A note on Burchnall-Chaundy polynomials and differential subresultants

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Introduction

Commutative Ordinary Differential Operators.

By J. L. Burchnall and T. W. Chaundy.

(Communicated by A. L. Dixon, F.R.S.—Received December 22, 1926.—Revised February 1, 1928.)

The paper is conveniently divided into two sections. The first contains the general argument and the main propositions unencumbered by proof: in the first paragraph of this section is collected material already published; the succeeding paragraphs of the section are devoted to new results. The second section of the paper includes proofs of these results, together with certain corollaries not essential to the main argument.

Part 1.

I.—*Preamble.*

We make certain notational conventions. There is a single independent variable x; the arbitrary dependent variable of a differential equation or operation is written y. With these exceptions Greek letters denote functions of x and English " lower-case " letters denote constants.

The distinctions extend to symbols of functional form, which will represent polynomials, unless the contrary is stated. Thus

$$f(t) \equiv a_0 t^n + a_1 t^{n-1} + \ldots + a_n$$

Introduction

Burchnall and Chaundy 1928 established a correspondence between commutative pairs of ordinary differential operators and algebraic curves.

(Gardner, Greene, Kruskal, Miura, 1967) With the discovery of *solitons* and the integrability of KdV equation, their theory was applied to partial differential equations called integrable (or with solitonic type solutions: Sine-Gordon, Schrödinger no lineal, etc).

Algebraic approach to handling the inverse spectral problem for the finite-gap operators, with the spectral data being encoded in the spectral curve and an associated line bundle (Krichever 1977).

The spectral curve was later computed by E. Previato (1991) using differential resultants.

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Differential resultant of 2 ODO's

Defined by Ritt (1932), Berkovich and Tsirulik (1986) and studied by Chardin (1991), Li (1998).

(K, ∂) differential field with a derivation
$P, Q \in K[\partial]$, $\mathrm{ord}(P) = n$, $\mathrm{ord}(Q) = m$

The Sylvester matrix $S(P, Q)$ is the coefficient matrix of the extended system

$$\{P, \partial P, \dots, \partial^{m-1} P, Q, \partial Q, \dots, \partial^{n-1} Q\}.$$

$S(P, Q)$ squared matrix of size $n + m$ and entries in K.

Differential resultant of P and Q,

$$\partial\mathrm{Res}(P, Q) := \det(S(P, Q))$$

Example

$$P = a_2\partial^2 + a_1\partial + a_0, \quad Q = b_3\partial^3 + b_2\partial^2 + b_1\partial + b_0$$

$$\partial\mathrm{Res}(P, Q) =$$

$$\begin{vmatrix} a_2 & a_1 + 2\partial(a_2) & a_0 + 2\partial(a_1) + \partial^2(a_2) & 2\partial(a_0) + \partial^2(a_1) & \partial^2(a_0) \\ 0 & a_2 & a_1 + \partial(a_2) & a_0 + \partial(a_1) & \partial(a_0) \\ 0 & 0 & a_2 & a_1 & a_0 \\ b_3 & b_2 + \partial(b_3) & b_1 + \partial(b_2) & b_0 + \partial(b_1) & \partial(b_0) \\ 0 & b_3 & b_2 & b_1 & b_0 \end{vmatrix}$$

Main properties

- $\partial\mathrm{Res}(P,Q) = AP + BQ$ with $A, B \in K[\partial]$, $\mathrm{ord}(A) < m$, $\mathrm{ord}(B) < n$.
$$\partial\mathrm{Res}(P,Q) \in (A,B) \cap K.$$
- $$\partial\mathrm{Res}(P,Q) = 0$$
$$\Updownarrow$$
$$P = P_1 R,\ Q = Q_1 R, \text{ with } \mathrm{ord}(R) > 0,\ P_1, Q_1, R \in K[\partial].$$

Poisson formula

(Poisson formula: Chardin (1991), Previato (1991)) Given monic $P, Q \in K[\partial]$ with respective orders n and m and fundamental systems of solutions $y_1, \ldots, y_n$ and $z_1, \ldots, z_m$ respectively.

$$\partial\mathrm{Res}(P, Q) = \frac{w(Q(y_1), \ldots, Q(y_n))}{w(y_1, \ldots, y_n)} = \frac{w(P(z_1), \ldots, P(z_m))}{w(z_1, \ldots, z_m)}$$

with Wronskian

$$w(y_1, \ldots, y_n) = \det \begin{bmatrix} y_1 & \cdots & y_n \\ \partial y_1 & \cdots & \partial y_n \\ \vdots & \vdots & \vdots \\ \partial^{n-1} y_1 & \cdots & \partial^{n-1} y_n \end{bmatrix}.$$

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Let C be the field of constants of (K, ∂), C algebraically closed and of characteristic zero.

(Burchnall-Chaundy, 1928)
Given $P, Q \in K[\partial]$, if $[P, Q] = PQ - QP = 0$ then there exists $f(\lambda, \mu) \in C[\lambda, \mu]$ such that $f(P, Q) = 0$, called a **Burchnall-Chaundy polynomial**.

$$g(\lambda, \mu) = \partial\mathrm{Res}(P - \lambda, Q - \mu) = a_n^m \mu^n - b_m^n \lambda^m + \dots$$

a non trivial polynomial

in $(P - \lambda, Q - \mu) \cap K[\lambda, \mu]$

$$g(\lambda, \mu) = A(P - \lambda) + B(Q - \mu) \text{ with } A, B \in K[\lambda, \mu][\partial]$$

$$\text{if } [P, Q] = 0 \text{ then } g(P, Q) = 0.$$

(Previato (1991)) Given $P, Q \in K[\partial]$ such that $[P, Q] = 0$ then

$$g(\lambda, \mu) = \partial\mathrm{Res}(P - \lambda, Q - \mu) \in C[\lambda, \mu]$$

and $g(P, Q) = 0$.

$P - \lambda$ and $Q - \mu$ are differential operators with coefficients in $(K(\lambda, \mu), \partial)$, whose field of constants is $C_{\lambda,\mu} := \overline{C(\lambda, \mu)}$.

$y_1, \dots, y_n$ a fundamental system of solutions of $P - \lambda$ over $C_{\lambda,\mu}$, a basis of the $C_{\lambda,\mu}$-vector space $V_{\lambda,\mu} := Ker(P - \lambda)$.

$$W((Q - \mu)(y_1), \dots, (Q - \mu)(y_n)) = W(y_1, \dots, y_n)M$$

M is an $n \times n$ matrix with entries in $C_{\lambda,\mu}$.

$$\partial\mathrm{Res}(P - \lambda, Q - \mu) = \frac{w((Q - \mu)(y_1), \dots, (Q - \mu)(y_n))}{w(y_1, \dots, y_n)} = \det(M).$$

KdV (Korteweg de Vries equation)

Let us consider the Schrödinger operator L in the stationary case

$$L = -\frac{d^2}{dx^2} + u(x)$$
$$A_3 = -\frac{d^3}{dx^3} + \frac{3}{4}\left(u\frac{d}{dx} + \frac{d}{dx}u\right) + \frac{d}{dx}$$

This Lax pair has commutator equal to

$$KdV_1 : [L, A_3] = LA_3 - A_3L = \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x - u_x$$

In particular, for the Rosen-Morse potential

$$u = \frac{-2}{\cosh^2(x)} \text{ then } [L, A_3] = 0$$

Lax pair of KdV

For $u = \frac{-2}{\cosh^2(x)}$, the spectral curve of the Lax pair $\{L, A_3\}$ is

$$\partial\mathrm{Res}(L-\lambda, A_3-\mu) = -\mu^2 - \lambda(\lambda+1)^2 =$$

$$\begin{bmatrix} -1 & 0 & \frac{-2}{(\cosh(x))^2} - \lambda & 8\,\frac{\sinh(x)}{(\cosh(x))^3} & \frac{4}{(\cosh(x))^2} - 12\,\frac{(\sinh(x))^2}{(\cosh(x))^4} \\ 0 & -1 & 0 & \frac{-2}{(\cosh(x))^2} - \lambda & 4\,\frac{\sinh(x)}{(\cosh(x))^3} \\ 0 & 0 & -1 & 0 & \frac{-2}{(\cosh(x))^2} - \lambda \\ -1 & 0 & \frac{-3}{(\cosh(x))^2} + 1 & 9\,\frac{\sinh(x)}{(\cosh(x))^3} - \mu & \frac{3}{(\cosh(x))^2} - 9\,\frac{(\sinh(x))^2}{(\cosh(x))^4} \\ 0 & -1 & 0 & \frac{-3}{(\cosh(x))^2} + 1 & 3\,\frac{\sinh(x)}{(\cosh(x))^3} - \mu \end{bmatrix}$$

(Gesztesy, Holden, 2003) Given $L(u) = -\partial^2 + u(x)$ there exist a family of differential operators $A_{2s+1}(u, c^s)$, $c^s = (c_1^s, \dots, c_s^s)$, $s \in \mathbb{N}$ such that

$$[L(u), A_{2s+1}(u, c^s)] = KdV_s(u, c^s).$$

$$A_3(u, c^1) = \frac{3}{4}u_x + \frac{(3}{2}\partial u - \partial^3 + c_1^1\partial$$

$$A_5(u, c^2) = -\frac{15}{16}u_{xxx} + \frac{15}{8}uu_x + \partial^5 - \frac{25}{8}\partial u_{xx}$$

$$+ \frac{15}{8}\partial u^2 - \frac{15}{4}u_x\partial^2 - \frac{5}{2}\partial^3 u$$

$$+ c_1^2(\frac{3}{4}u_x + \frac{(3}{2}\partial u - \partial^3) + c_2^2\partial$$

For example $u_s = \frac{-s(s+1)}{\cosh^2(x)}$ verifies $KdV_s(u_s, \bar{c}^s) = 0$ for

$$\bar{c}^1 = (1), \;\; \bar{c}^2 = (5, 4), \;\; \bar{c}^3 = (14, 49, 36), \dots$$

By (Goodearl, 1983), we prove that, given u_1 and $\bar{c}^1$ such that $[L(u_1), A_3(u_1, \bar{c}^1)] = KdV_1(u_1, \bar{c}^1) = 0$, the centralizer

$$\begin{aligned} C(L(u_1)) &= \{Q \in \mathcal{C}^\infty[\partial] \mid [L, Q] = 0\} = \mathbb{C}[L]\langle 1, A_3\rangle \\ &= \{p_0(L) + p_1(L)A_3 \mid p_0(L), p_1(L) \in \mathbb{C}[L]\}. \end{aligned}$$

Given u_s and $\bar{c}^s$ such that $KdV_s(u_s, \bar{c}^s) = 0$ we define

$$P_{2n+1} = L^{(n-s)} A_{2s+1}, \quad n \geq s.$$

Recursive formulas

Observe that $P_{2s+1} = A_{2s+1}(u_s, \bar{c}^s)$, $P_{2n+1} = LP_{2n-1}$ and therefore

$$[L, P_{2n+1}] = 0, \quad n \geq s.$$

Let $L_s = L(u_s)$ be a (stationary) Schrödinger operator defined by a potential u_s verifying $[L_s, P_{2n+1}] = KdV_n(u_s, \bar{c}^s) = 0$ for $n \geq s$. Then

$$\partial\mathrm{Res}(L_s - \lambda, P_{2n+1} - \mu) = \mu^2 - \lambda^{2(n-s)} \partial\mathrm{Res}(L_s - \lambda, A_{2s+1}).$$

1. **For** $u_s = s(s+1)/x^2$, $\bar{c}^s = (0,\ldots,0)$, $s \geq 1$ **let** $L_s := L(u_s)$.

$$\partial\mathrm{Res}(L_1 - \lambda, P_{2n+1} - \mu) = \mu^2 - \lambda^{2(n-1)}\lambda^3, \quad n \geq 1$$
$$\partial\mathrm{Res}(L_2 - \lambda, P_{2n+1} - \mu) = \mu^2 - \lambda^{2(n-2)}\lambda^5, \quad n \geq 2.$$

2. **For** $u_s = \frac{-s(s+1)}{\cosh^2(x)}$, $\bar{c}^s$, $s \geq 1$ **let** $L_s := L(u_s)$.

$$\partial\mathrm{Res}(L_1 - \lambda, P_{2n+1} - \mu) = \mu^2 - \lambda^{2(n-1)}\lambda(\lambda-1)^2, \quad n \geq 1$$
$$\partial\mathrm{Res}(L_2 - \lambda, P_{2n+1} - \mu) = \mu^2 - \lambda^{2(n-2)}\lambda(\lambda-1)^2(\lambda-4)^2,$$
$$n \geq 2.$$

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As differential operators in $K[\lambda, \mu][\partial]$, the operators $P - \lambda$ and $Q - \mu$ have no common nontrivial solution.

Let $f(\lambda, \mu)$ be the square free part of $\partial\mathrm{Res}(P - \lambda, Q - \mu)$. The algebraic curve

$$\Gamma := \{(\lambda, \mu) \in C^2 \mid f(\lambda, \mu) = 0\}$$

is known as the **spectral curve**.

Let $K(\Gamma)$ be the fraction field of the domain $\frac{K[\lambda,\mu]}{(f(\lambda,\mu))}$. As elements of $K(\Gamma)[\partial]$, the differential operators $P - \lambda$, $Q - \mu$ have a common non constant factor.

$$\mathcal{L}_r = \mathrm{gcrd}(P - \lambda, Q - \mu).$$

Subresultant

Given u_s and $\bar{c}^s$ such that $K(u_s, \bar{c}^s) = 0$, the spectral curve Γ_s is defined by

$$f_s = \partial\mathrm{Res}(L(u_s) - \lambda, A_{2s+1}(u_s, \bar{c}^s) - \mu)$$

.

The Sylvester matrix $S(L_s - \lambda, A_{2s+1} - \mu)$ is the coefficient matrix of

$$\{L_s - \lambda, \ldots, \partial^{2s}(L_s - \lambda), A_{2s+1} - \mu, \partial(A_{2s+1} - \mu)\}.$$

Subresultant

Given u_s and $\bar{c}^s$ such that $K(u_s, \bar{c}^s) = 0$, the spectral curve Γ_s is defined by

$$f_s = \partial \mathrm{Res}(L(u_s) - \lambda, A_{2s+1}(u_s, \bar{c}^s) - \mu)$$

.

Let S^1 be the $(2s+1) \times (2s+2)$ coefficient matrix of

$$\{L_s - \lambda, \ldots, \partial^{2s-1}(L_s - \lambda), A_{2s+1} - \mu\}.$$

$$S_0^1 := \mathrm{submatrix}(S^1, 1 \ldots 2s, [1 \ldots 2s, 2s+2]), \quad \phi_1 = \det(S_0^1)$$
$$S_1^1 := \mathrm{submatrix}(S^1, 1 \ldots 2s, [1 \ldots 2s, 2s+1]), \quad \phi_2 = \det(S_1^1)$$

The **subresultant** is $\phi_1 + \phi_2 \partial$

$$\mathrm{gcrd}_{K(\Gamma_s)}(L_s - \lambda, A_{2s+1} - \mu) = \partial - \phi = \partial + \frac{\phi_1}{\phi_2}$$

Subresultant

Given u_s and $\bar{c}^s$ such that $K(u_s, \bar{c}^s) = 0$, the spectral curve Γ_s is defined by

$$f_s = \partial\mathrm{Res}(L(u_s) - \lambda, A_{2s+1}(u_s, \bar{c}^s) - \mu)$$

.

Let S^1 be the $(2s+1) \times (2s+2)$ coefficient matrix of

$$\{L_s - \lambda, \ldots, \partial^{2s-1}(L_s - \lambda), A_{2s+1} - \mu\}.$$

$$S^1_0 := \mathrm{submatrix}(S^1, 1 \ldots 2s, [1 \ldots 2s, 2s+2]), \quad \phi_1 = \det(S^1_0)$$
$$S^1_1 := \mathrm{submatrix}(S^1, 1 \ldots 2s, [1 \ldots 2s, 2s+1]), \quad \phi_2 = \det(S^1_1)$$

The **subresultant** is $\phi_1 + \phi_2\partial$

$$\mathrm{gcrd}_{K(\Gamma_s)}(L_s - \lambda, A_{2s+1} - \mu) = \partial - \phi = \partial + \frac{\phi_1}{\phi_2}$$

Factorization Algorithm

- Given $s \in \mathbb{N}$ and a potential $u_s(x)$ (solution of KdV_s).
- Return a factor $\partial - \phi(\tau, x)$ of $L_s - \lambda = -\partial^2 + u_s - \lambda$ in $K(\Gamma_s)$.

1. Obtain the vector of constants $\bar{c}^s$ so that $A_{2n+1}(u_s, \bar{c}^s)$ commutes with L_s.
2. Compute $f_s := \partial\mathrm{Res}(L_s - \lambda, A_{2n+1} - \mu)$, the defining polynomial of Γ_s.
3. Check if Γ_s admits a global parametrization (for example, if it is a rational curve). If not return No global factorization obtained.
4. Compute a (rational) parametrization $\mathcal{P}(\tau) = (\mathcal{P}_1(\tau), \mathcal{P}_2(\tau))$ of Γ_s.
5. Return the monic greatest common right divisor $\partial - \phi(\tau, x)$ of $L_s - \mathcal{P}_1(\tau)$ and $A_{2n+1} - \mathcal{P}_2(\tau)$.

Examples: Rational Potentials $u_s(x) = \frac{s(s+1)}{x^2}$

Factorization of $L - \lambda = -\partial^2 + u(x) - \lambda = (-\partial - \phi)(\partial - \phi)$

ϕ verifies the Ricatti $\phi_x + \phi^2 = u_s - \lambda$.

s	$f(\lambda,\mu)$	$\mathcal{P}(\tau)$	$-\phi(\tau, x)$
1	$-\mu^2 - \lambda^3$	$[-\tau^2, -\tau^3]$	$-\frac{\tau^2x^2 - x\tau + 1}{(x\tau - 1)x}$
2	$-\mu^2 - \lambda^5$	$[-\tau^2, -\tau^5]$	$\frac{x^3\tau^3 + 3\tau^2x^2 + 6x\tau + 6}{(\tau^2x^2 + 3x\tau + 3)x}$
3	$-\mu^2 - \lambda^7$	$[-\tau^2, \tau^7]$	$\frac{\tau^4x^4 + 6x^3\tau^3 + 21\tau^2x^2 + 45x\tau + 45}{(x^3\tau^3 + 6\tau^2x^2 + 15x\tau + 15)x}$
4	$-\mu^2 - \lambda^9$	$[-\tau^2, -\tau^9]$	$\frac{x^5\tau^5 + 10\tau^4x^4 + 55x^3\tau^3 + 195\tau^2x^2 + 420x\tau + 420}{(\tau^4x^4 + 10x^3\tau^3 + 45\tau^2x^2 + 105x\tau + 105)x}$

Examples: Rosen-Morse Potentials $u(x) = \frac{-1(1+1)}{\cosh^2(x)}$

$$f = -\mu^2 - \lambda(\lambda+1)^2$$

$$\mathcal{P}(\tau) = \left[-\frac{\tau^2}{\tau^2 + 2\,\tau + 1}, \frac{\tau\,(2\,\tau+1)}{3\,\tau^2 + 3\,\tau + \tau^3 + 1}\right]$$

Factorization of $L - \lambda = -\partial^2 + u(x) - \lambda = (-\partial - \phi)(\partial - \phi)$ with

$$\phi(\tau, x) = \frac{-\phi_1(\tau, x)}{\phi_2(\tau, x)}$$

$$\begin{aligned}\phi_1 = & -\sinh(x)\,\tau^2\cosh(x) - \cosh(x)\,\tau\,\sinh(x) - 2\,\tau + \tau^2\,(\cosh(x))^2 \\ & - \tau^2 - 1\end{aligned}$$

$$\phi_2 = (1+\tau)\,\cosh(x)\,(\cosh(x)\,\tau - \sinh(x)\,\tau - \sinh(x))$$

Examples: Rosen-Morse Potentials $u(x) = \frac{-2(2+1)}{\cosh^2(x)}$

$$f = -\mu^2 - \lambda\,(\lambda+4)^2\,(\lambda+1)^2\,,\ \ \mathcal{P}(\tau) = [\mathcal{P}_1(\tau), \mathcal{P}_2(\tau)]$$

$$\mathcal{P}(\tau) = [\mathcal{P}_1(\tau), \mathcal{P}_2(\tau)]$$

$$\mathcal{P}_1 = -\frac{\tau^2}{256\,\tau^2 + 32\,\tau + 1}$$

$$\mathcal{P}_2 = -\frac{\tau\ \left(4 + 256\,\tau + 6139\,\tau^2 + 65376\,\tau^3 + 260865\,\tau^4\right)}{80\,\tau + 2560\,\tau^2 + 40960\,\tau^3 + 327680\,\tau^4 + 1048576\,\tau^5 + 1}$$

Factorization of $L-\lambda = -\partial^2 + u(x) - \lambda = (-\partial - \phi)(\partial - \phi)$ with

$$\phi(\tau, x) = \frac{-\phi_1(\tau, x)}{\phi_2(\tau, x)}$$

$$\begin{aligned}
\phi_1 = & -288\sinh(x)\,\tau - 4608\sinh(x)\,\tau^2 - 24576\sinh(x)\,\tau^3 \\
& + 3\,\tau^2(\cosh(x))^2\sinh(x) + 48\sinh(x)(\cosh(x))^2\tau^3 - 6\sinh(x) \\
& + 513\,(\cosh(x))^3\tau^3 - 1536\cosh(x)\,\tau^3 + 64\,\tau^2(\cosh(x))^3 \\
& - 192\cosh(x)\,\tau^2 + 2\,(\cosh(x))^3\tau - 6\cosh(x)\,\tau \\
\phi_2 = & (16\,\tau + 1)\cosh(x)\Big(513\,\tau^2(\cosh(x))^2 + 64\,(\cosh(x))^2\tau \\
& + 2\,(\cosh(x))^2 + 48\sinh(x)\,\tau^2\cosh(x) + 3\cosh(x)\,\tau\sinh(x) \\
& - 768\,\tau^2 - 96\,\tau - 3
\end{aligned}$$

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