Resultants and Subresultants through Evaluation

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OUTLINE

- Resultants and Subresultants: basics.
- Resultants and Subresultants in terms of roots.
- Resultants and Subresultants through evaluation.
- Resultants and Subresultants through evaluation (the motivation).
- Conclusions and further work.

Resultants and Subresultants

Resultant

$$P(x) = \sum_{i=0}^{p} a_i x^i \qquad Q(x) = \sum_{i=0}^{q} b_i x^i$$

$$\mathbf{Resultant}(P,Q) = \begin{vmatrix} a_p & \cdots & a_0 & & \\ & \ddots & & \ddots & \\ & & a_p & \cdots & a_0 \\ b_q & \cdots & b_0 & & \\ & \ddots & & \ddots & \\ & & b_q & \cdots & b_0 \end{vmatrix}$$

P(x) and *Q(x)* have a common root if and only if *Resultant(P,Q)=0*

Subresultants

$$\mathbf{Sylv}_i(P,Q) = \overbrace{\begin{pmatrix} a_p & \dots & a_0 & & \\ & \ddots & & \ddots & \\ & & a_p & \dots & a_0 \\ b_q & \dots & b_0 & & \\ & \ddots & & \ddots & \\ & & b_q & \dots & b_0 \end{pmatrix}}^{p+q-i} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ \ddots \\ a \end{matrix}}\right\} q-i \\ \left.\vphantom{\begin{matrix} b \\ \ddots \\ b \end{matrix}}\right\} p-i \end{matrix}$$

$$\mathbf{Sres}_i(P,Q) = \sum_{k=0}^{i} M_k x^k$$

$P := a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$ $\quad$ $Q := b_3 x^3 + b_2 x^2 + b_1 x + b_0$

$$M_0 := \begin{bmatrix} a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & a_4 & a_3 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 & 0 & 0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 & 0 & 0 \\ 0 & 0 & b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & 0 & b_3 & b_2 & b_1 & b_0 \end{bmatrix}$$

Resultant:=det(M_0)=$Sres_0$(P,Q)

$$M_1 := \begin{bmatrix} a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & a_4 & a_3 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 & 0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & b_3 & b_2 & b_1 & b_0 \end{bmatrix}$$

$Sres_1$(P,Q):=det(M_{11})x+det(M_{10})

$$M_2 := \begin{bmatrix} a_4 & a_3 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 \end{bmatrix}$$

$Sres_2$(P,Q):= det(M_{22})x^2+ det(M_{21})x+ det(M_{20})

$$P := a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$$

$$Q := b_3 x^3 + b_2 x^2 + b_1 x + b_0$$

$$M_0 := \begin{bmatrix} a_4 & a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & a_4 & a_3 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 & 0 & 0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 & 0 & 0 \\ 0 & 0 & b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & 0 & b_3 & b_2 & b_1 & b_0 \end{bmatrix}$$

$sres_0(P,Q)$

Resultant:=det(M_0)=$Sres_0$(P,Q)

$$M_1 := \begin{bmatrix} a_4 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & a_4 & a_3 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 & 0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & b_3 & b_2 & b_1 & b_0 \end{bmatrix}$$

$Sres_1$(P,Q):=det(M_{11})x+det(M_{10})

$sres_1(P,Q)$

$$M_2 := \begin{bmatrix} a_4 & a_3 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 \end{bmatrix}$$

$sres_2(P,Q)$

$Sres_2$(P,Q):= det(M_{22})x^2+ det(M_{21})x+ det(M_{20})

Subresultants

$$
\mathbf{Sres}_i(P,Q) = (-1)^{i(q-i+1)}
\left|\begin{array}{ccccccccc}
a_p & a_{p-1} & a_{p-2} & \dots & \dots & a_0 & & & \\
 & \ddots & \ddots & \ddots & & & \ddots & & \\
 & & a_p & a_{p-1} & a_{p-2} & \dots & \dots & a_0 & \\
 & & & & & 1 & -x & & \\
 & & & & & & \ddots & \ddots & \\
 & & & & & & & 1 & -x \\
b_q & b_{q-1} & b_{q-2} & \dots & \dots & \dots & b_0 & & \\
 & \ddots & \ddots & \ddots & & & & \ddots & \\
 & & b_q & b_{q-1} & b_{q-2} & \dots & \dots & \dots & b_0
\end{array}\right|
\begin{array}{l}
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} p-i \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} i \\
\left.\begin{array}{c} \\ \\ \\ \end{array}\right\} q-i
\end{array}
$$

Subresultants

$$\gcd(P,Q) = \mathbf{Sres}_i(P,Q)$$

$$\Longleftrightarrow$$

$$\mathbf{sres}_0(P,Q) = \mathbf{sres}_1(P,Q) = \cdots = \mathbf{sres}_{i-1}(P,Q) = 0, \ \mathbf{sres}_i(P,Q) \neq 0$$

Used in (avoid divisions: parameters !)

- Cylindrical Algebraic Decomposition
- Computer Aided Geometric Design
- Real Algebraic Geometry
- ....

Subresultants

$$\begin{array}{lcc}
S_{j+1} & \rightarrow & (\deg(S_{j+1}) = j+1) \\
S_j & \rightarrow & (\deg(S_j) = k \leq j) \\
S_{j-1} & \rightarrow & 0 \\
\vdots & & \vdots \\
S_{k+1} & \rightarrow & 0 \\
S_k & \rightarrow & \left(\frac{\operatorname{lcof}(S_j)}{R_{j+1}}\right)^{j-k} S_j \\
S_{k-1} & \rightarrow & \frac{\operatorname{Prem}(S_{j+1}, S_j)}{(-R_{j+1})^{j-k+2}}
\end{array}$$

Resultants & Subresultants In terms of the roots

In terms of the roots: Resultant

$$P(x) = \sum_{i=0}^{p} a_i x^i = a_p \prod_{i=1}^{p} (x - \alpha_i) \qquad Q(x) = \sum_{j=0}^{q} b_j x^j = b_q \prod_{j=1}^{q} (x - \beta_i)$$

In terms of the roots: Resultant

$$P(x) = \sum_{i=0}^{p} a_i x^i = a_p \prod_{i=1}^{p} (x - \alpha_i) \qquad Q(x) = \sum_{j=0}^{q} b_j x^j = b_q \prod_{j=1}^{q} (x - \beta_i)$$

$$\mathbf{Res}(P, Q) = a_p^q \prod_{i=1}^{p} Q(\alpha_i) = b_q^p \prod_{j=1}^{q} P(\beta_j) = a_p^q b_q^p \prod_{i=1}^{p} \prod_{j=1}^{q} (\alpha_i - \beta_j)$$

In terms of the roots: Subresultants

If $P(x)$ is monic and squarefree

$k \in \{0, \ldots, \min(p, q) - 1\}$ and $N = \{1, \ldots, p\}$,

$$P_I(x) = \prod_{i \in I} (x - \alpha_i)$$

$$\mathbf{Sres}_k(P, Q) = \sum_{\substack{I \biguplus J = N \\ |J| = k}} \frac{\mathrm{res}(P_I, Q)}{\mathrm{res}(P_I, P_J)} P_J(x) = \sum_{\substack{I \biguplus J = N \\ |J| = k}} \frac{\prod_{i \in I} Q(\alpha_i)}{\prod_{i \in I, j \in J} (\alpha_i - \alpha_j)} P_J(x)$$

A. Lascoux, P. Pragacz: *Double Sylvester sums for subresultants and multi-Schur functions*. Journal of Symbolic Computation 35, 689-710, 2003.

C. D'Andrea, H. Hong, T. Krick, A. Szanto: *An elementary proof of Sylvester's double sums for subresultants*. Journal of Symbolic Computation 42, 290-297, 2007.

C. D'Andrea, H. Hong, T. Krick, A. Szanto: *Sylvester's double sums: The general case*. Journal of Symbolic Computation 44, 1164-1175, 2009.

M.-F. Roy, A. Szpirglas: *Sylvester double sums and subresultants*. Journal of Symbolic Computation 46, 385-395, 2011.

Resultants & Subresultants Through Evaluation (not in the roots)

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Through evaluation: Resultant (I)

The Bezoutian associated to $P(T)$ and $Q(T)$ is the symmetric matrix

$$\mathrm{Bez}(P,Q) = \begin{bmatrix} c_{0,0} & \cdots & \cdots & c_{0,n-1} \\ \vdots & & & \vdots \\ c_{n-1,0} & \cdots & \cdots & c_{n-1,n-1} \end{bmatrix}$$

where the $c_{i,j}$ are defined by the Cayley quotient:

$$\frac{P(T)Q(Z) - P(Z)Q(T)}{T - Z} = \sum_{i,j=0}^{n-1} c_{i,j} T^i Z^j$$

Through evaluation: Resultant (I)

The Bezoutian associated to $P(T)$ and $Q(T)$ is the symmetric matrix

$$\mathrm{Bez}(P,Q) = \begin{bmatrix} c_{0,0} & \cdots & \cdots & c_{0,n-1} \\ \vdots & & & \vdots \\ c_{n-1,0} & \cdots & \cdots & c_{n-1,n-1} \end{bmatrix}$$

where the $c_{i,j}$ are defined by the Cayley quotient:

$$\frac{P(T)Q(Z) - P(Z)Q(T)}{T - Z} = \sum_{i,j=0}^{n-1} c_{i,j} T^i Z^j$$

$$\det(\mathrm{Bez}(P,Q)) = a_p^{p-q} \cdot \mathrm{Resultant}(P,Q)$$

Through evaluation: Resultant (I)

$p, q \in \mathbb{P}_n$ with $n = \max(\deg(p), \deg(q))$.
$\boldsymbol{\tau} = (\tau_1, \dots, \tau_{n+1}) \in \mathbb{C}^{n+1}$ and $\widetilde{\boldsymbol{\tau}} = (\tau_1, \dots, \tau_n) \in \mathbb{C}^n$.
$\mathbf{p} = (p_1, \dots, p_{n+1}) = (p(\tau_1), \dots, p(\tau_{n+1}))$.
$\mathbf{q} = (q_1, \dots, q_{n+1}) = (q(\tau_1), \dots, q(\tau_{n+1}))$.
$1 \le i \le n$: $p'_i = p'(\tau_i)$ and $q'_i = q'(\tau_i)$.

The Bezout matrix $\mathbf{Bez}_{p,q}$ in the Lagrange basis $\mathbf{L}(t; \widetilde{\boldsymbol{\tau}}) \in [\mathbb{P}_{n-1}]^n$ is the matrix:

$$\begin{bmatrix} p'_1 q_1 - p_1 q'_1 & \dfrac{p_1 q_2 - p_2 q_1}{\tau_1 - \tau_2} & \cdots\cdots & \dfrac{p_1 q_n - p_n q_1}{\tau_1 - \tau_n} \\ \dfrac{p_2 q_1 - p_1 q_2}{\tau_2 - \tau_1} & p'_2 q_2 - p_2 q'_2 & \cdots\cdots & \dfrac{p_2 q_n - p_n q_2}{\tau_2 - \tau_n} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \dfrac{p_n q_1 - p_1 q_n}{\tau_n - \tau_1} & \dfrac{p_n q_2 - p_2 q_n}{\tau_n - \tau_2} & \cdots\cdots & p'_n q_n - p_n q'_n \end{bmatrix}$$

Through evaluation: Resultant (II)

Let $\{x_1, \ldots, x_{p+q}\}$ be a set with $p+q$ elements

$[p+q] = \{1, 2, \ldots, p+q\}$

$$\mathrm{Res}(P,Q) = \sum_{\substack{I \subset_q [p+q],\, J \subset_p [p+q] \\ I \cup J = [p+q]}} \frac{\prod_{i \in I} P(x_i) \prod_{j \in J} Q(x_j)}{\prod_{i \in I, j \in J} (x_i - x_j)}$$

Advantages: no conditions on *P* or *Q*.

Disadvantages: exponential number of summands.

F. Apery, J.-P. Jouanolou: Élimination. Le cas d'une variable. Collection Methodes. Hermann, 2006.

Through evaluation: Subresultants

Let $\{x_1, x_2, \ldots, x_{p+q-t}\}$ be a set with $p+q-t$ elements.

$E = \{1, 2, \ldots, p+q-t\}$

$$\mathbf{Sres}_t(P,Q) =$$

$$\sum_{\substack{I \subset_{q-t} E,\, J \subset_{p-t} E \\ K \subset_t E,\, I \uplus J \uplus K = E}} \frac{\prod_{i \in I} P(x_i) \prod_{j \in J} Q(x_j)}{\prod_{i \in I, j \in J} (x_i - x_j) \prod_{i \in I, k \in K} (x_i - x_k) \prod_{j \in J, k \in K} (x_j - x_k)} \prod_{k \in K} (x - x_k)$$

Advantages: no conditions on *P* and *Q*.

Disadvantages: exponential number of summands.

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Resultants & Subresultants Through Evaluation (Our Formulae)

Through evaluation: Resultant

Let $\{x_1, \ldots, x_{p+q}\}$ be a set with $p+q$ elements $\qquad [p+q] = \{1, 2, \ldots, p+q\}$

$$\mathbf{Res}(P,Q) = \frac{\begin{vmatrix} x_1^{q-1}P(x_1) & \ldots & x_{p+q}^{q-1}P(x_{p+q}) \\ \vdots & & \vdots \\ P(x_1) & \ldots & P(x_{p+q}) \\ x_1^{p-1}Q(x_1) & \ldots & x_{p+q}^{p-1}Q(x_{p+q}) \\ \vdots & & \vdots \\ Q(x_1) & \ldots & Q(x_{p+q}) \end{vmatrix}}{V(x_1, \ldots, x_{p+q})}$$

Advantages:

- No conditions on *P* and *Q*.
- Derivatives of *P* and *Q* are not evaluated (but a bigger matrix).
- No exponential number of summands.

Through evaluation: Resultant (the proof)

$$\mathbf{Res}(P,Q) = \begin{vmatrix} a_p & \dots & a_0 & & \\ & \ddots & & \ddots & \\ & & a_p & \dots & a_0 \\ b_q & \dots & b_0 & & \\ & \ddots & & \ddots & \\ & & b_q & \dots & b_0 \end{vmatrix} = \begin{vmatrix} a_p & \dots & \dots & \dots & \star & x^{q-1}P(x) \\ & \ddots & & & \vdots & \vdots \\ & & a_p & \dots & a_1 & P(x) \\ b_q & \dots & \dots & \dots & \star & x^{p-1}Q(x) \\ & \ddots & & & \vdots & \vdots \\ & & b_q & \dots & b_1 & Q(x) \end{vmatrix}.$$

Through evaluation: Resultant (the proof)

$$\mathbf{Res}(P,Q)=\begin{vmatrix} a_p & \dots & a_0 & & \\ & \ddots & & \ddots & \\ & & a_p & \dots & a_0 \\ b_q & \dots & b_0 & & \\ & \ddots & & \ddots & \\ & & b_q & \dots & b_0 \end{vmatrix}=\begin{vmatrix} a_p & \dots & \dots & \dots & \star & x^{q-1}P(x) \\ & \ddots & & & \vdots & \vdots \\ & & a_p & \dots & a_1 & P(x) \\ b_q & \dots & \dots & \dots & \star & x^{p-1}Q(x) \\ & \ddots & & & \vdots & \vdots \\ & & b_q & \dots & b_1 & Q(x) \end{vmatrix}.$$

$$\begin{vmatrix} x_1^{q-1}P(x_1) & \dots & x_{p+q}^{q-1}P(x_{p+q}) \\ \vdots & & \vdots \\ P(x_1) & \dots & P(x_{p+q}) \\ x_1^{p-1}Q(x_1) & \dots & x_{p+q}^{p-1}Q(x_{p+q}) \\ \vdots & & \vdots \\ Q(x_1) & \dots & Q(x_{p+q}) \end{vmatrix}=\begin{vmatrix} a_p & \dots & \dots & \dots & \star & x_{p+q}^{q-1}P(x_{p+q}) \\ & \ddots & & & \vdots & \vdots \\ & & a_p & \dots & a_1 & P(x_{p+q}) \\ b_q & \dots & \dots & \dots & \star & x_{p+q}^{p-1}Q(x_{p+q}) \\ & \ddots & & & \vdots & \vdots \\ & & b_q & \dots & b_1 & Q(x_{p+q}) \end{vmatrix}.$$

$$\cdot(x_1^{p+q-1}x_2^{p+q-2}\cdots x_{p+q-1})+\mathrm{ldt}=\mathrm{Res(P,Q)}\cdot(x_1^{p+q-1}x_2^{p+q-2}\cdots x_{p+q-1})+\mathrm{ldt}$$

Through evaluation: Resultant (the proof)

$$\mathbf{Res}(P,Q) = \begin{vmatrix} a_p & \dots & a_0 & & \\ & \ddots & & \ddots & \\ & & a_p & \dots & a_0 \\ b_q & \dots & b_0 & & \\ & \ddots & & \ddots & \\ & & b_q & \dots & b_0 \end{vmatrix} = \begin{vmatrix} a_p & \dots & \dots & \dots & \star & x^{q-1}P(x) \\ & \ddots & & & \vdots & \vdots \\ & & a_p & \dots & a_1 & P(x) \\ b_q & \dots & \dots & \dots & \star & x^{p-1}Q(x) \\ & \ddots & & & \vdots & \vdots \\ & & b_q & \dots & b_1 & Q(x) \end{vmatrix} .$$

$$\begin{vmatrix} x_1^{q-1}P(x_1) & \dots & x_{p+q}^{q-1}P(x_{p+q}) \\ \vdots & & \vdots \\ P(x_1) & \dots & P(x_{p+q}) \\ x_1^{p-1}Q(x_1) & \dots & x_{p+q}^{p-1}Q(x_{p+q}) \\ \vdots & & \vdots \\ Q(x_1) & \dots & Q(x_{p+q}) \end{vmatrix} = \begin{vmatrix} a_p & \dots & \dots & \dots & \star & x_{p+q}^{q-1}P(x_{p+q}) \\ & \ddots & & & \vdots & \vdots \\ & & a_p & \dots & a_1 & P(x_{p+q}) \\ b_q & \dots & \dots & \dots & \star & x_{p+q}^{p-1}Q(x_{p+q}) \\ & \ddots & & & \vdots & \vdots \\ & & b_q & \dots & b_1 & Q(x_{p+q}) \end{vmatrix} .$$

$$\cdot(x_1^{p+q-1}x_2^{p+q-2}\cdots x_{p+q-1}) + \mathrm{ldt} = \mathrm{Res(P,Q)} \cdot (x_1^{p+q-1}x_2^{p+q-2}\cdots x_{p+q-1}) + \mathrm{ldt}$$

Dividing this determinant by

$$V(x_1,\dots,x_{p+q}) = x_1^{p+q-1}x_2^{p+q-2}\cdots x_{p+q-1} + \text{lowest degree terms} ,$$

the corresponding degree comparison produces the searched equality.

Through evaluation: Subresultants

Let $\{x_1, x_2, \ldots, x_{p+q-t}\}$ be a set with $p+q-t$ elements. $E = \{1, 2, \ldots, p+q-t\}$

$$\beta_{\underline{F}}(y_1, y_2, \ldots, y_{p+q-2t}) = \frac{\begin{vmatrix} y_1^{q-t-1}P(y_1) & y_2^{q-t-1}P(y_2) & \ldots & y_r^{q-t-1}P(y_r) \\ \vdots & \vdots & & \vdots \\ P(y_1) & P(y_2) & \ldots & P(y_r) \\ y_1^{p-t-1}Q(y_1) & y_2^{p-t-1}Q(y_2) & \ldots & y_r^{m-t-1}Q(y_r) \\ \vdots & \vdots & & \vdots \\ Q(y_1) & Q(y_2) & \ldots & Q(y_r) \end{vmatrix}}{V(y_1, y_2, \ldots, y_{p+q-2t})}$$

$$\mathbf{Sres}_t(P, Q) = \sum_{\substack{[p+q-t] = I \uplus J \\ |I| = p+q-2t, |J| = t}} \beta_{\underline{F}}((x_i)_{i \in I}) \cdot \frac{\prod_{j \in J}(x - x_j)}{\prod_{i \in I, j \in J}(x_i - x_j)}$$

Advantages: no conditions on *P* and *Q*.
Disadvantages: exponential number of summands (but smaller).

Resultants & Subresultants Through Evaluation (The Motivation)

MOTIVATION

The output of an algorithm may produce as solution an algebraic expression (a polynomial or a rational function) whose expansion:

- either is very costly, or
- produces a very complicated object to deal with,
- or requires a change of basis.

This happens, for example,

- when computing implicit equations for solving intersection problems in CAGD, or
- when numerically solving initial value problems for ordinary differential equations.

Algorithms for geometric entities (curves and surfaces) presented by polynomials given by values:

Symbolic Computation for polynomials presented, for example, “à la Lagrange” or “á la Hermite” when dealing with curves, surfaces, … .

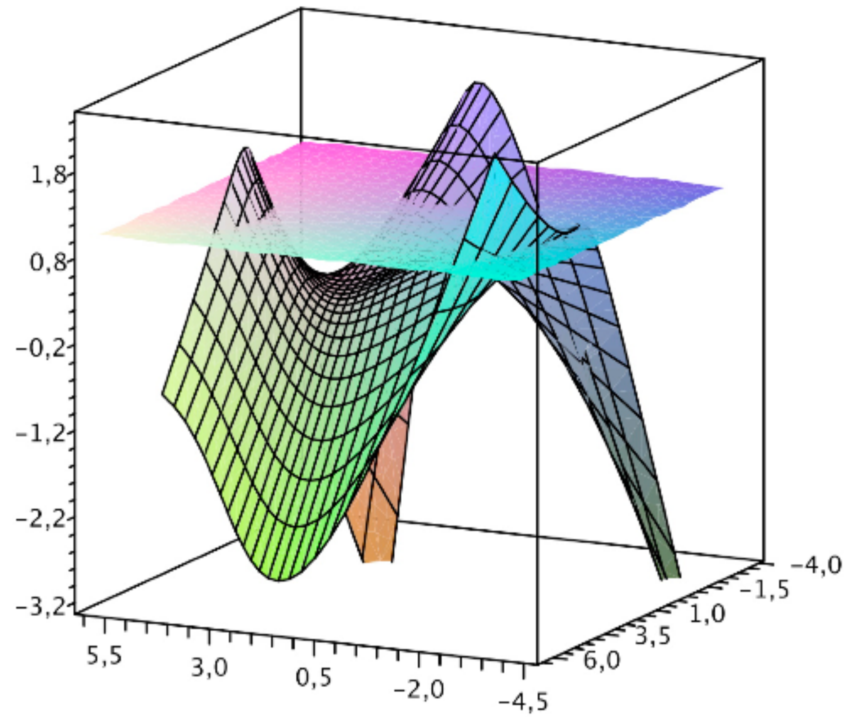
Motivating Example

$$\begin{aligned} x(u,v) &= 3v(v-1)^2 + (u-1)^3 + 3u \\ y(u,v) &= 3u(u-1)^2 + v^3 + 3v \\ z(u,v) &= -3u(u^2-5u+5)v^3 - 3(u^3+6u^2-9u+1)v^2 \\ &= +v(6u^3+9u^2-18u+3) - 3u(u-1). \end{aligned}$$

$$\mathcal{H}_{\mathcal{B}}(x,y,z) = z^9 + \sum_{i=1}^{9} r_i(x,y) z^{9-i}$$

$$\begin{aligned} \mathbf{r}_1(x,y) \quad = \quad & -\frac{233469x}{2048} + \frac{188595y}{2048} - \frac{112832595}{262144} - \frac{81x^2}{64} \\ & + \frac{135xy}{32} - \frac{81y^2}{64} \end{aligned}$$

$$\begin{aligned} \mathbf{r}_2(x,y) \quad = \quad & -\frac{20972672709381x}{536870912} + \frac{17975329363179y}{536870912} - \frac{729y^4}{8192} \\ & - \frac{729x^4}{8192} + \frac{1215x^3y}{2048} + \frac{1215xy^3}{2048} - \frac{4105971x^3}{65536} \\ & + \frac{3129597y^3}{65536} + \frac{14456151x^2y}{65536} - \frac{13181049xy^2}{65536} \\ & + \frac{48101467761xy}{8388608} - \frac{38812918311y^2}{16777216} \\ & - \frac{1}{2}\left(\frac{233469x}{2048} + \frac{112832595}{262144} + \frac{81x^2}{64} - \frac{188595y}{2048}\right. \\ & \left. - \frac{135xy}{32} + \frac{81y^2}{64}\right)\left(-\frac{233469x}{2048} + \frac{188595y}{2048} + \frac{135xy}{32}\right. \\ & \left. - \frac{81y^2}{64} - \frac{112832595}{262144} - \frac{81x^2}{64}\right) - \frac{4779x^2y^2}{4096} \\ & - \frac{22656991982391171}{137438953472} - \frac{54187594407x^2}{16777216} + \ldots\ldots \\ = \quad & -\frac{20972672709381x}{536870912} + \frac{17975329363179y}{536870912} - \frac{729y^4}{8192} \\ & - \frac{729x^4}{8192} + \frac{1215x^3y}{2048} + \frac{1215xy^3}{2048} - \frac{4105971x^3}{65536} \\ & + \frac{3129597y^3}{65536} + \frac{14456151x^2y}{65536} - \frac{13181049xy^2}{65536} \\ & + \frac{48101467761xy}{8388608} - \frac{38812918311y^2}{16777216} \\ & - \frac{1}{2}\left(\frac{233469x}{2048} + \frac{112832595}{262144} + \frac{81x^2}{64} - \frac{188595y}{2048}\right. \\ & \left. - \frac{135xy}{32} + \frac{81y^2}{64}\right) \boxed{\cdot\, \mathbf{r}_1(x,y)} - \frac{4779x^2y^2}{4096} \\ & - \frac{22656991982391171}{137438953472} - \frac{54187594407x^2}{16777216} + \ldots\ldots \end{aligned}$$



Motivating Example

$f(x,y)=0$
$g(x,y)=0$

Coefficients not available directly

$R(x)=det(M(x))=0$

Linearization: GEP
(without computing the determinant !)

Solving
$f(\alpha,y)=0$
$g(\alpha,y)=0$
with $R(\alpha)=0$

Nullspace computations: SVD
Subresultants

G.-M. Diaz-Toca, M.Fioravanti, L. Gonzalez-Vega, A. Shakoori: *Using implicit equations of parametric curves and surfaces without computing them: Polynomial algebra by values.* Computer Aided Geometric Design 30, 116-139, 2013.
R. M. Corless, G.-M. Diaz-Toca, M. Fioravanti, L. Gonzalez-Vega, I. F. Rua, A. Shakoori: *Computing the topology of a real algebraic plane curve whose defining equations are available only "by values".* Computer Aided Geometric Design 30, 675-706, 2013.

Motivating Example

u(t) - x = 0
v(t) - y = 0

Coefficients of *u(t)* and *v(t)* not available directly

R(x,y) = det(M(x,y)) = 0

Implicit equation in determinant form

Solving
R(w(s),z(s)) = 0

Generalized Eigenvalue Problem
(without computing the determinant !)

G.-M. Diaz-Toca, M.Fioravanti, L. Gonzalez-Vega, A. Shakoori: *Using implicit equations of parametric curves and surfaces without computing them: Polynomial algebra by values*. Computer Aided Geometric Design 30, 116-139, 2013.

R. M. Corless, G.-M. Diaz-Toca, M. Fioravanti, L. Gonzalez-Vega, I. F. Rua, A. Shakoori: *Computing the topology of a real algebraic plane curve whose defining equations are available only "by values"*. Computer Aided Geometric Design 30, 675-706, 2013.

Resultants & Subresultants Through Evaluation (Conclusions and ...)

Conclusions and further work

Resultants and Subresultants through evaluation “à la Hermite

$x_1 = x_2$

$$\mathbf{Res}(P,Q) = \frac{\begin{vmatrix} [x^{q-1}P(x)]'_{x=x_1} & x_1^{q-1}P(x_1) & x_3^{q-1}P(x_3) & \dots & x_{p+q}^{q-1}P(x_{p+q}) \\ \vdots & \vdots & \vdots & & \vdots \\ P'(x_1) & P(x_1) & P(x_3) & \dots & P(x_{p+q}) \\ [x^{p-1}Q(x)]'_{x=x_1} & x_1^{p-1}Q(x_1) & x_3^{p-1}Q(x_3) & \dots & x_{p+q}^{p-1}Q(x_{p+q}) \\ \vdots & \vdots & \vdots & & \vdots \\ Q'(x_1) & Q(x_1) & Q(x_3) & \dots & Q(x_{p+q}) \end{vmatrix}}{V(x_3,\dots,x_{p+q})\prod_{k\geq 3}(x_1-x_k)^2}$$

C. D'Andrea, T. Krick, A. Szanto: *Subresultants in multiple roots*. Linear Algebra and Its Applications 438, 1969-1989, 2013.

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Conclusions and further work

Reducing the complexity of the expressions: replacing the exponential number of summands by a single determinant.

deg(P) - *rank(M(P,Q))* = *deg(gcd(P,Q))* .

Looking for a Barnett's factorization for *M(P,Q):*

$$p_0^m \mathrm{Bez}(P,Q) = \widetilde{Q}(\Delta_P)\mathrm{Bez}(P,1) = \mathrm{Bez}(P,1)\widetilde{Q}(\Delta_P^t).$$

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GRACIAS
THANKS
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