A numeric-symbolic algorithm for computing the Liouvillian solutions of linear differential equations and systems

Alberto Llorente & Jorge Mozo-Fernández

Universidad Isabel I & Universidad de Valladolid

XV Encuentro de Álgebra Computacional y Aplicaciones (EACA)

Logroño, June 22nd, 2016

Introduction

Abstract

We give an algorithm for deciding if an explicitable higher-order system of linear differential equations over the complex rational functions, given symbolically, admits non-null Liouvillian solutions, computing one in the positive case, by numeric-symbolic methods in the sense of J. van der Hoeven.

This brings up the following questions I will answer in the following slides.

- What differential equations?
- What kind of solutions?
- By what methods?

Abstract

We give an algorithm for deciding if an explicitable higher-order system of linear differential equations over the complex rational functions, given symbolically, admits non-null Liouvillian solutions, computing one in the positive case, by numeric-symbolic methods in the sense of J. van der Hoeven.

This brings up the following questions I will answer in the following slides.

- What differential equations?
- What kind of solutions?
- By what methods?

Abstract

We give an algorithm for deciding if an explicitable higher-order system of linear differential equations over the complex rational functions, given symbolically, admits non-null Liouvillian solutions, computing one in the positive case, by numeric-symbolic methods in the sense of J. van der Hoeven.

This brings up the following questions I will answer in the following slides.

- What differential equations?
- What kind of solutions?
- By what methods?

Introduction

Abstract

We give an algorithm for deciding if an explicitable higher-order system of linear differential equations over the complex rational functions, given symbolically, admits non-null Liouvillian solutions, computing one in the positive case, by numeric-symbolic methods in the sense of J. van der Hoeven.

This brings up the following questions I will answer in the following slides.

- What differential equations?
- What kind of solutions?
- By what methods?

Abstract

We give an algorithm for deciding if an explicitable higher-order system of linear differential equations over the complex rational functions, given symbolically, admits non-null Liouvillian solutions, computing one in the positive case, by numeric-symbolic methods in the sense of J. van der Hoeven.

This brings up the following questions I will answer in the following slides.

- What differential equations?
- What kind of solutions?
- By what methods?

What differential equations?

We work with explicitable higher-order systems of linear homogeneous differential equations over the complex rational functions.

Explicitable higher-order systems of linear differential equations

A higher-order system of linear homogeneous differential equations

$$\mathtt{A}_r(x)\mathbf{y}^{(r)} + \mathtt{A}_{r-1}(x)\mathbf{y}^{(r-1)} + \cdots + \mathtt{A}_0(x)\mathbf{y} = \mathbf{0},$$

with $\mathtt{A}_0(x), \mathtt{A}_1(x) \ldots, \mathtt{A}_r(x)$ matrices $n \times n$, is explicitable if $\det \mathtt{A}_r(x) \not\equiv 0$.

"Explicitable" because they can be made explicit in $\mathbf{y}^{(r)}$.

This deals with both scalar linear differential equations (for $n = 1$) and first-order linear systems of differential equations (for $r = 1$) at once.

The matrices $\mathtt{A}_0(x), \mathtt{A}_1(x) \ldots, \mathtt{A}_r(x)$ consist of complex rational functions symbolically given.

What differential equations?

We work with explicitable higher-order systems of linear homogeneous differential equations over the complex rational functions.

Explicitable higher-order systems of linear differential equations

A higher-order system of linear homogeneous differential equations

$$\mathtt{A}_r(x)\mathbf{y}^{(r)} + \mathtt{A}_{r-1}(x)\mathbf{y}^{(r-1)} + \cdots + \mathtt{A}_0(x)\mathbf{y} = \mathbf{0},$$

with $\mathtt{A}_0(x), \mathtt{A}_1(x) \ldots, \mathtt{A}_r(x)$ matrices $n \times n$, is explicitable if $\det \mathtt{A}_r(x) \not\equiv 0$.

"Explicitable" because they can be made explicit in $\mathbf{y}^{(r)}$.

This deals with both scalar linear differential equations (for $n = 1$) and first-order linear systems of differential equations (for $r = 1$) at once.

The matrices $\mathtt{A}_0(x), \mathtt{A}_1(x) \ldots, \mathtt{A}_r(x)$ consist of complex rational functions symbolically given.

What differential equations?

We work with explicitable higher-order systems of linear homogeneous differential equations over the complex rational functions.

Explicitable higher-order systems of linear differential equations

A higher-order system of linear homogeneous differential equations

$$\mathtt{A}_r(x)\mathbf{y}^{(r)} + \mathtt{A}_{r-1}(x)\mathbf{y}^{(r-1)} + \cdots + \mathtt{A}_0(x)\mathbf{y} = \mathbf{0},$$

with $\mathtt{A}_0(x), \mathtt{A}_1(x) \ldots, \mathtt{A}_r(x)$ matrices $n \times n$, is explicitable if $\det \mathtt{A}_r(x) \not\equiv 0$.

"Explicitable" because they can be made explicit in $\mathbf{y}^{(r)}$.

This deals with both scalar linear differential equations (for $n = 1$) and first-order linear systems of differential equations (for $r = 1$) at once.

The matrices $\mathtt{A}_0(x), \mathtt{A}_1(x) \ldots, \mathtt{A}_r(x)$ consist of complex rational functions symbolically given.

What differential equations?

We work with explicitable higher-order systems of linear homogeneous differential equations over the complex rational functions.

Explicitable higher-order systems of linear differential equations

A higher-order system of linear homogeneous differential equations

$$\mathtt{A}_r(x)\mathbf{y}^{(r)} + \mathtt{A}_{r-1}(x)\mathbf{y}^{(r-1)} + \cdots + \mathtt{A}_0(x)\mathbf{y} = \mathbf{0},$$

with $\mathtt{A}_0(x), \mathtt{A}_1(x) \ldots, \mathtt{A}_r(x)$ matrices $n \times n$, is explicitable if $\det \mathtt{A}_r(x) \not\equiv 0$.

"Explicitable" because they can be made explicit in $\mathbf{y}^{(r)}$.

This deals with both scalar linear differential equations (for $n = 1$) and first-order linear systems of differential equations (for $r = 1$) at once.

The matrices $\mathtt{A}_0(x), \mathtt{A}_1(x) \ldots, \mathtt{A}_r(x)$ consist of complex rational functions symbolically given.

What kind of solutions?

Liouvillian solutions

A function is Liouvillian if it is built up by the use of

- rational operations
- derivatives
- exponentials
- quadratures
- solution of polynomials

over the rational functions.

By what methods?

I speak of numeric-symbolic methods in the sense of J. van der Hoeven's *Around the numeric-symbolic computation of differential Galois groups* (JSC 2007), where he proposes an algorithm for decomposing differential operators.

Global structure

- symbolic preprocess
- numerical computation at a given precision
  - it may give false positives, but not false negatives
  - the result is correct for precision fine enough
- symbolic reconstruction of the candidate solution and symbolic testing
  - if the test successes, the candidate solution is true
  - if the test fails, we repeat the numerical computation at finer precision

These properties of the numerical step grant that the procedure terminates

- if the answer is positive, with a true solution at the symbolic step
- if the answer is negative, with a true negative at the numerical step

By what methods?

I speak of numeric-symbolic methods in the sense of J. van der Hoeven's *Around the numeric-symbolic computation of differential Galois groups* (JSC 2007), where he proposes an algorithm for decomposing differential operators.

Global structure

- symbolic preprocess
- numerical computation at a given precision
  - it may give false positives, but not false negatives
  - the result is correct for precision fine enough
- symbolic reconstruction of the candidate solution and symbolic testing
  - if the test successes, the candidate solution is true
  - if the test fails, we repeat the numerical computation at finer precision

These properties of the numerical step grant that the procedure terminates

- if the answer is positive, with a true solution at the symbolic step
- if the answer is negative, with a true negative at the numerical step

By what methods?

I speak of numeric-symbolic methods in the sense of J. van der Hoeven's *Around the numeric-symbolic computation of differential Galois groups* (JSC 2007), where he proposes an algorithm for decomposing differential operators.

Global structure

- symbolic preprocess
- numerical computation at a given precision
  - it may give false positives, but not false negatives
  - the result is correct for precision fine enough
- symbolic reconstruction of the candidate solution and symbolic testing
  - if the test successes, the candidate solution is true
  - if the test fails, we repeat the numerical computation at finer precision

These properties of the numerical step grant that the procedure terminates

- if the answer is positive, with a true solution at the symbolic step
- if the answer is negative, with a true negative at the numerical step

The solutions we look for

Applying the Cyclic Vector Lemma, we can extend the following theorem of Singer to systems.

Singer's theorem

There exists an arithmetic function $I : \mathbb{N} \to \mathbb{N}$ such that, if a scalar differential equations of order r has a non-null Liouvillian solution, then there exists a solution $y \neq 0$ such that y'/y is algebraic of degree $I(r)$ at most.

I called such solutions Singerian solutions.

Galoisian characterization

If G is the differential Galois group of an $n \times n$ system of order r, and G° its identity component, a solution $\mathbf{y} \neq \mathbf{0}$ is Singerian if and only if the line $\mathbb{C}\mathbf{y}$ is invariant by G° and its orbit by the action of G has length $I(nr)$ at most.

The solutions we look for

Applying the Cyclic Vector Lemma, we can extend the following theorem of Singer to systems.

Singer's theorem

There exists an arithmetic function $I : \mathbb{N} \to \mathbb{N}$ such that, if a scalar differential equations of order r has a non-null Liouvillian solution, then there exists a solution $y \neq 0$ such that y'/y is algebraic of degree $I(r)$ at most.

I called such solutions Singerian solutions.

Galoisian characterization

If G is the differential Galois group of an $n \times n$ system of order r, and G° its identity component, a solution $\mathbf{y} \neq \mathbf{0}$ is Singerian if and only if the line $\mathbb{C}\mathbf{y}$ is invariant by G° and its orbit by the action of G has length $I(nr)$ at most.

The solutions we look for

Form of the Singerian solutions

For any Singerian solution $\mathbf{y}$, there exist an integer $p > 0$, $\alpha \in \mathbb{C}$, a polynomial q with $q(0) = 0$ and convergent series $f_1, f_2, \ldots, f_n$ such that $\mathbf{y} = \exp\left(q(x^{-1/p})\right) x^{\alpha} \left(f_1(x^{1/p}), f_2(x^{1/p}), \ldots, f_n(x^{1/p})\right)^{\mathsf{T}}$.

Compared with Fabry-Hukuhara-Turrittin form, Singerian solutions are free of logarithms and divergence, so they present a special behavior under Ramis generators of the differential Galois group.

Ramis generators of the local Galois group

- monodromy
- Stokes automorphisms
- exponential torus

The solutions we look for

Form of the Singerian solutions

For any Singerian solution $\mathbf{y}$, there exist an integer $p > 0$, $\alpha \in \mathbb{C}$, a polynomial q with $q(0) = 0$ and convergent series $f_1, f_2, \ldots, f_n$ such that $\mathbf{y} = \exp\left(q(x^{-1/p})\right) x^{\alpha} \left(f_1(x^{1/p}), f_2(x^{1/p}), \ldots, f_n(x^{1/p})\right)^{\mathsf{T}}$.

Compared with Fabry-Hukuhara-Turrittin form, Singerian solutions are free of logarithms and divergence, so they present a special behavior under Ramis generators of the differential Galois group.

Ramis generators of the local Galois group

- monodromy
- Stokes automorphisms
- exponential torus

The solutions we look for

Form of the Singerian solutions

For any Singerian solution $\mathbf{y}$, there exist an integer $p > 0$, $\alpha \in \mathbb{C}$, a polynomial q with $q(0) = 0$ and convergent series $f_1, f_2, \ldots, f_n$ such that $\mathbf{y} = \exp\left(q(x^{-1/p})\right) x^{\alpha} \left(f_1(x^{1/p}), f_2(x^{1/p}), \ldots, f_n(x^{1/p})\right)^{T}$.

Compared with Fabry-Hukuhara-Turrittin form, Singerian solutions are free of logarithms and divergence, so they present a special behavior under Ramis generators of the differential Galois group.

Ramis generators of the local Galois group

- monodromy
- Stokes automorphisms (trivial)
- exponential torus

The solutions we look for

Form of the Singerian solutions

For any Singerian solution $\mathbf{y}$, there exist an integer $p > 0$, $\alpha \in \mathbb{C}$, a polynomial q with $q(0) = 0$ and convergent series $f_1, f_2, \ldots, f_n$ such that $\mathbf{y} = \exp\left(q(x^{-1/p})\right) x^{\alpha} \left(f_1(x^{1/p}), f_2(x^{1/p}), \ldots, f_n(x^{1/p})\right)^{\mathsf{T}}$.

Compared with Fabry-Hukuhara-Turrittin form, Singerian solutions are free of logarithms and divergence, so they present a special behavior under Ramis generators of the differential Galois group.

Ramis generators of the local Galois group

- monodromy
- Stokes automorphisms (trivial)
- exponential torus (scalar)

Computation of Ramis generators

- Centered at a singularity, we can compute symbolically the ramification index p and the admissible pairs (q, α) of the Fabry-Hukuhara-Turrittin formal solutions.
- For each pair (q, α), we can compute numerically the solutions in $\exp\left(q(x^{-1/p})\right) x^{\alpha}\, \mathbb{C}\{x^{1/p}\}$. We need only numerical analytic continuation and linear algebra.
- This structure (a union of vector subspaces whose sum is direct) contains the Singerian solutions, can be moved (by numerical analytic continuation) to a common point for all the singularities and (by numerical linear algebra) can be intersected getting another structure of this kind.
- By numerical analytic continuation we can compute the generators of the monodromy group.

Computation of Ramis generators

- Centered at a singularity, we can compute symbolically the ramification index p and the admissible pairs (q, α) of the Fabry-Hukuhara-Turrittin formal solutions.
- For each pair (q, α), we can compute numerically the solutions in $\exp\left(q(x^{-1/p})\right) x^{\alpha}\, \mathbb{C}\{x^{1/p}\}$. We need only numerical analytic continuation and linear algebra.
- This structure (a union of vector subspaces whose sum is direct) contains the Singerian solutions, can be moved (by numerical analytic continuation) to a common point for all the singularities and (by numerical linear algebra) can be intersected getting another structure of this kind.
- By numerical analytic continuation we can compute the generators of the monodromy group.

Computation of Ramis generators

- Centered at a singularity, we can compute symbolically the ramification index p and the admissible pairs (q, α) of the Fabry-Hukuhara-Turrittin formal solutions.
- For each pair (q, α), we can compute numerically the solutions in $\exp\left(q(x^{-1/p})\right) x^{\alpha}\,\mathbb{C}\{x^{1/p}\}$. We need only numerical analytic continuation and linear algebra.
- This structure (a union of vector subspaces whose sum is direct) contains the Singerian solutions, can be moved (by numerical analytic continuation) to a common point for all the singularities and (by numerical linear algebra) can be intersected getting another structure of this kind.
- By numerical analytic continuation we can compute the generators of the monodromy group.

Computation of Ramis generators

- Centered at a singularity, we can compute symbolically the ramification index p and the admissible pairs (q, α) of the Fabry-Hukuhara-Turrittin formal solutions.
- For each pair (q, α), we can compute numerically the solutions in $\exp\left(q(x^{-1/p})\right) x^{\alpha}\, \mathbb{C}\{x^{1/p}\}$. We need only numerical analytic continuation and linear algebra.
- This structure (a union of vector subspaces whose sum is direct) contains the Singerian solutions, can be moved (by numerical analytic continuation) to a common point for all the singularities and (by numerical linear algebra) can be intersected getting another structure of this kind.
- By numerical analytic continuation we can compute the generators of the monodromy group.

Effective numerics

For these computations of the Ramis generators, we will not use ordinary numerics, but effective numerics.

Effective complex numbers

A complex number a is effective if it is endowed with an algorithm $f : \mathbb{N}2^{\mathbb{Z}} \to (\mathbb{Z} + \mathrm{i}\mathbb{Z})2^{\mathbb{Z}}$ such that $|f(\varepsilon) - a| < \varepsilon$ for any $\varepsilon \in \mathbb{N}2^{\mathbb{Z}}$.

- The rational operations among effective complex numbers are effective, provided we have a lower bound of the denominator.
- J. van der Hoeven (1999) develops the numerical analytic continuation of the solutions of a linear differential equation with polynomial coefficients, which can be extended to explicitable systems.
- Numerical linear algebra is a bit problematic, as we cannot decide if a small number is zero, so we must resort to the following device.

Effective numerics

For these computations of the Ramis generators, we will not use ordinary numerics, but effective numerics.

Effective complex numbers

A complex number a is effective if it is endowed with an algorithm $f : \mathbb{N}2^{\mathbb{Z}} \to (\mathbb{Z} + \mathrm{i}\mathbb{Z})2^{\mathbb{Z}}$ such that $|f(\varepsilon) - a| < \varepsilon$ for any $\varepsilon \in \mathbb{N}2^{\mathbb{Z}}$.

- The rational operations among effective complex numbers are effective, provided we have a lower bound of the denominator.
- J. van der Hoeven (1999) develops the numerical analytic continuation of the solutions of a linear differential equation with polynomial coefficients, which can be extended to explicitable systems.
- Numerical linear algebra is a bit problematic, as we cannot decide if a small number is zero, so we must resort to the following device.

Effective numerics

For these computations of the Ramis generators, we will not use ordinary numerics, but effective numerics.

Effective complex numbers

A complex number a is effective if it is endowed with an algorithm $f : \mathbb{N}2^{\mathbb{Z}} \to (\mathbb{Z} + \mathrm{i}\mathbb{Z})2^{\mathbb{Z}}$ such that $|f(\varepsilon) - a| < \varepsilon$ for any $\varepsilon \in \mathbb{N}2^{\mathbb{Z}}$.

- The rational operations among effective complex numbers are effective, provided we have a lower bound of the denominator.
- J. van der Hoeven (1999) develops the numerical analytic continuation of the solutions of a linear differential equation with polynomial coefficients, which can be extended to explicitable systems.
- Numerical linear algebra is a bit problematic, as we cannot decide if a small number is zero, so we must resort to the following device.

Effective numerics

For these computations of the Ramis generators, we will not use ordinary numerics, but effective numerics.

Effective complex numbers

A complex number a is effective if it is endowed with an algorithm $f : \mathbb{N}2^{\mathbb{Z}} \to (\mathbb{Z} + \mathrm{i}\mathbb{Z})2^{\mathbb{Z}}$ such that $|f(\varepsilon) - a| < \varepsilon$ for any $\varepsilon \in \mathbb{N}2^{\mathbb{Z}}$.

- The rational operations among effective complex numbers are effective, provided we have a lower bound of the denominator.
- J. van der Hoeven (1999) develops the numerical analytic continuation of the solutions of a linear differential equation with polynomial coefficients, which can be extended to explicitable systems.
- Numerical linear algebra is a bit problematic, as we cannot decide if a small number is zero, so we must resort to the following device.

Effective numerics

- In order to deal with the fact that we cannot decide if a small number is zero, J. van der Hoeven (2007) introduces a global parameter tol $\in \mathbb{N}2^{\mathbb{Z}}$ such that, for an effective complex number (a, f), we deem $a = 0$ if $|f(\text{tol})| < \text{tol}$.
- The numerical version of Gaussian elimination may deem zero a non-zero row/column, and thus underestimate the rank, but it cannot overestimate the rank.
- Nevertheless, for tol small enough, this approximate zero-test works correctly.
- The algorithm of this work is devised in such a way that the space of Singerian solutions may be overestimated, but never underestimated.
- So, it satisfies the conditions to fit in a numeric-symbolic method as exposed at the beginning.

Effective numerics

- In order to deal with the fact that we cannot decide if a small number is zero, J. van der Hoeven (2007) introduces a global parameter tol $\in \mathbb{N}2^{\mathbb{Z}}$ such that, for an effective complex number (a, f), we deem $a = 0$ if $|f(\text{tol})| < \text{tol}$.
- The numerical version of Gaussian elimination may deem zero a non-zero row/column, and thus underestimate the rank, but it cannot overestimate the rank.
- Nevertheless, for tol small enough, this approximate zero-test works correctly.
- The algorithm of this work is devised in such a way that the space of Singerian solutions may be overestimated, but never underestimated.
- So, it satisfies the conditions to fit in a numeric-symbolic method as exposed at the beginning.

Effective numerics

- In order to deal with the fact that we cannot decide if a small number is zero, J. van der Hoeven (2007) introduces a global parameter tol $\in \mathbb{N}2^{\mathbb{Z}}$ such that, for an effective complex number (a, f), we deem $a = 0$ if $|f(\text{tol})| < \text{tol}$.
- The numerical version of Gaussian elimination may deem zero a non-zero row/column, and thus underestimate the rank, but it cannot overestimate the rank.
- Nevertheless, for tol small enough, this approximate zero-test works correctly.
- The algorithm of this work is devised in such a way that the space of Singerian solutions may be overestimated, but never underestimated.
- So, it satisfies the conditions to fit in a numeric-symbolic method as exposed at the beginning.

Effective numerics

- In order to deal with the fact that we cannot decide if a small number is zero, J. van der Hoeven (2007) introduces a global parameter tol $\in \mathbb{N}2^{\mathbb{Z}}$ such that, for an effective complex number (a, f), we deem $a = 0$ if $|f(\mathrm{tol})| < \mathrm{tol}$.
- The numerical version of Gaussian elimination may deem zero a non-zero row/column, and thus underestimate the rank, but it cannot overestimate the rank.
- Nevertheless, for tol small enough, this approximate zero-test works correctly.
- The algorithm of this work is devised in such a way that the space of Singerian solutions may be overestimated, but never underestimated.
- So, it satisfies the conditions to fit in a numeric-symbolic method as exposed at the beginning.

Effective numerics

- In order to deal with the fact that we cannot decide if a small number is zero, J. van der Hoeven (2007) introduces a global parameter $\text{tol} \in \mathbb{N}2^{\mathbb{Z}}$ such that, for an effective complex number (a, f), we deem $a = 0$ if $|f(\text{tol})| < \text{tol}$.
- The numerical version of Gaussian elimination may deem zero a non-zero row/column, and thus underestimate the rank, but it cannot overestimate the rank.
- Nevertheless, for tol small enough, this approximate zero-test works correctly.
- The algorithm of this work is devised in such a way that the space of Singerian solutions may be overestimated, but never underestimated.
- So, it satisfies the conditions to fit in a numeric-symbolic method as exposed at the beginning.

Eurymeric closure of the Galois group

- H. Derksen (2005) and J. van der Hoeven (2007) devised an algorithm for computing the algebraic group generated by a finite family of matrices, but it relies on some exact computations which would require infinite time in effective numerics.
- In order to avoid this impossibility, we define the eurymeric closure of a linear algebraic group, which can be computed in effective numerics up to testing if a number is zero or a root of unity.
- What we can effectively compute is a truncation of the eurymeric closure, which is enough for the computation of Singerian solutions.

Eurymeric closure of the Galois group

- H. Derksen (2005) and J. van der Hoeven (2007) devised an algorithm for computing the algebraic group generated by a finite family of matrices, but it relies on some exact computations which would require infinite time in effective numerics.
- In order to avoid this impossibility, we define the eurymeric closure of a linear algebraic group, which can be computed in effective numerics up to testing if a number is zero or a root of unity.
- What we can effectively compute is a truncation of the eurymeric closure, which is enough for the computation of Singerian solutions.

Eurymeric closure of the Galois group

- H. Derksen (2005) and J. van der Hoeven (2007) devised an algorithm for computing the algebraic group generated by a finite family of matrices, but it relies on some exact computations which would require infinite time in effective numerics.
- In order to avoid this impossibility, we define the eurymeric closure of a linear algebraic group, which can be computed in effective numerics up to testing if a number is zero or a root of unity.
- What we can effectively compute is a truncation of the eurymeric closure, which is enough for the computation of Singerian solutions.

Sketch of the global algorithm

1. Compute the truncation of the eurymeric group G generated by the Ramis generators, on a space containing all the Singerian solutions, as the Lie algebra $\mathfrak{g}$ and generators of G/G°.
2. Choose a common eigenvector of $\mathfrak{g}$ with the indications of my thesis and in such a determined way that the same Lie algebra yields always the same eigenvector.
   - If there is no common eigenvector, this proves that there is no non-null Liouvillian solution.
3. Construct the Darboux polynomial corresponding to the chosen eigenvector and reconstruct symbolically its coefficients.
4. Check if the reconstruction of the Darboux polynomial is an actual Darboux polynomial and if it satisfies the Brill equations.
   - In case of failure, we restart the numerical computation with finer precision, which will be explained later.

Sketch of the global algorithm

1. Compute the truncation of the eurymeric group G generated by the Ramis generators, on a space containing all the Singerian solutions, as the Lie algebra $\mathfrak{g}$ and generators of G/G°.
2. Choose a common eigenvector of $\mathfrak{g}$ with the indications of my thesis and in such a determined way that the same Lie algebra yields always the same eigenvector.
   - If there is no common eigenvector, this proves that there is no non-null Liouvillian solution.
3. Construct the Darboux polynomial corresponding to the chosen eigenvector and reconstruct symbolically its coefficients.
4. Check if the reconstruction of the Darboux polynomial is an actual Darboux polynomial and if it satisfies the Brill equations.
   - In case of failure, we restart the numerical computation with finer precision, which will be explained later.

Sketch of the global algorithm

1. Compute the truncation of the eurymeric group G generated by the Ramis generators, on a space containing all the Singerian solutions, as the Lie algebra $\mathfrak{g}$ and generators of G/G°.
2. Choose a common eigenvector of $\mathfrak{g}$ with the indications of my thesis and in such a determined way that the same Lie algebra yields always the same eigenvector.
   - If there is no common eigenvector, this proves that there is no non-null Liouvillian solution.
3. Construct the Darboux polynomial corresponding to the chosen eigenvector and reconstruct symbolically its coefficients.
4. Check if the reconstruction of the Darboux polynomial is an actual Darboux polynomial and if it satisfies the Brill equations.
   - In case of failure, we restart the numerical computation with finer precision, which will be explained later.

Sketch of the global algorithm

1. Compute the truncation of the eurymeric group G generated by the Ramis generators, on a space containing all the Singerian solutions, as the Lie algebra $\mathfrak{g}$ and generators of G/G°.
2. Choose a common eigenvector of $\mathfrak{g}$ with the indications of my thesis and in such a determined way that the same Lie algebra yields always the same eigenvector.
   - If there is no common eigenvector, this proves that there is no non-null Liouvillian solution.
3. Construct the Darboux polynomial corresponding to the chosen eigenvector and reconstruct symbolically its coefficients.
4. Check if the reconstruction of the Darboux polynomial is an actual Darboux polynomial and if it satisfies the Brill equations.
   - In case of failure, we restart the numerical computation with finer precision, which will be explained later.

Sketch of the global algorithm

1. Compute the truncation of the eurymeric group G generated by the Ramis generators, on a space containing all the Singerian solutions, as the Lie algebra $\mathfrak{g}$ and generators of G/G°.
2. Choose a common eigenvector of $\mathfrak{g}$ with the indications of my thesis and in such a determined way that the same Lie algebra yields always the same eigenvector.
   - If there is no common eigenvector, this proves that there is no non-null Liouvillian solution.
3. Construct the Darboux polynomial corresponding to the chosen eigenvector and reconstruct symbolically its coefficients.
4. Check if the reconstruction of the Darboux polynomial is an actual Darboux polynomial and if it satisfies the Brill equations.
   - In case of failure, we restart the numerical computation with finer precision, which will be explained later.

Sketch of the global algorithm

1. Compute the truncation of the eurymeric group G generated by the Ramis generators, on a space containing all the Singerian solutions, as the Lie algebra $\mathfrak{g}$ and generators of G/G°.
2. Choose a common eigenvector of $\mathfrak{g}$ with the indications of my thesis and in such a determined way that the same Lie algebra yields always the same eigenvector.
   - If there is no common eigenvector, this proves that there is no non-null Liouvillian solution.
3. Construct the Darboux polynomial corresponding to the chosen eigenvector and reconstruct symbolically its coefficients.
4. Check if the reconstruction of the Darboux polynomial is an actual Darboux polynomial and if it satisfies the Brill equations.
   - In case of failure, we restart the numerical computation with finer precision, which will be explained later.

Symbolic reconstruction

- Given a rational function $f(x)$ as an effective power series, we reconstruct symbolically the numerator and the denominator of $f(x)$ by means of Padé approximation.
- We proved that the constants in the final result are algebraic over the constants of the original data.
- In order to reconstruct the minimal polynomial of such constants, J. van der Hoeven proposes using the LLL algorithm, and I added HJLS and PSLQ to the repertoire, whose terminations proofs I had to refine for effective numerics.

Symbolic reconstruction

- Given a rational function $f(x)$ as an effective power series, we reconstruct symbolically the numerator and the denominator of $f(x)$ by means of Padé approximation.
- We proved that the constants in the final result are algebraic over the constants of the original data.
- In order to reconstruct the minimal polynomial of such constants, J. van der Hoeven proposes using the LLL algorithm, and I added HJLS and PSLQ to the repertoire, whose terminations proofs I had to refine for effective numerics.

Symbolic reconstruction

- Given a rational function $f(x)$ as an effective power series, we reconstruct symbolically the numerator and the denominator of $f(x)$ by means of Padé approximation.
- We proved that the constants in the final result are algebraic over the constants of the original data.
- In order to reconstruct the minimal polynomial of such constants, J. van der Hoeven proposes using the LLL algorithm, and I added HJLS and PSLQ to the repertoire, whose terminations proofs I had to refine for effective numerics.

Several steps of the algorithm require truncation parameters in order to keep the computations finite.

Giving a suitable meaning to "refining the precision" for all these truncation parameters together, we get the following result.

The main theorem of the work

The main algorithm of the work terminates with a non-zero Liouvillian solution, if such a solution exists, or with the statement that zero is the only Liouvillian solution if this is the case.

Several steps of the algorithm require truncation parameters in order to keep the computations finite.
Giving a suitable meaning to "refining the precision" for all these truncation parameters together, we get the following result.

The main theorem of the work

The main algorithm of the work terminates with a non-zero Liouvillian solution, if such a solution exists, or with the statement that zero is the only Liouvillian solution if this is the case.

Examples

- The simplest non-trivial examples are second-order equations or systems, studied by Kovacic (1979).
- An interesting higher-order example can be built up as direct sum of second-order ones.
- If we do not know the coordinates that split the system as direct sum, we cannot use the techniques designed for second-order equations.
- So, we shall review how our algorithm works with some significant second-order equations.

Examples

- The simplest non-trivial examples are second-order equations or systems, studied by Kovacic (1979).
- An interesting higher-order example can be built up as direct sum of second-order ones.
- If we do not know the coordinates that split the system as direct sum, we cannot use the techniques designed for second-order equations.
- So, we shall review how our algorithm works with some significant second-order equations.

Examples

- The simplest non-trivial examples are second-order equations or systems, studied by Kovacic (1979).
- An interesting higher-order example can be built up as direct sum of second-order ones.
- If we do not know the coordinates that split the system as direct sum, we cannot use the techniques designed for second-order equations.
- So, we shall review how our algorithm works with some significant second-order equations.

Examples

- The simplest non-trivial examples are second-order equations or systems, studied by Kovacic (1979).
- An interesting higher-order example can be built up as direct sum of second-order ones.
- If we do not know the coordinates that split the system as direct sum, we cannot use the techniques designed for second-order equations.
- So, we shall review how our algorithm works with some significant second-order equations.

Example: Euler equation

Example (Euler equation)

The differential equation,

$$x^3y'' + (x^2 + x)y' - y = 0,$$

has as a fundamental system of solutions

$$f = \exp\left(\frac{1}{x}\right) \quad \text{and} \quad \hat{g} = \sum_{n=0}^{\infty}(-1)^n n!\, x^{n+1}.$$

- This equation has two singularities (at the origin and at infinity) and hence one monodromy.
- The ramification index is 1, so the Singerian solutions must be invariant by the monodromy.
- If g is any sum of $\hat{g}$, we have that f is invariant by the monodromy and g is not.

Example: Euler equation

Example (Euler equation)

The differential equation,

$$x^3y'' + (x^2 + x)y' - y = 0,$$

has as a fundamental system of solutions

$$f = \exp\left(\frac{1}{x}\right) \quad \text{and} \quad \hat{g} = \sum_{n=0}^{\infty} (-1)^n n!\, x^{n+1}.$$

- This equation has two singularities (at the origin and at infinity) and hence one monodromy.
- The ramification index is 1, so the Singerian solutions must be invariant by the monodromy.
- If g is any sum of $\hat{g}$, we have that f is invariant by the monodromy and g is not.

Example: Euler equation

Example (Euler equation)

The differential equation,

$$x^3y'' + (x^2 + x)y' - y = 0,$$

has as a fundamental system of solutions

$$f = \exp\left(\frac{1}{x}\right) \quad \text{and} \quad \hat{g} = \sum_{n=0}^{\infty}(-1)^n n!\, x^{n+1}.$$

- This equation has two singularities (at the origin and at infinity) and hence one monodromy.
- The ramification index is 1, so the Singerian solutions must be invariant by the monodromy.
- If g is any sum of $\hat{g}$, we have that f is invariant by the monodromy and g is not.

Example: Euler equation

- The admissible pairs are $\{q(t) = t, \alpha = 0\}$ and $\{q(t) = 0, \alpha = 0\}$, yielding $\exp\big(q(x^{-1})\big)x^{\alpha} = f$ and $\exp\big(q(x^{-1})\big)x^{\alpha} = 1$ respectively.
- Then we check if $f^{-1}y$ or y annihilate $y \mapsto \int_{\circlearrowleft 0} y(z)\, z^k \, \mathrm{d}z$ for $k \geqslant 0$.
- All these integrals vanish for $f^{-1}f = 1$, but none of them does for f, so the only candidate space is $\mathbb{C}f$ with $q(t) = t$ and $\alpha = 0$.
- As $\mathbb{C}f$ is invariant by the Galoisian action, it is a candidate Singerian orbit.
- Then we reconstruct symbolically the Darboux polynomial, getting that the candidate Singerian solution is $\exp(1/x)$ and that it is indeed a solution.

Example: Euler equation

- The admissible pairs are $\{q(t) = t, \alpha = 0\}$ and $\{q(t) = 0, \alpha = 0\}$, yielding $\exp\left(q(x^{-1})\right)x^{\alpha} = f$ and $\exp\left(q(x^{-1})\right)x^{\alpha} = 1$ respectively.
- Then we check if $f^{-1}y$ or y annihilate $y \mapsto \int_{\circlearrowleft 0} y(z)\, z^k \,\mathrm{d}z$ for $k \geqslant 0$.
- All these integrals vanish for $f^{-1}f = 1$, but none of them does for f, so the only candidate space is $\mathbb{C}f$ with $q(t) = t$ and $\alpha = 0$.
- As $\mathbb{C}f$ is invariant by the Galoisian action, it is a candidate Singerian orbit.
- Then we reconstruct symbolically the Darboux polynomial, getting that the candidate Singerian solution is $\exp(1/x)$ and that it is indeed a solution.

Example: Euler equation

- The admissible pairs are $\{q(t) = t, \alpha = 0\}$ and $\{q(t) = 0, \alpha = 0\}$, yielding $\exp\big(q(x^{-1})\big)x^{\alpha} = f$ and $\exp\big(q(x^{-1})\big)x^{\alpha} = 1$ respectively.
- Then we check if $f^{-1}y$ or y annihilate $y \mapsto \int_{\circlearrowleft 0} y(z)\, z^k \,\mathrm{d}z$ for $k \geqslant 0$.
- All these integrals vanish for $f^{-1}f = 1$, but none of them does for f, so the only candidate space is $\mathbb{C}f$ with $q(t) = t$ and $\alpha = 0$.
- As $\mathbb{C}f$ is invariant by the Galoisian action, it is a candidate Singerian orbit.
- Then we reconstruct symbolically the Darboux polynomial, getting that the candidate Singerian solution is $\exp(1/x)$ and that it is indeed a solution.

Example: Euler equation

- The admissible pairs are $\{q(t) = t, \alpha = 0\}$ and $\{q(t) = 0, \alpha = 0\}$, yielding $\exp\left(q(x^{-1})\right)x^{\alpha} = f$ and $\exp\left(q(x^{-1})\right)x^{\alpha} = 1$ respectively.
- Then we check if $f^{-1}y$ or y annihilate $y \mapsto \int_{\circlearrowleft 0} y(z)\, z^k \,\mathrm{d}z$ for $k \geqslant 0$.
- All these integrals vanish for $f^{-1}f = 1$, but none of them does for f, so the only candidate space is $\mathbb{C}f$ with $q(t) = t$ and $\alpha = 0$.
- As $\mathbb{C}f$ is invariant by the Galoisian action, it is a candidate Singerian orbit.
- Then we reconstruct symbolically the Darboux polynomial, getting that the candidate Singerian solution is $\exp(1/x)$ and that it is indeed a solution.

Example: Euler equation

- The admissible pairs are $\{q(t) = t, \alpha = 0\}$ and $\{q(t) = 0, \alpha = 0\}$, yielding $\exp\big(q(x^{-1})\big)x^{\alpha} = f$ and $\exp\big(q(x^{-1})\big)x^{\alpha} = 1$ respectively.
- Then we check if $f^{-1}y$ or y annihilate $y \mapsto \int_{\circlearrowleft 0} y(z)\, z^k \, \mathrm{d}z$ for $k \geqslant 0$.
- All these integrals vanish for $f^{-1}f = 1$, but none of them does for f, so the only candidate space is $\mathbb{C}f$ with $q(t) = t$ and $\alpha = 0$.
- As $\mathbb{C}f$ is invariant by the Galoisian action, it is a candidate Singerian orbit.
- Then we reconstruct symbolically the Darboux polynomial, getting that the candidate Singerian solution is $\exp(1/x)$ and that it is indeed a solution.

Example (Airy equation)

The differential equation,

$$y'' = x\,y,$$

has as a fundamental system of solutions

$$h_{\pm} = \exp\left(\pm\frac{2}{3}x^{3/2}\right)\, x^{-1/4}\,\hat{\varphi}(\pm x^{-3/2}),$$

where $\hat{\varphi}$ is a formal series with no null term.

- This equation has one singularity (at infinity) and hence no monodromy.
- We center the variable $u = x^{-1}$ at infinity, where the ramification index is 4.
- The admissible pairs are $\{q_{\pm}(t) = \pm\frac{2}{3}t^{6}, \alpha = \frac{1}{4}\}$.

Example: Airy equation

Example (Airy equation)

The differential equation,

$$y'' = x\,y,$$

has as a fundamental system of solutions

$$h_{\pm} = \exp\left(\pm \tfrac{2}{3} x^{3/2}\right)\, x^{-1/4}\, \hat{\varphi}(\pm x^{-3/2}),$$

where $\hat{\varphi}$ is a formal series with no null term.

- This equation has one singularity (at infinity) and hence no monodromy.
- We center the variable $u = x^{-1}$ at infinity, where the ramification index is 4.
- The admissible pairs are $\{q_{\pm}(t) = \pm\frac{2}{3}t^{6}, \alpha = \frac{1}{4}\}$.

Example: Airy equation

Example (Airy equation)

The differential equation,

$$y'' = x\,y,$$

has as a fundamental system of solutions

$$h_{\pm} = \exp\left(\pm\tfrac{2}{3}x^{3/2}\right)\, x^{-1/4}\,\hat{\varphi}(\pm x^{-3/2}),$$

where $\hat{\varphi}$ is a formal series with no null term.

- This equation has one singularity (at infinity) and hence no monodromy.
- We center the variable $u = x^{-1}$ at infinity, where the ramification index is 4.
- The admissible pairs are $\{q_{\pm}(t) = \pm\frac{2}{3}t^6, \alpha = \frac{1}{4}\}$.

Example: Airy equation

- Then we check if any solution divided by $\exp\left(q(u^{-1/4})\right)u^{\alpha}$ annihilates $y \mapsto \int_{\circlearrowleft\infty} y(z)\, z^{-k/4}\,\mathrm{d}z$ for $k \geqslant -3$.
- As some integrals fail to vanish for both exponential parts and both fundamental solution, we conclude that zero is the only Liouvillian solution of Airy equation.

Example: Airy equation

- Then we check if any solution divided by $\exp\left(q(u^{-1/4})\right) u^{\alpha}$ annihilates $y \mapsto \int_{\circlearrowleft\infty} y(z)\, z^{-k/4}\, \mathrm{d}z$ for $k \geqslant -3$.
- As some integrals fail to vanish for both exponential parts and both fundamental solution, we conclude that zero is the only Liouvillian solution of Airy equation.

Thank you for the attention

A numeric-symbolic algorithm for computing the Liouvillian solutions of linear differential equations and systems

Alberto Llorente & Jorge Mozo-Fernández

Universidad Isabel I & Universidad de Valladolid

XV Encuentro de Álgebra Computacional y Aplicaciones (EACA)

Logroño, June 22nd, 2016