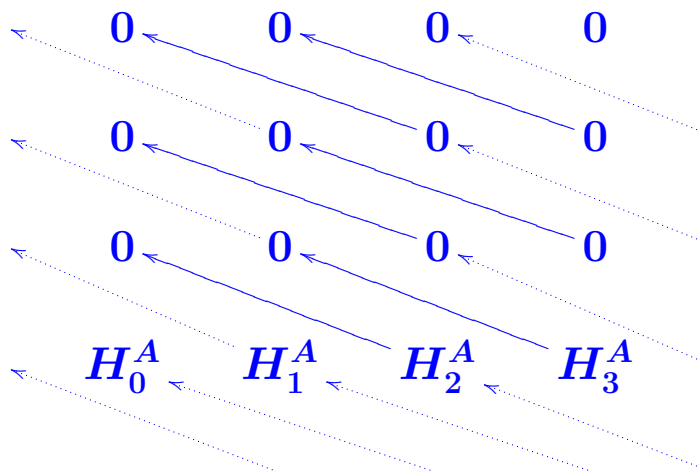
$$\leftarrow \cdots 0 \leftarrow 0 \leftarrow 0 \leftarrow 0 \leftarrow \cdots$$

$$\leftarrow \cdots 0 \leftarrow 0 \leftarrow 0 \leftarrow 0 \leftarrow \cdots$$

$$\leftarrow \cdots A_0 \xleftarrow{d'} A_1 \xleftarrow{d'} A_2 \xleftarrow{d'} A_3 \leftarrow \cdots$$

$\Rightarrow$

E^2 -page of the spectral sequence:

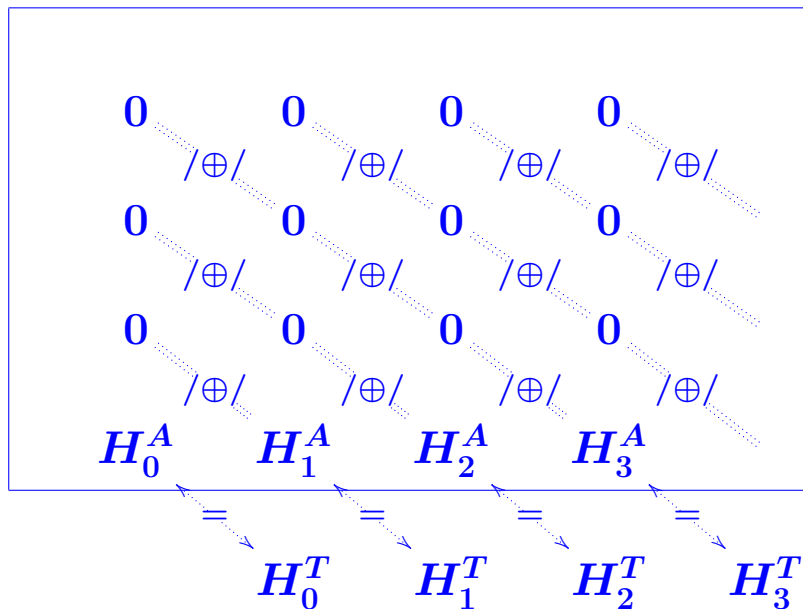


$\Rightarrow r \geq 2 \Rightarrow$

$E_{pq}^r = 0$ for $q > 0$

and canonical isomorphism $E_{p0}^r \cong H_p^A$.

$\Rightarrow E^\infty$ -page of the spectral sequence:



$\Rightarrow$

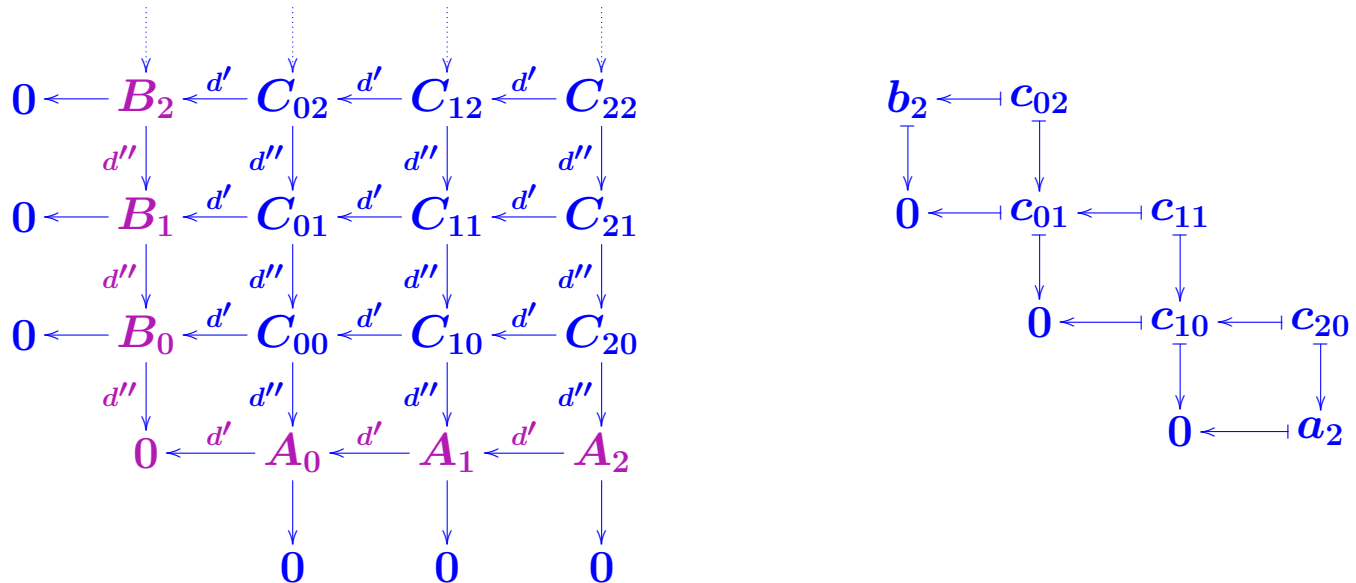
$$H_i^A \cong H_i^T \cong H_i^B$$

QED

Spectral sequence proof **easy** but **not effective**!

What about an **effective proof**?

Diagram chasing proof: $H_2^A \ni h_2^A \xrightarrow{???} h_2^B \in H_2^B$



Proof details: **terribly awful!!**

More convenient:

Use the **homological perturbation “lemma”**.

1. An exact sequence:

$$0 \leftarrow A_i \leftarrow C_{i0} \leftarrow C_{i1} \leftarrow \dots$$

can be viewed as a reduction:

$$\rho_i : \boxed{h_i \hookrightarrow C_{i*} \begin{matrix} \xleftarrow{g_i} \\ \xrightarrow{f_i} \end{matrix} A_i}$$

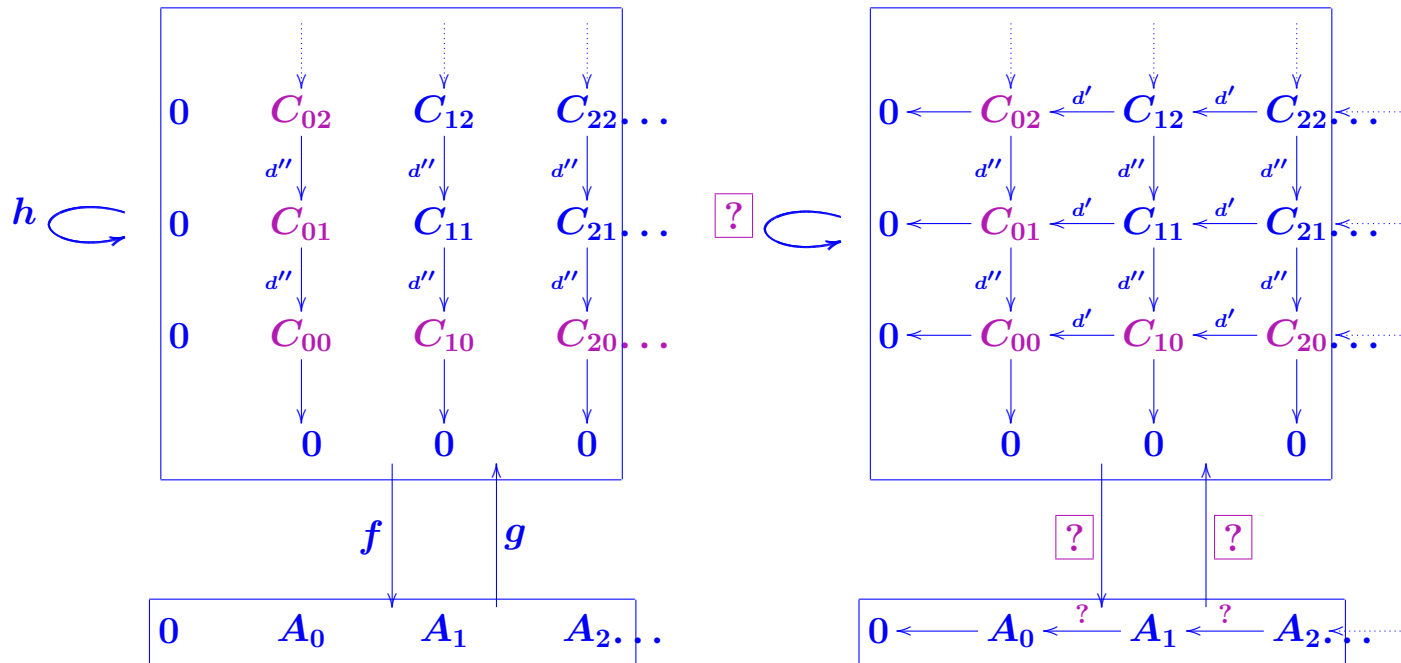
2. Direct sum with respect to i :

$$\rho : \boxed{h \hookrightarrow \oplus_i C_{i*} \begin{matrix} \xleftarrow{g} \\ \xrightarrow{f} \end{matrix} \oplus_i A_i}$$

$\Rightarrow \dots / \dots$

$\Rightarrow \dots / \dots$

Homological Perturbation



Basic Perturbation “Lemma” (BPL):

Given:

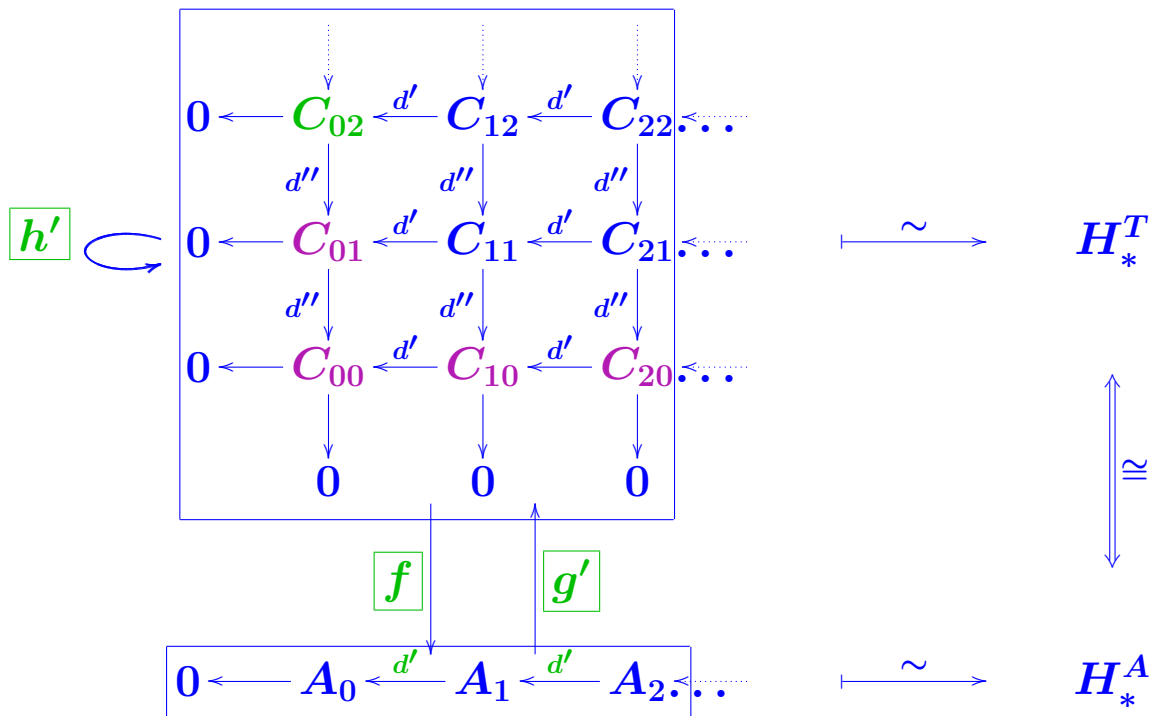
$$+ \quad \boxed{\hat{C}_* \rightrightarrows \hat{d}} \quad \text{satisfying:}$$

1. $\hat{d}$ is a **perturbation** of the differential $\hat{d}$;
2. The **operator** $h \circ \hat{d}$ is **pointwise nilpotent**.

Then a **general algorithm** *BPL* constructs:

$$+ \quad \boxed{\hat{C}_* \rightrightarrows \hat{d}} \quad \xrightarrow{\text{BPL}} \quad \boxed{\begin{array}{c} h+\delta_h \rightrightarrows \hat{C}_* \rightrightarrows \hat{d}+\hat{\delta} \\ f+\delta_f \uparrow \downarrow g+\delta_g \\ C_* \rightrightarrows d+\delta_d \end{array}}$$

$\Rightarrow$



Doing the **symmetric work**

in the horizontal direction

constructs an **equivalence**:

$$\begin{array}{c}
 \begin{array}{ccc}
 & \text{\textit{lh}} \circlearrowleft & T(C_{**}) \circlearrowright \text{\textit{rh}} \\
 \text{\textit{lf}} \nearrow & & \nwarrow \text{\textit{rg}} \\
 A_* & & B_* \\
 \nwarrow \text{\textit{lg}} & & \nearrow \text{\textit{rf}}
 \end{array}
 \end{array}
 =
 \boxed{A_* \rightleftarrows B_*}$$

which **effectively** induces an **isomorphism**:

$$H_*^A \xrightleftharpoons[\text{\textit{[lf,lg,rf,rg]} }]{\cong} H_*^B$$

QED

Example of **Application** in **Commutative Algebra**:

Let $\mathfrak{R} = \mathfrak{k}[x_1, \dots, x_m]$ = common polynomial ring.

Let $M = \mathfrak{R}$ -module of finite type.

Strong known relations between:

1. Free $\mathfrak{R}$ -resolution of M :

$$0 \leftarrow M \leftarrow M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \dots$$

2. Homology of the Koszul complex $\text{Ksz}(M)$:

$$H_*(0 \leftarrow M \leftarrow M \otimes \mathfrak{k}^m \leftarrow M \otimes \wedge^2 \mathfrak{k}^m \leftarrow M \otimes \wedge^3 \mathfrak{k}^m \leftarrow \dots)$$

Usually: Resolution $\mapsto$ Homology of Koszul Complex:

$$\text{Rsl}(M) \mapsto \text{Rsl}(M) \otimes \mathfrak{k} \mapsto H_*(\text{Rsl}(M) \otimes \mathfrak{k}) = H_*(\text{Ksz}(M)).$$

Effective Homology produces a reverse process.

Theorem: A simple algorithm:

Input: Effective Homology $H_*^{\text{Eff}}(\text{Ksz}(M))$.

Output: An effective resolution $\text{Rsl}^{\text{Eff}}(M)$.

Combined with elementary methods of Effective Homology:

$$\text{GB}(M) \mapsto H_*^{\text{Eff}}(\text{Ksz}(M)) \mapsto \text{Rsl}^{\text{Eff}}(M)$$


New simple understanding

Koszul complex and its differential:

$$\mathrm{Ksz}(M) := \{\cdots \leftarrow M \otimes \wedge^{i-1} \mathfrak{k}^m \leftarrow M \otimes \wedge^i \mathfrak{k}^m \leftarrow \cdots\}$$

$$d(\mu \cdot dx_2 \wedge dx_3 \wedge dx_5) := \\ x_2 \mu \cdot dx_3 \wedge dx_5 - x_3 \mu \cdot dx_2 \wedge dx_5 + x_5 \mu \cdot dx_2 \wedge dx_3$$

Two $\mathfrak{R}$ -modules $M_1, M_2 \mapsto$ a “double” Koszul complex:

$$M_1 \otimes \wedge^i \mathfrak{k}^m \otimes M_2$$

with two differentials:

$$d'(\mu_1 \cdot (dx_2 \wedge dx_5) \cdot \mu_2) := x_2 \mu_1 \cdot dx_5 \cdot \mu_2 - x_5 \mu_1 \cdot dx_2 \cdot \mu_2$$

$$d''(\mu_1 \cdot (dx_2 \wedge dx_5) \cdot \mu_2) := -\mu_1 \cdot dx_5 \cdot (x_2 \mu_2) + \mu_1 \cdot dx_2 \cdot (x_5 \mu_2)$$

$\Rightarrow$ Double-complex.

$M = \mathfrak{R}$ -module of finite type.

$H_*^{\text{Eff}}(\text{Ksz}(M))$ given:

Equivalence: $\boxed{d \hookrightarrow M \otimes \wedge \mathfrak{k}^m} \rightleftarrows \boxed{EC_M \hookrightarrow d}$

EC_M = chain complex of finite dimensional $\mathfrak{k}$ -vector spaces.

Particular case: $M = \mathfrak{R}$. Koszul's theorem:

Reduction: $\boxed{d \hookrightarrow \wedge \mathfrak{k}^m \otimes \mathfrak{R}} \rightleftarrows \boxed{\mathfrak{k} \hookrightarrow 0}$

We intend to play with both equivalences

in the double Koszul complex:

$$M \otimes \wedge \mathfrak{k}^m \otimes \mathfrak{R}$$

Detailed description of $M \otimes \wedge^{\mathfrak{k}^m} \otimes \mathfrak{R}$

= **Aramova-Herzog** bicomplex:

$AH(M) :=$

$$\begin{array}{ccccccc}
 & \downarrow \cdots & & \downarrow \cdots & & \downarrow \cdots & & \downarrow \cdots \\
 M \otimes \wedge^3 \otimes \mathfrak{R}_0 & \xrightarrow{d''} & M \otimes \wedge^2 \otimes \mathfrak{R}_1 & \xrightarrow{d''} & M \otimes \wedge^1 \otimes \mathfrak{R}_2 & \xrightarrow{d''} & M \otimes \wedge^0 \otimes \mathfrak{R}_3 & \longrightarrow 0 \\
 \downarrow d' & & \downarrow d' & & \downarrow d' & & \downarrow & \\
 M \otimes \wedge^2 \otimes \mathfrak{R}_0 & \xrightarrow{d''} & M \otimes \wedge^1 \otimes \mathfrak{R}_1 & \xrightarrow{d''} & M \otimes \wedge^0 \otimes \mathfrak{R}_2 & \longrightarrow & 0 & \\
 \downarrow d' & & \downarrow d' & & \downarrow & & & \\
 M \otimes \wedge^1 \otimes \mathfrak{R}_0 & \xrightarrow{d''} & M \otimes \wedge^0 \otimes \mathfrak{R}_1 & \longrightarrow & 0 & & & \\
 \downarrow d' & & \downarrow & & & & & \\
 M \otimes \wedge^0 \otimes \mathfrak{R}_0 & \longrightarrow & 0 & & & & & \\
 \downarrow & & & & & & & \\
 0 & & & & & & &
 \end{array}$$

$$\mathfrak{R}_p = \mathfrak{k}[x_1, \dots, x_m]^{[p]}$$

$$\wedge^q = \wedge^q \mathfrak{k}^m = \wedge^q(\mathfrak{m}/\mathfrak{m}^2)$$

$$M = \mathfrak{R}\text{-module}$$

$$\otimes = \otimes_{\mathfrak{k}}$$

$$\text{Horizontal} = M \otimes \text{Ksz}(\mathfrak{R})_q$$

$$\text{Vertical} = \text{Ksz}(M) \otimes \mathfrak{R}_p$$

1. Horizontal reduction.

$\text{Row}_q =$

$$\begin{aligned} \{\cdots \rightarrow M \otimes \wedge^{q-p} \otimes \mathfrak{R}_p \rightarrow M \otimes \wedge^{q-p-1} \otimes \mathfrak{R}_{p+1} \rightarrow \cdots\} \\ = q\text{-homogeneous component } M \otimes [\wedge \otimes \mathfrak{R}]_q \\ = M \otimes [\text{Ksz}(\mathfrak{R})]_q \end{aligned}$$

$\Rightarrow$ acyclic!

except for $q = 0$ where $\text{Row}_0 = M \otimes \wedge^0 \otimes \mathfrak{R}_0 = M$.

$\Rightarrow$ Reduction: $\bigoplus_q \text{Row}_q \Rrightarrow M$

Adding the vertical differentials = BPL

$\Rightarrow$ Reduction $AH(M) \Rrightarrow M$.

2. Vertical equivalence:

$$\mathrm{Col}_p =$$

$$\begin{aligned} \{\cdots \rightarrow M \otimes \wedge^{q-p} \otimes \mathfrak{R}_p \rightarrow M \otimes \wedge^{q-p-1} \otimes \mathfrak{R}_p \rightarrow \cdots\} \\ = \mathrm{Ksz}(M) \otimes \mathfrak{R}_p \end{aligned}$$

$$\begin{aligned} \text{Available \textcolor{red}{equivalence}: } \mathrm{Ksz}(M) \rightleftarrows C_*^{\textcolor{red}{Eff}} = \\ = \text{chain complex of \textcolor{red}{finite}-dimensional } \mathfrak{k}\text{-vector spaces.} \end{aligned}$$

$$\Rightarrow \textcolor{green}{\text{Equivalence: }} \oplus_p \mathrm{Col}_p \rightleftarrows C_*^{\textcolor{red}{Eff}} \otimes \mathfrak{R}$$

Adding the horizontal differentials = $\textcolor{green}{\text{BPL}}$

$$\Rightarrow \boxed{\textcolor{green}{\text{Equivalence: }} AH(M) \rightleftarrows C_*^{\textcolor{red}{Eff}} \otimes \mathfrak{R} \curvearrowright d}$$

$$\text{where now } C_*^{\textcolor{red}{Eff}} \otimes \mathfrak{R} \curvearrowright d =$$

$$= \text{Chain complex of } \boxed{\textcolor{green}{\text{free}}} \mathfrak{R}\text{-modules.}$$

3. Composing Reduction 1 + Equivalence 2:

$$M \Leftrightarrow AH(M) \Leftrightarrow C_*^{\text{Eff}} \otimes \mathfrak{R}$$

gives an equivalence:

$$M \Leftrightarrow C_*^{\text{Eff}} \otimes \mathfrak{R}$$

Interpreting this equivalence $\Rightarrow$

$$C_*^{\text{Eff}} \otimes \mathfrak{R} = \boxed{\text{free effective } \mathfrak{R}\text{-resolution of } M}.$$

With $\boxed{\text{simple natural explicit formulas}}$

for the differentials $\boxed{\text{and}}$ the contraction of $C_*^{\text{Eff}} \otimes \mathfrak{R}$.

$$= \boxed{\text{Hodge decomposition}} \text{ of } C_*^{\text{Eff}} \otimes \mathfrak{R}$$

Toy example:

$$\mathfrak{R} = \mathbb{Q}[x, y, t] \quad M = \mathfrak{R} / \langle x - t^3, y - t^5 \rangle$$

$$\text{Groebner basis} = \langle xt^2 - y, t^3 - x, x^2 - yt \rangle$$

$\Rightarrow$ **Approximate** monomial module:

$$M' = \mathfrak{R} / \langle xt^2, t^3, x^2 \rangle$$

$\Rightarrow$ a **recursive process** produces an **equivalence**:

$$\text{Ksz}(M') \Leftrightarrow \{ \mathbb{Q} \xleftarrow{0} \mathbb{Q}^3 \xleftarrow{0} \mathbb{Q}^3 \xleftarrow{\neq 0} \mathbb{Q} \}$$

Applying **BPL** $\Rightarrow$ **Equivalence**:

$$\text{Ksz}(M) \Leftrightarrow \{ \mathbb{Q} \xleftarrow{0} \mathbb{Q}^3 \xleftarrow{\neq 0} \mathbb{Q}^3 \xleftarrow{\neq 0} \mathbb{Q} \}$$

Aramova-Herzog bicomplex + Two BPL applications:

$\Rightarrow$ Resolution:

$$M \xleftarrow{\varepsilon} \mathfrak{R} \xleftarrow{d_1} \mathfrak{R}^3 \xleftarrow{d_2} \mathfrak{R}^3 \xleftarrow{d_3} \mathfrak{R} \xleftarrow{\quad} 0$$

$$[x^2 - yt, -t^3 + x, -xt^2 + y] \quad \begin{matrix} d_1 & d_2 & d_3 \\ \begin{bmatrix} 0 & \mathbf{1} & t^2 \\ 0 & -x & -y \\ 0 & t & x \end{bmatrix} & \begin{bmatrix} \mathbf{-1} \\ 0 \\ 0 \end{bmatrix} \end{matrix}$$

Final obvious simplifications $\Rightarrow$ Minimal resolution:

$$M \xleftarrow{\varepsilon} \mathfrak{R} \xleftarrow{d'_1} \mathfrak{R}^2 \xleftarrow{d'_2} \mathfrak{R} \xleftarrow{\quad} 0$$

$$[-t^3 + x, -xt^2 + y] \quad \begin{matrix} d'_1 & d'_2 \\ \begin{bmatrix} xt^2 - y \\ -t^3 + x \end{bmatrix} \end{matrix}$$

First **Stillman** example: $\mathfrak{R} = \mathbb{Q}[a, b, c, d, e, f, g, h, i]$

$$M = \mathfrak{R} / \langle fh - ei, ch - bi, \dots \rangle$$

defined by the 9 2×2 -minors of the 3×3 -matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$$M' = \mathfrak{R} / \langle fh, ch, \dots \rangle$$

Equivalence: $\text{Ksz}(M') \Leftrightarrow \{0 \leftarrow \mathbb{Q} \xleftarrow{0} \mathbb{Q}^9 \xleftarrow{0} \mathbb{Q}^{18} \xleftarrow{1} \mathbb{Q}^{14} \xleftarrow{1} \mathbb{Q}^4 \xleftarrow{0} 0\}$

BPL application $\Rightarrow$

Equivalence: $\text{Ksz}(M) \Leftrightarrow \{0 \leftarrow \mathbb{Q} \xleftarrow{0} \mathbb{Q}^9 \xleftarrow{0} \mathbb{Q}^{18} \xleftarrow{2} \mathbb{Q}^{14} \xleftarrow{3} \mathbb{Q}^4 \xleftarrow{0} 0\}$

$\Rightarrow$ **Betti numbers** = $\{1, 9, 16, 9, 1\}$

Aramova-Herzog bicomplex + $2 \times \text{BPL} \Rightarrow \text{Resolution:}$

$$0 \longleftarrow M \longleftarrow \mathfrak{R} \longleftarrow \mathfrak{R}^9 \longleftarrow \mathfrak{R}^{18} \xleftarrow{2} \mathfrak{R}^{14} \xleftarrow{3} \mathfrak{R}^4 \longleftarrow 0$$

Obvious simplifications $\Rightarrow$ Minimal resolution:

$$0 \longleftarrow M \longleftarrow \mathfrak{R} \longleftarrow \mathfrak{R}^9 \longleftarrow \mathfrak{R}^{16} \longleftarrow \mathfrak{R}^9 \longleftarrow \mathfrak{R} \longleftarrow 0$$

But the methods of effective homology produce

an effective resolution:

with a Hodge decomposition:

$$\text{id} = dh + hd$$

.

Interpretation.

$$0 \longleftarrow M \longleftarrow \mathfrak{R} \longleftarrow \mathfrak{R}^9 \begin{matrix} \xleftarrow{d} \\ \xrightarrow{h} \end{matrix} \mathfrak{R}^{16} \begin{matrix} \xleftarrow{d} \\ \xrightarrow{h} \end{matrix} \mathfrak{R}^9 \longleftarrow \mathfrak{R} \longleftarrow 0$$

Let $\mu \in \mathfrak{R}^{16} \in \text{Rsl}_2(M)$.

Then:

$\mu - hd\mu$ is the **projection** of μ

over the **syzygy** subspace of $\mathfrak{R}^{16}$

$h\mu \in \mathfrak{R}^9 = \text{Rsl}_3(M)$ is the **preimage** of this **syzygy** in $\mathfrak{R}^9$.

Hodge decomposition of μ :

$$\mu = hd\mu + dh\mu$$

Example of **calculation**.

$b_{2,1}$ = first generator of $\mathbf{Rsl}_2 = \mathfrak{R}^{16}$.

$$\mu = b_{2,1} \Rightarrow \mu - h d\mu = 0$$

$$\begin{aligned} \mu' = fg \cdot b_{2,1} \Rightarrow \mu' - h d\mu' = \\ -di \cdot g_{2,1} - ci \cdot g_{2,3} + ei \cdot g_{2,6} - fi \cdot g_{2,7} + i^2 \cdot g_{2,15}. \end{aligned}$$

$$h\mu' = h(\mu' - h d\mu') = i \cdot g_{3,4}$$

Hodge decomposition:

$$\begin{aligned} \mu' = fg \cdot b_{2,1} = h d\mu' + d h\mu' = \\ (fg + di) \cdot b_{2,1} + ci \cdot b_{2,3} - ei \cdot b_{2,6} + fi \cdot b_{2,7} - i^2 \cdot b_{2,15} \\ - di \cdot b_{2,1} - ci \cdot b_{2,3} + ei \cdot b_{2,6} - fi \cdot b_{2,7} + i^2 \cdot b_{2,15} \end{aligned}$$

Final scheme for **computing** a **resolution** by this method.

1. Given $M = \text{finite type } \mathfrak{R}\text{-module}$.
2. $\Rightarrow$ **Groebner basis** $\text{Gb}(M)$.
3. $\Rightarrow$ **Monomial approximation** $M' \sim M$.
4. $\Rightarrow H_*^{\text{Eff}}(\text{Ksz}(M'))$.
5. **BPL** $\Rightarrow H_*^{\text{Eff}}(\text{Ksz}(M))$.
6. **Aramova-Herzog bicomplex** + **BPL** $\Rightarrow \text{Rsl}^{\text{Eff}}(M)$.

The END

```
;; Clock  
Computing  
<TnPr <TnPr  
End of computing.
```

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;; Clock -> 2002-01-17, 19h 25m 36s.  
Computing the boundary of the generator 19 (dimension 7) :  
<TnPr <TnPr <TnPr S3 <<Abar[2 S1][2 S1]>>> <<Abar>>> <<Abar>>>  
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Homology in dimension 6 :

Component Z/12Z

---done---

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Talk in honour of Jacques Calmet

*Francis Sergeraert, Institut Fourier, Grenoble, France
Logroño, February 28, 2008*