Is Every Secant Variety of a Segre Product Arithmetically Cohen Macaulay?

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Secant varieties and tensors

Let $V_1, \ldots, V_d$, be $\mathbb{C}$-vector spaces, then the tensor product $V_1 \otimes \ldots \otimes V_d$ is the vector space with elements $(T_{i_1,\ldots,i_d})$ considered as hyper-matrices or tensors.

- Segre variety (rank 1 tensors): Defined by

$$\begin{array}{rcl} \mathrm{Seg} : \mathbb{P}V_1 \times \cdots \times \mathbb{P}V_d & \longrightarrow & \mathbb{P}\big(V_1 \otimes \cdots \otimes V_d\big) \\ ([v_1], \ldots, [v_d]) & \longmapsto & [v_1 \otimes \cdots \otimes v_d]. \end{array}$$

  In coordinates: $T_{i_1,\ldots,i_d} = v_{1,i_1} \cdot v_{2,i_2} \cdots v_{d,i_d}$.
- The r^{th} secant variety of a variety $X \subset \mathbb{P}^N$:

$$\sigma_r(X) := \overline{\bigcup_{x_1,\ldots,x_r \in X} \mathbb{P}(\mathrm{span}\{x_1, \ldots, x_r\})} \subset \mathbb{P}^N.$$

General points of $\sigma_r(\mathrm{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_d))$ have the form

$$\left[\sum_{s=1}^{r} v_1^s \otimes v_2^s \otimes \ldots \otimes v_d^s\right],$$

or in coordinates: $T_{i_1,\ldots,i_n} = \sum_{s=1}^r v_{1,i_1}^s \cdot v_{2,i_2}^s \cdots v_{d,i_d}^s$.

Some Applications of Secant Varieties

- Classical Algebraic Geometry: When can a given projective variety $X \subset \mathbb{P}^n$ be isomorphically projected into $\mathbb{P}^{n-1}$?

  Determined by the dimension of the secant variety $\sigma_2(X)$.

- Algebraic Complexity Theory: Bound the border rank of algorithms via equations of secant varieties. Berkeley-Simons program Fall'14

- Algebraic Statistics and Phylogenetics:
  Given contingency tables for DNA of several species, determine the correct statistical model for their evolution.

  Find invariants (equations) of mixture models (secant varieties).

  For star trees / bifurcating trees this is the salmon conjecture.

- Signal Processing: Blind identification of under-determined mixtures, analogous to CDMA technology for cell phones.

  A given signal is the sum of many signals, one for each user.

  Decompose the signal uniquely to recover each user's signal.

- Computer Vision, Neuroscience, Quantum Information Theory, Chemistry...

First algebraic / geometric questions for tensors

Let $X \subset \mathbb{PC}^N$, with $N = n_1 \times \cdots \times n_d$, denote the set of rank-one tensors, and let $\sigma_r(X)$ denote the Zariski closure of the set of rank-r tensors.

1. [Dimensions] What is the dimension of $\sigma_r(X)$?
   – When does $\sigma_r(X)$ fill the ambient $\mathbb{PC}^N$? (defectivity)
2. [Equations] What are the polynomial defining equations of $\sigma_r(X)$?
3. [Decomposition] Given $\mathcal{T} \in \mathbb{C}^N$, can you find an expression of $\mathcal{T}$ as a sum of points from X?
4. [Specific Identifiability] Does a given $\mathcal{T} \in \mathbb{C}^N$ have an essentially unique decomposition?
5. [Generic Identifiability] Do *generic* $\mathcal{T} \in \mathbb{C}^N$ have essentially unique decompositions?

Knowing equations of secant varieties can help with all of these questions, especially if they're determinantal.

Often some equations for secant varieties are known, but the difficult question is to show when the known equations suffice.

Basic algebraic question: Show that a given ideal is prime.

Typical situation for implicitization problems

- Given a parametrized (irreducible) variety $X \subset \mathbb{P}^N$.
- Found: candidate minimal generators $f_1, \ldots, f_t$. Set $J := \langle f_1, \ldots, f_t \rangle$
- Shown: $\mathcal{V}(J) = X$ as a set.
- Often: Numerical computations in Bertini and a (symbolic) degree computation can often indicate that J is reduced in its top dimension.
- Unknown: If $J = \mathcal{I}(X)$ (perhaps there are lower dimensional embedded primes)
- Attempt to show that J is prime. The set-theoretic result and the fact that $I(X)$ is prime then would imply that $J = I(X)$.

Sometimes symmetry and knowing a list of orbits can provide enough information about the primary decomposition of J to rule out embedded primes (see [Aholt-Oeding 2014]).

Cohen-Macaulay-ness and showing an ideal is prime

Here's the typical situation:

- Given a parametrized (irreducible) variety $X \subset \mathbb{P}^N$.
- Found: candidate minimal generators $f_1, \ldots, f_t$.
- Shown: $\mathcal{V}(f_1, \ldots, f_t) = X$ as a set.
- Unknown: If $\langle f_1, \ldots, f_t \rangle = \mathcal{I}(X)$.

Standard argument to show primeness: If an ideal J in a polynomial ring R is aCM and the affine scheme it defines is generically reduced, then it is everywhere reduced, and if the zero set $\mathcal{V}(J)$ agrees with X, then $J = \mathcal{I}(X)$. (Used in [Landsberg–Weyman 2007] for some secant varieties)

Definition

Suppose R is a polynomial ring in finitely many variables, and let I be an ideal of R. Then R/I is *Cohen-Macaulay* if $\operatorname{depth} I = \operatorname{codim} I$. We say that $X = \mathcal{V}(I)$ is arithmetically Cohen-Macaulay (aCM) if R/I is Cohen-Macaualy.

Known examples of aCM secant varieties

Segre varieties: $X = \mathrm{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_d)$ (also any homogeneous variety).

Ambient spaces: If k is such that $\sigma_k(X) = \mathbb{P}^N$ - obviously $\sigma_k(X)$ is aCM.

Hypersurfaces: If k is such that $\sigma_k(X) \subset \mathbb{P}^N$ has codimension 1 then it is aCM.

Determinantal varieties: If $X = \mathrm{Seg}(\mathbb{P}V_1 \times \mathbb{P}V_2)$, then $\sigma_k(X)$ is a determinantal variety and aCM [Eagon, Eagon-Hochster].

(Secretly) Determinantal varieties: Suppose $X = \mathrm{Seg}(\mathbb{P}V_1 \times \cdots \mathbb{P}V_d)$ and $Y = \mathrm{Seg}(\mathbb{P}^a \times \mathbb{P}^b)$. If $\sigma_k(X) = \sigma_k(Y)$ (see Thm. 2.4 [CGG08]), then $\sigma_k(X)$ is a determinantal variety and aCM.

Subspace varieties: [Weyman]
$\mathrm{Sub}_{r_1,\dots,r_n} = \{T \in V_1 \otimes \cdots \otimes V_n \mid \exists V_i' \subset V_i,\ \dim V_i' = r_i,\ T \in V_1' \otimes \cdots \otimes V_n'\}$.

Special case [Landsberg–Weyman]: $\sigma_2(\mathrm{Seg}(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3} \times \mathbb{P}^{n_4}))$

Special case [Landsberg–Weyman]: $\sigma_3(\mathrm{Seg}(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3}))$

Defective dimension: $\sigma_3(\mathrm{Seg}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1))$ (complete intersection of 2 quartics).

Secants of RNC: [Thm. 1.56, Iarrobino-Kanev, Kanev'98] $\sigma_s(\nu_d\mathbb{P}^n)$ is aCM if either $d = 2$, or $n = 1$ or $s \leq 2$, . (dim., deg., and, sing. loc. also given).

Geramita's conjecture: $\sigma_s(\nu_d\mathbb{P}^n)$ is aCM for all d, n, s, [p55 Geramita Lectures].

Known examples of locally CM secant varieties

Theorem (Michalek-Oeding-Zwiernik, 2014)

$\sigma_2(\mathrm{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_d))$ is covered by open normal toric varieties, and in particular is (locally) Cohen Macaulay. The only Gorenstein cases are:

$$\sigma_2(\mathbb{P}^a \times \mathbb{P}^a), \quad \sigma_2(\mathbb{P}^1 \times \mathbb{P}^k) = \mathbb{P}^{2k+1} \text{ (known)},$$
$$\sigma_2(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3), \quad \sigma_2(\mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{P}^3), \quad \sigma_2(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3),$$
$$\sigma_2(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1).$$

Note $\sigma_2(\mathbb{P}^3 \times \mathbb{P}^3) \cong \sigma_2(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3)$ is arithmetically Gorenstein. What about the others?
Using LW lifting, can lift the resolution of $\sigma_2(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3)$, to $\sigma_2(\mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{P}^3)$, and $\sigma_2(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$, and find that they are aG.
Still not sure about $\sigma_2(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$.

A new example of an aCM secant variety

B. Sturmfels's "algebraic fitness session" at the Simon's Institute, Fall 2014:

Theorem⋆ (Oeding-Sam 2015)

The affine cone of $\sigma_5(\mathrm{Seg}(\mathbb{P}^{1\times 5}))$ is a complete intersection of a degree 6 and a degree 16 equation.

In particular (up to our belief in the careful numerical, sometimes probablistic computations in our proofs), $\sigma_5(\mathrm{Seg}(\mathbb{P}^{1\times 5}))$ is arithmetically Cohen Macaulay. Our computations took approximately *two weeks of human/computer time.*

Inspiration:

- [Classical] $2\times 2\times 2\times 2$ tensors are defective in rank 3.
- [Bocci-Chiantini 2014]: $2\times 2\times 2\times 2\times 2$ tensors are not identifiable in rank 5.
  — the generic tensor of that format has exactly 2 decompositions.
- [Bocci-Chiantini-Ottaviani 2014]: For ≥ 6 factors, the Segre is almost always k-identifiable.

Theorem* (Oeding-Sam 2015)

The affine cone of $\sigma_5(\mathrm{Seg}(\mathbb{P}^{1\times 5}))$ is a complete intersection of a degree 6 and a degree 16 equation.

- Use `Bertini` (with Jon Hauenstein's help) to discover that $\deg \sigma_5(\mathrm{Seg}(\mathbb{P}^{1\times 5})) = 96$
- Known codim 2, so we suspect complete intersection of two polynomials.
- Compute the only degree 6 invariant, and show that it vanishes on any number of pseudorandom points of X.
  It turns out that there are 5 standard tableaux of shape $(3,3)$ and content $\{1,2,\dots,6\}$ and the following Schur module, which uses one of each of the 5 standard fillings, realizes the non-trivial copy of $\bigotimes_{i=1}^{5}(\mathbf{S}_{3,3}V_i)$ inside of $\mathrm{Sym}^6(\mathbf{V})$:

$$\mathbf{S}_{\begin{smallmatrix}1&3&5\\2&4&6\end{smallmatrix}}V_1 \otimes \mathbf{S}_{\begin{smallmatrix}1&3&4\\2&5&6\end{smallmatrix}}V_2 \otimes \mathbf{S}_{\begin{smallmatrix}1&2&5\\3&4&6\end{smallmatrix}}V_3 \otimes \mathbf{S}_{\begin{smallmatrix}1&2&4\\3&5&6\end{smallmatrix}}V_4 \otimes \mathbf{S}_{\begin{smallmatrix}1&2&3\\4&5&6\end{smallmatrix}}V_5.$$

- Check invariants of degree 8,10,12,14, 16. In degree 16, $\dim U_{16} = 1313$.

  Find a basis of the 10-dimensional space $U_{16}^{\mathfrak{S}_5, sgn}$ of skew invariants via sums of Young symmetrizers. We discovered that $U_{16}^{\mathfrak{S}_5, sgn} \cap I(X)$ is full-dimensional.

  Find a basis of the 39-dimensional space $U_{16}^{\mathfrak{S}_5}$ of invariants via sums of Young symmetrizers. We discovered that $U_{16}^{\mathfrak{S}_5} \cap I(X)$ has dimension 36 (pseudorandom). $f_6 \cdot U_{10}^{\mathfrak{S}_5, sgn}$ is 2-dimensional, so there is ≥ 1 mingen of degree 16 in $I(X)$.

- $Y = V(f_6, f_{16})$, a complete intersection. Also $X \subseteq Y$ and $\deg X \geq \deg Y = 96$. Since X is irreducible of codimension 2, and Y is equidimensional, Y is also irreducible (otherwise the degree inequality would be violated). So X is the reduced subscheme of Y.

- Also, this implies that $\deg(X) = \deg(Y)$, so Y is generically reduced. Since Y is Cohen–Macaulay, generically reduced is equivalent to reduced. Hence $X = Y$ is a complete intersection.

An adaptation of Weyman's geometric technique

Theorem (Landsberg–Weyman 2007)

Suppose $X := \sigma_r(\mathrm{Seg}(\mathbb{P}^{r-1\times d}))$ is aCM, with "a resolution by small partitions." If $n_i \geq r-1$ for all $1 \leq i \leq d$, then $\sigma_r(\mathrm{Seg}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_d}))$ is aCM and its ideal is generated by the equations inherited from X and the $(r+1)\times(r+1)$-minors of flattenings.

New cases found by [LW] using this result:

- Direct computation: $\sigma_2(\mathrm{Seg}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1))$ is aCM with small partitions, and its ideal is defined by 3×3 minors of flattenings.
- [LW] result implies $\sigma_2(\mathrm{Seg}(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3} \times \mathbb{P}^{n_4}))$ is aCM, and ideal defined by 3×3 minors of flattenings.
- Direct computation: $\sigma_3(\mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2))$ is aCM with small partitions, and its ideal is defined by (Strassen's) 27 quartic equations.
- [LW] result implies that $\sigma_3(\mathrm{Seg}(\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3}))$ is aCM and ideal defined by quartic equations: those inherited from Strassen's and the 4×4 minors of flattenings.

An adaptation of Landsberg–Weyman 2007

What about the cases of $\sigma_r(\mathrm{Seg}(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_d}))$ when $n_i < r-1$ for some i?

Suppose $A_i' \subseteq A_i$ for $1 \le i \le n$. We will say that a G-variety Y has an (s_j)-small resolution if for every module $S_\pi A$ occurring in the resolution has the property

$$\text{for each } j \text{ the first part of } \pi^j \text{ is not greater than } s_j.$$

Let $\widehat{a_j} := \frac{a_1 \ldots a_n}{a_j}$, $\widehat{r_j} := \frac{r_1 \ldots r_n}{r_j}$, $G = \mathrm{GL}(A_1) \times \cdots \times \mathrm{GL}(A_n)$ and $G' = \mathrm{GL}(A_1') \times \cdots \times \mathrm{GL}(A_n')$.

Theorem

If a G'-variety Y is an aCM with an resolution that is $(\widehat{r_j} - r_j)$-small for every j for which $0 < r_j < a_j$, then $\overline{G.Y}$ is aCM.
Moreover we obtain a (not necessarily minimal) resolution of $\overline{G.Y}$ that is (s_j)-small with

$$s_j = \max_\pi \begin{cases} \widehat{a_j} - r_j, & \text{if } r_j < a_j \\ \widehat{a_j} - \widehat{r_j} + \pi_1^j, & \text{if } r_j = a_j, \end{cases}$$

where the max is taken over all multi-partitions π occurring in the resolution of $\mathbb{C}[Y]$.

Applying the adaptation of the Landsberg–Weyman inheritance result, we have the following:

Proposition (Oeding)

$\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$	is aCM, deg. 9 hypersurface	[Strassen]
$GL(4).\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$	is aCM and codim 3 in	$\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$.
$GL(4)^{\times 2}.\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$	is aCM and codim 4 in	$\sigma_4(\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^3)$.
$GL(4)^{\times 3}.\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)$	is aCM and codim 5 in	$\sigma_4(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$.
$GL(4).\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$	is aCM and codim 1 in	$\sigma_4(\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^3)$.
$GL(4)^{\times 2}.\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$	is aCM and codim 2 in	$\sigma_4(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$.
$GL(4).\sigma_4(\mathbb{P}^2 \times \mathbb{P}^3 \times \mathbb{P}^3)$	has codim 1 in	$\sigma_4(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$.

Missing: Don't know how to lift the aCM property further (or if it's possible).

Obsessing over salmon

E. Allman gave half of an Alaskan salmon as a prize for the following result:

Theorem (Friedland 2010)

Suppose $p, q, r \geq 4$. The set of tensors of border rank 4 in $\mathbb{C}^p \otimes \mathbb{C}^q \otimes \mathbb{C}^r$,

$$\text{the secant variety } \sigma_4(\mathbb{P}^{p-1} \times \mathbb{P}^{q-1} \times \mathbb{P}^{r-1}),$$

is defined by certain equations of degrees 5, 9 and 16.

This theorem relied on the following reduction:

Theorem (Landsberg–Manivel 2008, Friedland 2010)

$\sigma_4\left(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3\right)$ is the zero set of:

1. $M_5 = \{$ (Strassen's [1983] degree 5 commutation conditions) $\}$
2. Equations inherited from $\sigma_4\left(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3\right)$

Would like to turn these set-theoretic results into ideal-theoretic results.

Inheritance via an example

Proposition (example of Proposition 4.4 Landsberg–Manivel'04)

$$\tilde{M}_6 := S_{(2,2,2)}\mathbb{C}^4 \otimes S_{(2,2,2)}\mathbb{C}^4 \otimes S_{(3,1,1,1)}\mathbb{C}^4 \subset \mathcal{I}\big(\sigma_4(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)\big)$$

if and only if

$$M_6 := S_{(2,2,2)}\mathbb{C}^3 \otimes S_{(2,2,2)}\mathbb{C}^3 \otimes S_{(3,1,1,1)}\mathbb{C}^4 \subset \mathcal{I}\big(\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)\big).$$

Note: $\dim(\tilde{M}_6) = 10^3$ but $\dim(M_6) = 10$, and has basis of polynomials, each with 576 or 936 monomials.

At every stage we study the smallest module possible. This is a significant dimension reduction.

For $\sigma_4(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$ we only need to consider $S_{\pi_1}A \otimes S_{\pi_2}B \otimes S_{\pi_3}C$ where π_1, π_2, π_3 have 4 parts, and those equations we get from inheritance.

A result of Strassen

Theorem (Strassen 1988 (reinterpreted by Landsberg–Manivel))

The ideal of the hypersurface $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^{26}$ is generated in degree 9 by a nonzero vector in the 1 dimensional module

$$S_{(3,3,3)}\mathbb{C}^3 \otimes S_{(3,3,3)}\mathbb{C}^3 \otimes S_{(3,3,3)}\mathbb{C}^3$$

Since $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2) \subset \sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$, inheritance implies that
$M_9 := S_{(3,3,3)}\mathbb{C}^3 \otimes S_{(3,3,3)}\mathbb{C}^3 \otimes S_{(3,3,3)}\mathbb{C}^4 \subset \mathcal{I}(\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3))$

Strassen's polynomial *only* has 9,216 monomials on 27 variables.
$\dim(M_9) = 20$, with natural basis of polynomials with 9,216 or 25,488 or 43,668 monomials on 36 variables! 23 Mb file of polynomials... :-(

Numerical Algebraic Geometry: Bertini

Theorem* (Bates-Oeding 2011)

$\mathcal{V}(M_6 + M_9) = \sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$.

Suppose $x \in \mathcal{V}(M_6) = \sigma_4\left(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3\right) \cup Sub_{3,3,3}$.
If $x \notin \sigma_4\left(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3\right)$, then use M_9 and consider $x \in Sub_{3,3,3} \cap \mathcal{V}(M_9)$
$\Rightarrow$ x is in some $\sigma_4\left(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2\right) \subset \sigma_4\left(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3\right)$.

Theorem* (Bates-Oeding 2011, Cor. to Landsberg–Manivel '08, Friedland '10)

The salmon variety is cut out *set-theoretically* in degrees 5, 6, 9:

$$\mathcal{V}\left(M_5 + \tilde{M}_6 + \tilde{M}_9\right) = \sigma_4\left(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3\right)$$

Resolves the salmon problem set-theoretically.
Provides a more efficient set of equations than [Friedland 2010].
Sharpens the conjecture for the ideal-theoretic question.
Friedland–Gross 2011 make Theorem into Theorem.*

Another new example of an aCM secant variety

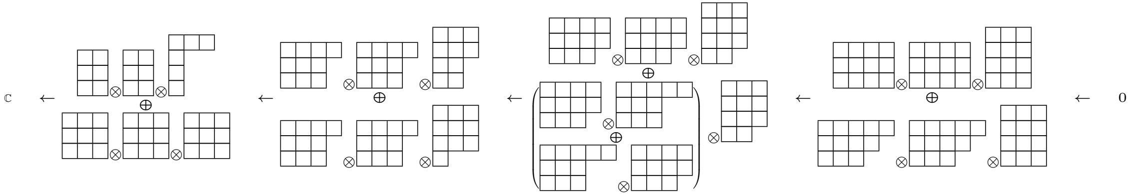
Using `Macaulay2` (and 30s of computational time on a 2009 MacBook Pro), found the minimal free resolution of $M_6 + M_9$, the set-theoretic defining equations for $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$, which has codim 4. The Betti table of this minimal free resolution is:

	0	1	2	3	4
total:	1	30	144	180	65
0:	1	.	.	.	.
1:	.	.	.	.	.
2:	.	.	.	.	.
3:	.	.	.	.	.
4:	.	.	.	.	.
5:	.	10	.	.	.
6:	.	.	.	.	.
7:	.	.	.	.	.
8:	.	20	144	180	65

$\operatorname{codim} \sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3) = 4$, the length 4 resolution implies it is aCM.
Daleo and Hauenstein were able to obtain the same result numerically using `Bertini`.

A nice equivariant resolution

Federico Galetto's `Macaulay2` package determines the G-module structure from the maps in a resolution (provided one can compute the resolution in the first place). Using this package, we obtain the following G-equivariant version of the resolution of $M_6 + M_9$.



We only record the Young tableau that index the G-modules in the resolution. The grading is captured by the number of boxes in each factor.

Theorem (Oeding)

- $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$ is codim 4, aCM, with resolution by small partitions, and ideal generated by equations of degrees 6 (Landsberg–Manivel's) and 9 (Strassen's).
- $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^n)$ is aCM, with resolution by small partitions, and ideal generated by equations of degrees 5(flattenings), and those of degrees 6 and 9 inherited from $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$.
- $\mathrm{GL}(n_1) \times \mathrm{GL}(n_2).\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^{n_3})$ is aCM, with resolution by small partitions, and ideal generated by equations of degrees 5 (flattenings), and those of degrees 6 and 9 inherited from $\sigma_4(\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^3)$.

2nd and 3rd secant varieties

Theorem (C. Raicu 2012)

Let X be the Segre-Veronese variety embedded by $\mathcal{O}(a_1, \ldots, a_n)$. The ideal of the secant variety $\sigma_2(X)$ is generated by 3×3 minors of flattenings.

Theorem (Michalek-Oeding-Zwiernik, 2014)

$\sigma_2(\mathrm{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_d))$ is (locally) Cohen Macaulay.

Unknown: Is σ_2 actually *arithmetically* Cohen-Macaulay?

Theorem (Yang Qi, 2013)

Let X be the Segre variety embedded by $\mathcal{O}(1, \ldots, 1)$. As a set, $\sigma_3(X)$ is cut out by the 4×4 minors of flattenings and Strassen's degree 4 commutation conditions.

It would be nice to turn Qi's result into an ideal-theoretic result. Possible if σ_3 is aCM.

Thanks!