Generators of multiple failure ideals of k-out-of-n and consecutive k-out-of-n systems

F. Mohammadi  E. Sáenz-de-Cabezón  H. Wynn

June 23, 2016
Algebraic reliability

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Canonical Example: Networks (communication, electrical,..)
Monomial Ideals and Reliability

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The importance of each component (system design problem) can be computed using Hilbert function of $I_S$ and related ideals.
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- These formulas give fast algorithms for computing the reliability of these systems. Include k-out-of-n and variants, series-parallel systems.
- The Hilbert function of $I_S$ has been used to define importance measures for optimal design of robust systems in terms of reliability.
- A generalization along the same principles have been used to study percolation on trees (of importance in probability theory) using asymptotic behavior of Betti numbers.
Multiple failure analysis

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Let $Y$ be the number of simultaneous elementary failure events. We want to study the probabilities $F(i) = \text{prob}\{Y \geq i\}$, i.e., the probability distribution as $i$ increases. Algebraically, we then want to study the ideals generated by lcm's of the generators of the system ideal.
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We are interested in the Hilbert series and free resolutions of all the ideals $I_i$ in the filtration.
Computational issues come from the fact that the number of generators of $I_i$ is potentially $\binom{r}{i}$ ($r$ is the number of generators of $I_S$).
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- We have to compute all lcm\s
- Autorreducing the generating set is expensive
- We have to compute the Hilbert series or resolutions for ideals with a big number of generators
Resolutions for $I_i$

- $\mathcal{T}(I_i)$, Taylor resolution based on the generating set $\langle \text{lcm}(m_\sigma) | \sigma \subset \{1, \ldots, r\} | |\sigma| = k \rangle$
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- $T(l_i)$, Taylor resolution based on the generating set $\langle \text{lcm}(m_{\sigma}) | \sigma \subset \{1, \ldots, r\} | |\sigma| = k \rangle$
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  $\mathcal{P}(I_i)$
- Mayer-Vietoris trees (computes ranks of the mapping cone resolution).
Example

Consecutive 2-out-of-n for n=10,11,12
\[ I = \langle x_1x_2, x_2x_3, \ldots, x_9x_{10} \rangle \]
Log of the sizes of the resolutions of \( I = I_1, I_2, \ldots, I_{n-1} \) (size= sum of all Betti numbers)
In green: Taylor with minimal generating set
In red: \( \mathbb{P}(I_i) \)
In blue: Minimal free resolution
$k$-out-of-$n$ and consecutive-$k$-out-of-$n$

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- Consecutive $k$-out-of-$n$: System with $n$ components that fails whenever $k$ consecutive components fail.
Two important cases:

- **$k$-out-of-$n$:** System with $n$ components that fails whenever $k$ components fail.

- **Consecutive $k$-out-of-$n$:** System with $n$ components that fails whenever $k$ consecutive components fail.

In these cases we can have a combinatorial description of the minimal generating set of the lcm-ideals.
Let $S_{k,n}$ be a $k$-out-of-$n$ system. The failure ideal of $S_{k,n}$ is given by $I_{k,n} = \langle \prod_{i \in \sigma} x_i | \sigma \subseteq \{1, \ldots, n\}, |\sigma| = k \rangle$. Let $I_{k,n,i}$ be the $i$-fold lcm-ideal of $I_{k,n}$.

**Theorem**

Let $k < j \leq n$. For all $(\begin{pmatrix} j - 1 \end{pmatrix} / k) < i \leq \begin{pmatrix} j \end{pmatrix} / k)$ we have that $I_{k,n,i} = \langle \prod_{s \in \sigma} x_s | \sigma \subseteq \{1, \ldots, n\}, |\sigma| = j \rangle = I_{j,n}$. 
Let \( S_{k,n} \) be a consecutive \( k \)-out-of-\( n \) system, its failure ideal is given by \( J_{k,n} = \langle x_1 \cdots x_k, x_2 \cdots x_{k+1}, \ldots, x_{n-k+1} \cdots x_n \rangle = \langle m_1, m_2, \ldots, m_{n-k+1} \rangle \). Let \( J_{k,n,i} \) be the \( i \)-fold \( lcm \)-ideal of \( J_{k,n} \).

Let us denote by \( S \) the set of subsets of \( \{1, \ldots, n - k + 1\} \), and let \( S^i \) the elements of \( S \) of cardinality \( i \). Let \( \sigma \subseteq \{1, \ldots, n - k + 1\} \). We say that \( \sigma \) has a gap of size \( s \) if there is a subset of \( s \) consecutive elements of \( \{\min(\sigma), \ldots, \max(\sigma)\} \) that are not in \( \sigma \).

Let \( S_a \) be the set of subsets \( \sigma \) of \( \{1, \ldots, n - k + 1\} \) such that the smallest gap in \( \sigma \) has size \( a \). Let \( S_a^i \) be the elements in \( S_a \) of cardinality \( i \).

**Theorem**

\( J_{k,n,i} \) is minimally generated by the monomials \( m_\sigma \) such that \( \sigma \in S_0^i \cup S_k^i \cup S_{k+1}^i \cup \cdots \cup S_{n-k+1}^i \) i.e. the minimal generators of \( J_{k,n,i} \) corresponds to the \( lcm \)’s of sets of monomials of cardinality \( i \) with no gaps of sizes between 1 and \( k - 1 \) both included.
**Example: $J_{2,9}$**

$J_{2,9}$ is generated by 8 monomials in 9 variables. $J_{2,9,4}$ is minimally generated by the 26 monomials that correspond to taking \( \text{lcm} \)'s of the following sets of generators of $I_{2,9}$.

Observe that e.g. 2345 means \( \text{lcm}(m_2, m_3, m_4, m_5) \).

<table>
<thead>
<tr>
<th>Pattern</th>
<th>sets</th>
<th>deg. of generators</th>
</tr>
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<tbody>
<tr>
<td>4</td>
<td>1234, 2345, 3456, 4567, 5678</td>
<td>5</td>
</tr>
<tr>
<td>3,1</td>
<td>1236, 1237, 1238, 2347, 2348, 3458, 1456, 1567, 2567, 1678, 2678, 3678</td>
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<td>2,2</td>
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<td>2,1,1</td>
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If we considered all possible subsets of 4 elements of \( \{1, \ldots, 8\} \) we would have considered 70 sets among which we should have made the corresponding finding and elimination of the 44 redundant ones.
Experiments

Using all i-subsets

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Sizes for the \( n=17 \) case:

| 16   | 120  | 560  | 1820 | 4368 | 8008 | 11440 | 12870 | 11440 | 8008 | 4368 | 1820 | 560  | 120  |
|------|------|------|------|------|------|-------|-------|-------|------|------|------|------|------|------|
| 16   | 106  | 390  | 916  | 1512 | 1882 | 1856  | 1500  | 1016  | 586  | 286  | 126  | 40   | 16   |
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These implementations have been done using Macaulay 2 v. 1.8.2 on an Intel Core i5 (2 cores) and 4GB RAM
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- B. Giglio and H. Wynn, Monomial ideals and the Scarf complex for coherent systems in reliability theory, Annals of Statistics, 2004
- E. SdC and H. Wynn, Betti numbers and minimal free resolutions for multi-state system reliability bounds, Journal of Symbolic Computation, 2009 (MEGA 2007 Special Issue)
- E. SdC and H. Wynn, Measuring the robustness of a network using minimal vertex covers, Mathematics and Computers in Simulation, 2014
- F. Mohammadi, Divisors on graphs, orientations, syzygies, and system reliability, arXiv:1405.7972