Canonical Representation of Constructible Sets

Antonio Montes / Josep M. Brunat

Universitat Politècnica de Catalunya

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Presentation

- Full paper: Josep M. Brunat, Antonio Montes.
  Computing the Canonical Representation of Constructible Sets.
  Math. Comput. Sci., 10:1 (2016) 165–178.
- Existence: J. O'Halloran, M. Schilmoeller.
  Gröbner bases for Constructible Sets.
  Comm. Algebra 30:11 (2002) 5479–5483.
- We develop Algorithms for effective computation.
- Can be used in many applications, particularly in the Gröbner Cover and loci computation.

Definitions

Ring: $R = K[u_1, \dots, u_n] = K[\mathbf{u}]$.
Computable field: $K = \mathbb{Q}$.
Use: $\mathbb{Q}$-Zariski topology of $\mathbb{C}^n$
Variety: $\mathbb{V}(P) = \{\mathbf{u} \in \mathbb{C}^n : g(\mathbf{u}) = 0, \text{ for all } g \in P\}$

Definition (Locally closed set)

A locally closed set is a difference of varieties,

$$S = \mathbb{V}(P) \setminus \mathbb{V}(Q)$$

Definition (Constructible set)

A Constructible set is the union of locally closed sets.

$$C = \bigcup_{i=1}^{s} V(P_i) \setminus \mathbb{V}(Q_i)$$

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Canonical representations of locally closed sets

Definition (Equivalent defintions of locally closed set)

(a) *The set S is locally closed;*

(b) *S is the intersection of an open and a closed set;*

(c) *the set S is the difference of two closed sets;*

(d) *the set S is open in the closure $\overline{S}$ of S;*

(e) *the set $\overline{S} \setminus S$ is closed.*

Canonical C-representation of a locally closed set

Variety:	$\mathbb{V}(P) = \{\mathbf{u} \in \mathbb{C}^n : g(\mathbf{u}) = 0,$ for all $g \in P\}$
Ideal of variety.	$\mathfrak{a} = \mathrm{RAD}(\langle P \rangle) = \mathbb{I}(\mathbb{V}(P))$ is canonical
	$\mathbb{V}(P) = \mathbb{V}(\langle P \rangle) = \mathbb{V}(\mathfrak{a})$

Given $S = \mathbb{V}(P) \setminus \mathbb{V}(Q)$ there exist two radical ideals $(\mathfrak{a}, \mathfrak{b})$ canonically associated to S such that

$$\begin{array}{ll} \mathfrak{a} = \mathbb{I}(\overline{S}) & \mathfrak{b} = \mathbb{I}(\overline{S} \setminus S) \\ \overline{S} = \mathbf{V}(\mathfrak{a}) & \overline{S} \setminus S = \mathbf{V}(\mathfrak{b}). \end{array}$$

$$S = \overline{S} \setminus (\overline{S} \setminus S) = \mathbf{V}(\mathfrak{a}) \setminus \mathbf{V}(\mathfrak{b}). \tag{1}$$

Taking into account the one-to-one correspondence between radical ideals and varieties, the ideals $\mathfrak{a} = \mathbf{I}(\overline{S})$ and $\mathfrak{b} = \mathbf{I}\left(\overline{S} \setminus S\right)$ are uniquely determined by S. The pair $\mathrm{CREP}(S) = [\mathfrak{a}, \mathfrak{b}]$ is called the *C-canonical representation* of the locally closed set S.
$\mathfrak{a}$ is the *top* and $\mathfrak{b}$ is the *hole*.

Canonical C-representation of a locally closed set

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Canonical C-representation of a locally closed set

Algorithm

Let $S = \mathbf{V}(P) \setminus \mathbf{V}(Q)$ be a locally closed set given by two ideals P and Q, and let $\{\mathfrak{p}'_1, \ldots, \mathfrak{p}'_s\}$ be the prime decomposition of P. Consider the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ of ideals $\mathfrak{p}'_i$ such that $\mathbf{V}(\mathfrak{p}'_i) \not\subseteq \mathbf{V}(Q)$. Then, the C-representation $[\mathfrak{a}, \mathfrak{b}]$ of S is

- $\mathfrak{a} = \bigcap_{i=1}^{r} \mathfrak{p}_i$,
- $\mathfrak{b} = \mathrm{RAD}(\mathfrak{a} + Q)$,

Proposition

- *if S is non-empty then $\mathfrak{a} \subset \mathfrak{b}$,*
- $\dim(\mathbb{V}(\mathfrak{b})) < \dim(\mathbb{V}(\mathfrak{a}))$.

Canonical P-representation of a locally closed set

We can further decompose $\textsc{CRep}(S) = [\mathfrak{a}, \mathfrak{b}]$ and obtain another representation of S.

Let $\{\mathfrak{p}_i : i \in [r]\}$ be the prime decomposition of $\mathfrak{a}$ and for $i \in [r]$ let $\{\mathfrak{p}_{ij} : j \in [r_i]\}$ be the prime decomposition of $\mathfrak{p}_i + \mathfrak{b}$. The set

$$\textsc{PRep}(S) = \{[\mathfrak{p}_i, \{\mathfrak{p}_{ij} : j \in [r_i]\}] : i \in [r]\} \tag{2}$$

is called the *P-representation* of S. Note that it only depends on S. Each $[\mathfrak{p}_i, \{\mathfrak{p}_{ij} : j \in [r_i]\}]$ is called a *component* of S, from which $\mathbf{V}(\mathfrak{p}_i)$ (or $\mathfrak{p}_i$) is the *top* and $\mathbf{V}(\mathfrak{p}_{ij})$ (or $\mathfrak{p}_{ij}$) with $j \in [r_i]$ its *holes*.
Moreover

$$\dim(\mathbb{V}(\mathfrak{p}_{ij})) < \dim(\mathbb{V}(\mathfrak{p}_i)) \text{ for all } i, j.$$

Canonical P-representation of a locally closed set

Proposition

Let $[\mathfrak{a},\mathfrak{b}]$ *and* $\{[\mathfrak{p}_i,\{\mathfrak{p}_{ij}: j\in[r_i]\}]: i\in[r]\}$ *be repectively the C-representation and P-representation of a locally closed set* S. *Then*

(i) *If* $S\neq\emptyset$*, then* $\mathfrak{p}_i\subset\mathfrak{p}_{ij}$*, for all* $i\in[r]$ *and* $j\in[r_i]$*;*

(ii) $\mathfrak{a}=\bigcap_{i=1}^{r}\mathfrak{p}_i,$

(iii) $\mathfrak{b}=\bigcap_{i=1}^{r}\bigcap_{j=1}^{r_i}\mathfrak{p}_{ij},$

(iv) $S=\bigcup_{i=1}^{r}\left(\mathbf{V}(\mathfrak{p}_i)\setminus\left(\bigcup_{j=1}^{r_i}\mathbf{V}(\mathfrak{p}_{ij})\right)\right).$

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Constructible sets

Definition (Constructible set)

A constructible set is an finite union of locally closed sets:

$$C = \bigcup_{i=1}^{s} \mathbb{V}(P_i) \setminus \mathbb{V}(Q_i).$$

Remark

(i) *If S_1 and S_2 are constructible, then $S_1 \cup S_2$, $S_1 \cap S_2$ and S_1^c are constructibles.*

(ii) *The set of constructible subsets of $\mathbb{C}^m$ is the minimal Boolean Algebra containing the locally closed subsets of $\mathbb{C}^m$.*

Canonical structure of constructible sets

Let $\mathcal{C}$ be the set of constructible subsets of $\mathbb{C}^m$ and
$\mathcal{L}$ the set of locally closed subsets of $\mathbb{C}^m$.

Definition

Define the maps:

$$\begin{array}{rcl} \mathbf{C}: \mathcal{C} & \to & \mathcal{C} \\ S & \mapsto & \mathbf{C}(S) = \overline{S} \setminus S \end{array} \qquad \begin{array}{rcl} \mathbf{L}: \mathcal{C} & \to & \mathcal{L} \\ S & \mapsto & \mathbf{L}(S) = \overline{S} \setminus \overline{C} \end{array}$$

- $\mathbf{L}(S) \subseteq S$ is the maximum locally closed set included in S.
- $\mathbf{L}(S)$ ist the FISTLEVEL(S).

Properties

Proposition

Let $S \neq \emptyset$ be a constructible set, $C = \mathbf{C}(S)$, $L = \mathbf{L}(S)$. Then,

(i) $\mathfrak{a} = \mathbf{I}(S)$ *and* $\mathfrak{b} = \mathbf{I}(C)$.

(ii) $L = \mathbb{V}(\mathfrak{a}) \setminus \mathbb{V}(\mathfrak{b})$;

(iii) $[\mathfrak{a}, \mathfrak{b}] = [\mathbf{I}(S), \mathbf{I}(C)] = [\mathbf{I}(\overline{S}), \mathbf{I}(\overline{C})]$ *is the C-representation of L.*

(iv) $\overline{C} \subset \overline{S}$;

(v) $\overline{S} = \overline{L}$;

(vi) $\dim C < \dim S$.

Computation of the complement

Proposition

Let $S = S_1 \cup \cdots \cup S_r$ *be a constructible set with each* S_i *locally closed. For* $i \in [r]$ *let* $\textsc{Crep}(S_i) = [\mathfrak{a}_i, \mathfrak{b}_i]$, $V_i = \mathbf{V}(\mathfrak{a}_i)$ *and* $W_i = \mathbf{V}(\mathfrak{b}_i)$. *Then,*

$$\begin{aligned} C &= \overline{S} \setminus S \\ &= \bigcup_{T \subset [r]} \left(\left(\bigcap_{j \in T} V_j^c \right) \cap \left(\bigcap_{j \notin T} W_j \right) \right) \\ &= \bigcup_{T \subset [r]} \left(\left(\bigcap_{j \notin T} W_j \right) \setminus \left(\bigcup_{j \in T} V_j \right) \right). \end{aligned}$$

Canonical seqüences

Definition

$$A_1 = S, \quad A_{i+1} = \mathbf{C}(A_i) = \overline{A_i} \setminus A_i$$

$S = A_1, A_2, \ldots, A_k, A_{k+1} = \emptyset$	constructibles
$\overline{S} = \overline{A_1} \supset \overline{A_2} \supset \cdots \supset \overline{A_{k+1}} = \emptyset,$	closed
$\dim(A_1) > \dim(A_2) > \ \ldots \ > \dim(A_{k+1}) = -1,$	
$L_1 = \overline{A_1} \setminus \overline{A_2},\ L_2 = \overline{A_2} \setminus \overline{A_3}\ , \ldots \ , L_k = \overline{A_k} \setminus \overline{A_{k+1}},$	locally closed

$$S = L_1 \uplus L_3 \uplus L_{2\ell\pm1}$$
$$C = L_2 \uplus L_4 \uplus L_{2\ell}$$

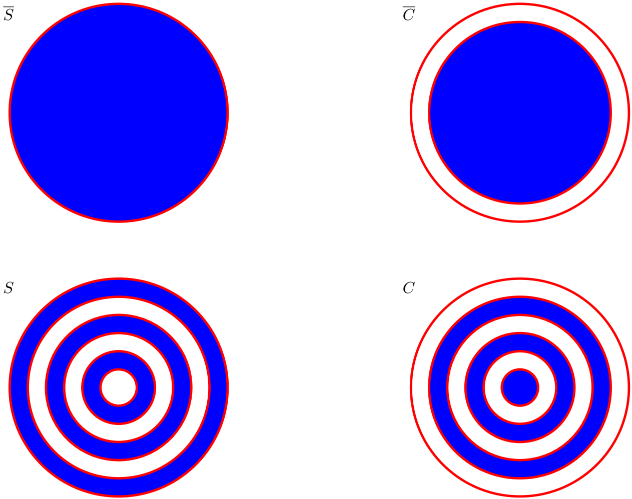

Figure: Scheme of the canonical structure of a constructible set

Algorithm FIRSTLEVEL

$[\mathfrak{a}, C] \leftarrow$ **FirstLevel**(A)
Input: $A = \{[\mathfrak{a}_1, \mathfrak{b}_1], \dots, [\mathfrak{a}_r, \mathfrak{b}_r]\}$
a set of CREP's of the segments defining a constructible set A
Output: $\mathfrak{a}$ (the closure of A) and
C (a set of CREP's of the segments defining $\mathbf{C}(A)$)
begin
$\mathfrak{a} = \bigcap_{i=1}^{r} \mathfrak{a}_i$; $\mathfrak{p} = \langle 0 \rangle$; $\mathfrak{q} = \langle 1 \rangle$; $C = \emptyset$;
for all $T \subset [r]$ **do**
for $j \in [r]$ **do**
if $j \in T$ **then** $\mathfrak{p} = \mathfrak{p} + \mathfrak{b}_j$ **else** $\mathfrak{q} = \mathfrak{q} \cap \mathfrak{a}_j$ **end if**
end for
$[\mathfrak{a}, \mathfrak{b}] = $ CREP$(\mathfrak{p}, \mathfrak{q})$
$C = $ APPEND$([\mathfrak{a}, \mathfrak{b}]$ to $C)$
end do
$C = $ SIMPLIFYUNION(C) # facility for reducing terms
return $([\mathfrak{a}, C])$
end

Algorithm ConsLevels

$L \leftarrow$ **ConsLevels**(A)
`Input`: $A = \{[\mathfrak{a}_1, \mathfrak{b}_1], \dots, [\mathfrak{a}_r, \mathfrak{b}_r]\}$
  a set of CRep's of the segments of a constructible set
`Output`: $L = [L_1, L_3, L_{2\ell\pm1}]$
  the set of CRep's of the canonical levels of S.
begin
  $L = \emptyset$; $\ell = 0$; $A' = A$;   # $\ell = \mathit{level}$
  while $A' \neq \emptyset$ **do**
    $\ell = \ell + 1$; $[\mathfrak{b}, C] =$ FirstLevel(A'); $A' = C$
    if $\ell \mod 2 = 1$ **then**
      $\mathfrak{a} = \mathfrak{b}$; **if** $C = \emptyset$ **then** $L =$ Append$([\mathfrak{a}, \langle 1 \rangle]$ to $L)$ **end if**
    else
      $L =$ Append$([\mathfrak{a}, \mathfrak{b}]$ to $L)$
    end if
  end while
  return(L)
end

Example of PREP and CREP

Use algorithms CREP and PREP with the following locally closed set:

1. $S_2 = \mathbb{V}(xyz) \setminus \mathbb{V}(x(z-1))$.
   Applying algorithms PREP and CREP the results is:

$$\begin{aligned}\text{PREP}(S_2) =& \{[\langle z\rangle, \{\langle z, x\rangle\}], [\langle y\rangle, \{\langle z-1, y\rangle, \langle y, x\rangle\}]\} \\ \text{CREP}(S_2) =& [\langle yz\rangle, \langle yz, xz-x, xy\rangle]\end{aligned}$$

The algorithm PTOCREP applied to PREP(S) returns the corresponding CREP(S) as expected.

Example 3. Simple constructible set

Consider the following locally closed sets:

$$S_1 = \mathbf{V}(x^2 + y^2 + z^2 - 1) \setminus \mathbf{V}(z, x^2 + y^2 - 1),$$
$$S_2 = \mathbf{V}(y, x^2 + z^2 - 1) \setminus \mathbf{V}(z(z+1), y, x + z + 1),$$
$$S_3 = \mathbf{V}(x) \setminus \mathbf{V}(5z - 4, 5y - 3, x).$$

Applying ConsLevels to them the result is the following two levels for the canonical representation of S

$$S = \left(\mathbf{V}(x(x^2 + y^2 + z^2 - 1)) \setminus \mathbf{V}(z, x^2 + y^2 - 1)\right) \uplus \mathbf{V}(z, x + y^2 - 1, xy, x^2 - x).$$

and expressed in Prep:

$$\begin{aligned} S =& (\mathbb{V}(x^2 + y^2 + z^2 - 1) \setminus \mathbb{V}(z, x^2 + y^2 - 1)) \\ & \cup (\mathbb{V}(x) \setminus (\mathbb{V}(x, y - 1, z)) \cup \mathbb{V}(x, y + 1, z)) \\ & \uplus (\mathbb{V}(x + 1, y, z) \cup \mathbb{V}(x, y + 1, z) \cup \mathbb{V}(x, y - 1, z)) \end{aligned}$$

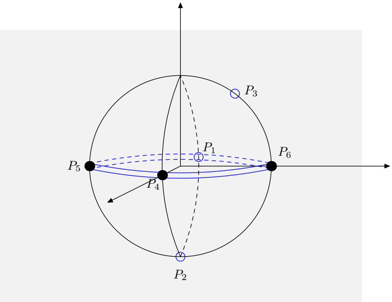

Figure: Constructible set S of Example 3.

$$\begin{aligned} S =& (\mathbb{V}(x^2+y^2+z^2-1) \setminus \mathbb{V}(z, x^2+y^2-1)) \\ & \cup (\mathbb{V}(x) \setminus (\mathbb{V}(x, y-1, z)) \cup \mathbb{V}(x, y+1, z)) \\ & \uplus (\mathbb{V}(x+1, y, z) \cup \mathbb{V}(x, y+1, z) \cup \mathbb{V}(x, y-1, z)) \end{aligned}$$

Alternative approach to Gröbner Cover

- Gröbner Cover is the canonical discussion of parametric ideals.
- It is based on Wibmer's Theorem for homogeneous parametric ideals stating that the set of parameter points for which the reduced Gröbner Basis has fixed lpp's (leading set of power products) is parametric and as consequence it is locally closed.
- Starting from a CGS of the homogenized ideal, adding together all the segments with the same lpps the result is guaranteed to be locally closed and canonical.
- Alternatively, without homogenizing, if we add together using CONSLEVELS all the segments with the same lpp, we obtain also a canonical set of segments.
- We verify that the resulting segments are the same as for the standard Gröbner Cover, except for the segments with basis 1 that are summarized into less segments using the alternative method.

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