Generators of multiple failure ideals of k-out-of-n and consecutive k-out-of-n systems

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Canonical Example: Networks (communication, electrical,..)

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- If we compute the Hilbert series by any resolution of I_S we also obtain bounds for the reliability of S the smaller the resolution, the tighter the bounds.
- The importance of each component (system design problem) can be computed using Hilbert function of I_S and related ideals.

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- A generalization along the same principles have been used to study percolation on trees (of importance in probability theory) using asymptotic behavior of Betti numbers.

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- We want to study the probabilities $F(i) = prob\{Y \geq i\}$ i.e. the probability distribution as i increases.
- Algebraically, we then want to study the ideals generated by lcm's of the generators of the system ideal

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We are interested in the Hilbert series and free resolutions of all the ideals I_i in the filtration.

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- Autorreducing the generating set is expensive
- We have to compute the Hilbert series or resolutions for ideals with a big number of generators

Resolutions for I_i

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- Mayer-Vietoris trees (computes ranks of the mapping cone resolution).

Example

Consecutive 2-out-of-n for n=10,11,12

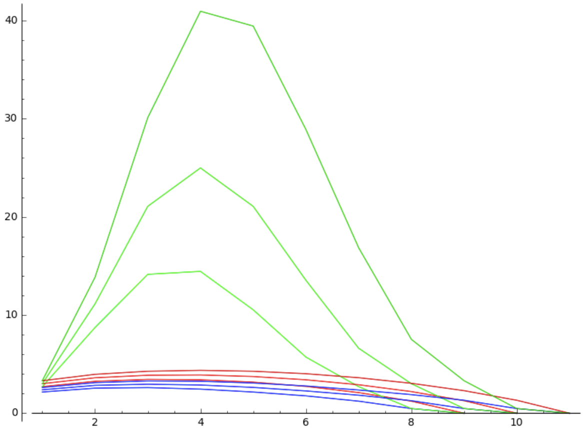
$I = \langle x_1x_2, x_2x_3, \ldots, x_9x_{10} \rangle$

Log of the sizes of the resolutions of $I = I_1, I_2, \ldots, I_{n-1}$ (size= sum of all Betti numbers)

In green: Taylor with minimal generating set

In red: $\mathbb{P}(I_i)$

In blue: Minimal free resolution



40
30
20
10
0
2
4
6
8
10

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In these cases we can have a combinatorial description of the minimal generating set of the lcm-ideals.

Let $S_{k,n}$ be a k-out-of-n system. The failure ideal of $S_{k,n}$ is given by $I_{k,n} = \langle \prod_{i \in \sigma} x_i | \sigma \subseteq \{1, \ldots, n\}, |\sigma| = k \rangle$. Let $I_{k,n,i}$ be the i-fold *lcm*-ideal of $I_{k,n}$.

Theorem

Let $k < j \leq n$. For all $\binom{j-1}{k} < i \leq \binom{j}{k}$ we have that $I_{k,n,i} = \langle \prod_{s \in \sigma} x_s | \sigma \subseteq \{1, \ldots, n\}, |\sigma| = j \rangle = I_{j,n}$.

Let $S_{k,n}$ be a consecutive k-out-of-n system, its failure ideal is given by $J_{k,n} = \langle x_1 \cdots x_k, x_2 \cdots x_{k+1}, \ldots, x_{n-k+1} \cdots x_n \rangle = \langle m_1, m_2, \ldots, m_{n-k+1} \rangle$. Let $J_{k,n,i}$ be the i-fold *lcm*-ideal of $J_{k,n}$. Let us denote by $\mathcal{S}$ the set of subsets of $\{1, \ldots, n-k+1\}$, and let $\mathcal{S}^i$ the elements of $\mathcal{S}$ of cardinality i. Let $\sigma \subseteq \{1, \ldots, n-k+1\}$. We say that σ has a gap of size s if there is a subset of s consecutive elements of $\{\min(\sigma), \ldots, \max(\sigma)\}$ that are not in σ. Let $\mathcal{S}_a$ be the set of subsets σ of $\{1, \ldots, n-k+1\}$ such that the smallest gap in σ has size a. Let $\mathcal{S}_a^i$ be the elements in $\mathcal{S}_a$ of cardinality i.

Theorem

$J_{k,n,i}$ is minimally generated by the monomials m_σ such that $\sigma \in \mathcal{S}_0^i \cup \mathcal{S}_k^i \cup \mathcal{S}_{k+1}^i \cup \cdots \cup \mathcal{S}_{n-k+1}^i$ i.e. the minimal generators of $J_{k,n,i}$ corresponds to the lcm's of sets of monomials of cardinality i with no gaps of sizes between 1 and $k-1$ both included.

Example: $J_{2,9}$

$J_{2,9}$ is generated by 8 monomials in 9 variables.
$J_{2,9,4}$ is minimally generated by the 26 monomials that correspond to taking *lcm*'s of the following sets of generators of $I_{2,9}$.
Observe that e.g. 2345 means $lcm(m_2, m_3, m_4, m_5)$.

Pattern	sets	deg. of generators
4	1234,2345,3456,4567,5678	5
3,1	1236,1237,1238,2347,2348,3458,1456,1567,2567,1678,2678,3678	6
2,2	1256,1267,1278,2367,2378,3478	6
2,1,1	1258,1458,1478	7

If we considered all possible subsets of 4 elements of $\{1, \ldots, 8\}$ we would have considered 70 sets among which we should have made the corresponding finding and elimination of the 44 redundant ones.

Experiments

Using all i-subsets

Ideal	sets	generators	ideals	total	size	hilbert	resolution
$J_{2,17}$	0.23094	37.1002	1.39358	38.72472	65535	2.14741	212.412
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Sizes for the n=17 case:

16	120	560	1820	4368	8008	11440	12870	11440	8008	4368	1820	560	120
16	106	390	916	1512	1882	1856	1500	1016	586	286	126	40	16

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Next version (in progress) **CoCoALib** implementation using bit set operations and optimized algorithms

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