Computing the Medial Axis for Closed Planar Domains Bounded by Finitely Many Segments and Conic Arcs

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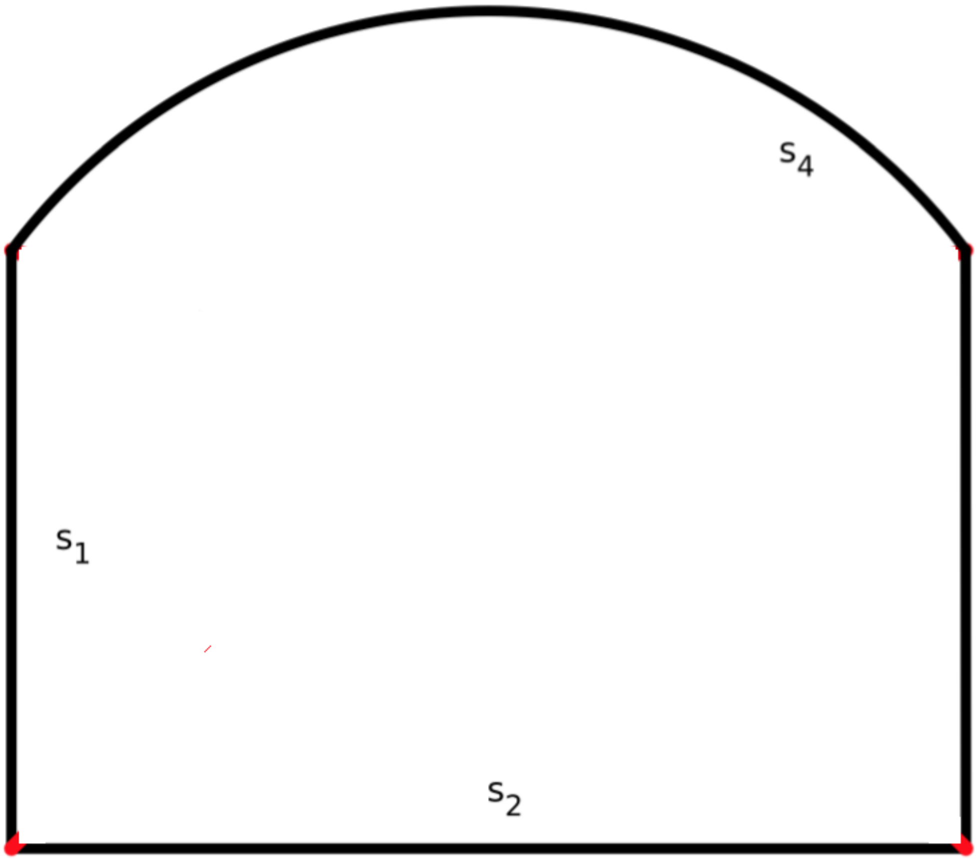
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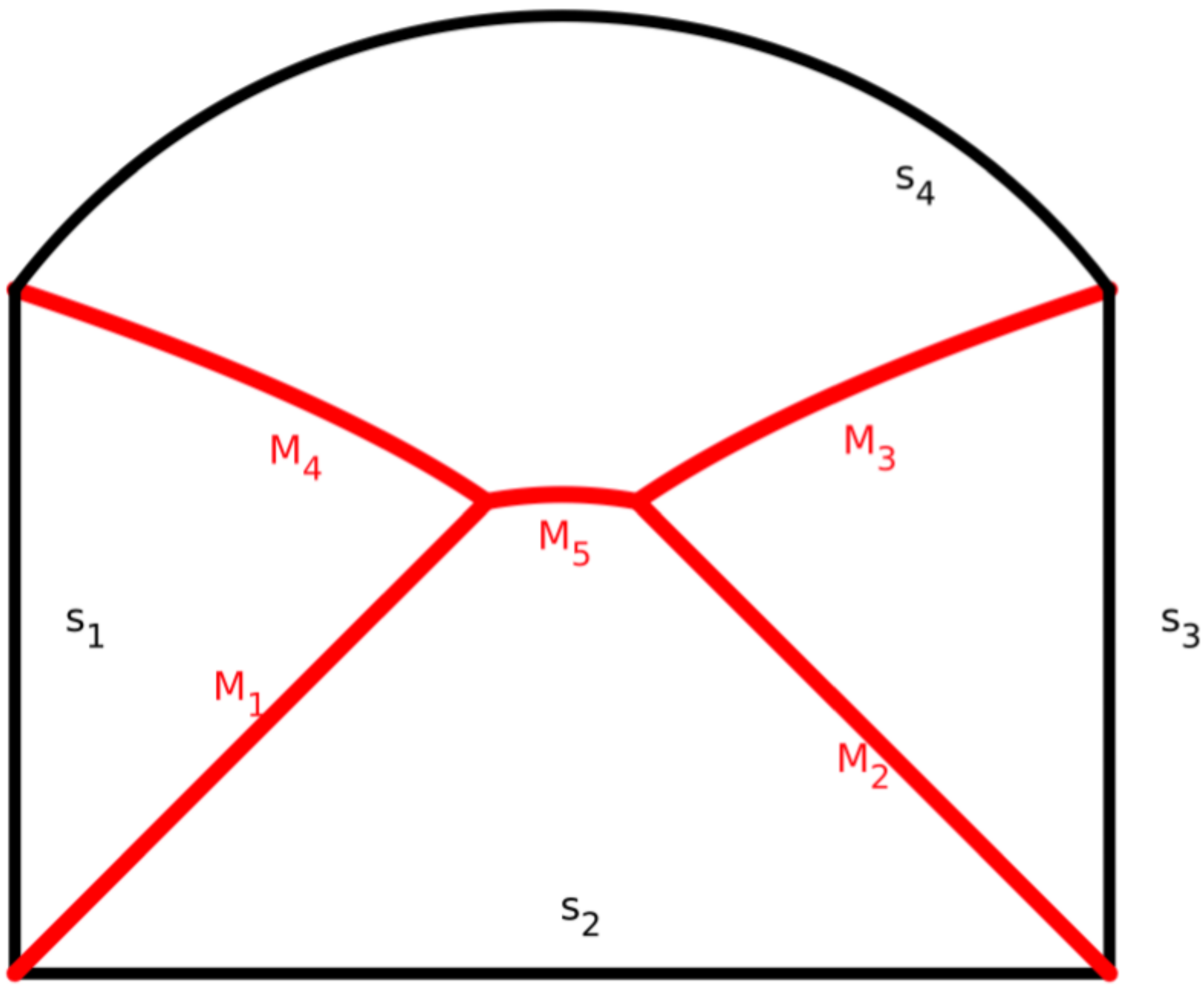
The **medial axis** of an object is the set of all (inside) points having more than one closest point on the object's boundary.



OUTLINE

- Medial Axis: basics.
- Computing the medial axis.
- Equations and topologies of the bisectors:
  - point & conic (including lines).
  - conic & conic (including lines).
- Bisector of curve segments (conics and lines).
- The algorithm.

The Medial Axis of a Planar Domain

Medial Axis: the definition

Let $\mathcal{D}$ be bounded domain in $\mathbb{R}^2$ with boundary $\mathcal{C}$ consisting of a finite number of curve segments.

The medial axis of $\mathcal{D}$, denoted $\mathcal{M}(\mathcal{D})$, can be geometrically defined as the closed locus of the centers of all maximal circles inside $\mathcal{D}$ which are tangent at least at two different points in the boundary of $\mathcal{D}$, i.e:

$$\mathcal{M}(\mathcal{D}) = \{P \in \mathcal{D}\colon \text{there exists } P_1, P_2 \in \mathcal{C} \text{ such that } P_1 \neq P_2,\ d(P, P_1) = d(P, P_2)\}.$$

If $\mathcal{C}$ is a curve given by a parametrization $\mathcal{C}(u)$ ($u \in [a, b]$, $\mathcal{C}(a) = \mathcal{C}(b)$ and $\mathcal{C}$ continuous and differentiable except in a finite number of points), $\mathcal{M}(\mathcal{D})$ can be defined by

$$\mathcal{M}(\mathcal{D}) = \{P \in \mathcal{D}\colon \text{there exists } u_1, u_2 \in [a, b] \text{ such that } u_1 \neq u_2,\ d(P, \mathcal{C}(u_1)) = d(P, \mathcal{C}(u_2))\}.$$

Medial Axis: the characterization

$P \in \mathcal{M}(\mathcal{D})$ if there exists parameter values $u_1, u_2 \in [a, b]$ such that

- P is at normals of $\mathcal{C}$ from $C_1 = \mathcal{C}(u_1)$ and $C_2 = \mathcal{C}(u_2)$:

$$\langle P - \mathcal{C}(u_1), \mathcal{C}'(u_1) \rangle = 0 \text{ and } \langle P - \mathcal{C}(u_2), \mathcal{C}'(u_2) \rangle = 0$$

- P is at equal distance from $C_1 = \mathcal{C}(u_1)$ and $C_2 = \mathcal{C}(u_2)$:

$$\langle P, 2(\mathcal{C}(u_2) - \mathcal{C}(u_1)) \rangle + \|\mathcal{C}(u_1)\|^2 - \|\mathcal{C}(u_2)\|^2 = 0$$

- The points $\mathcal{C}(u_1)$ and $\mathcal{C}(u_2)$ are not equal: $\mathcal{C}(u_2) \neq \mathcal{C}(u_1)$.

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Medial Axis: the characterization

$\mathcal{M}(\mathcal{D})$ is a collection of finitely many curve segments coming from the bisectors of any two curve segments in the boundary $\mathcal{C}$ of $\mathcal{D}$ (including the vertices).

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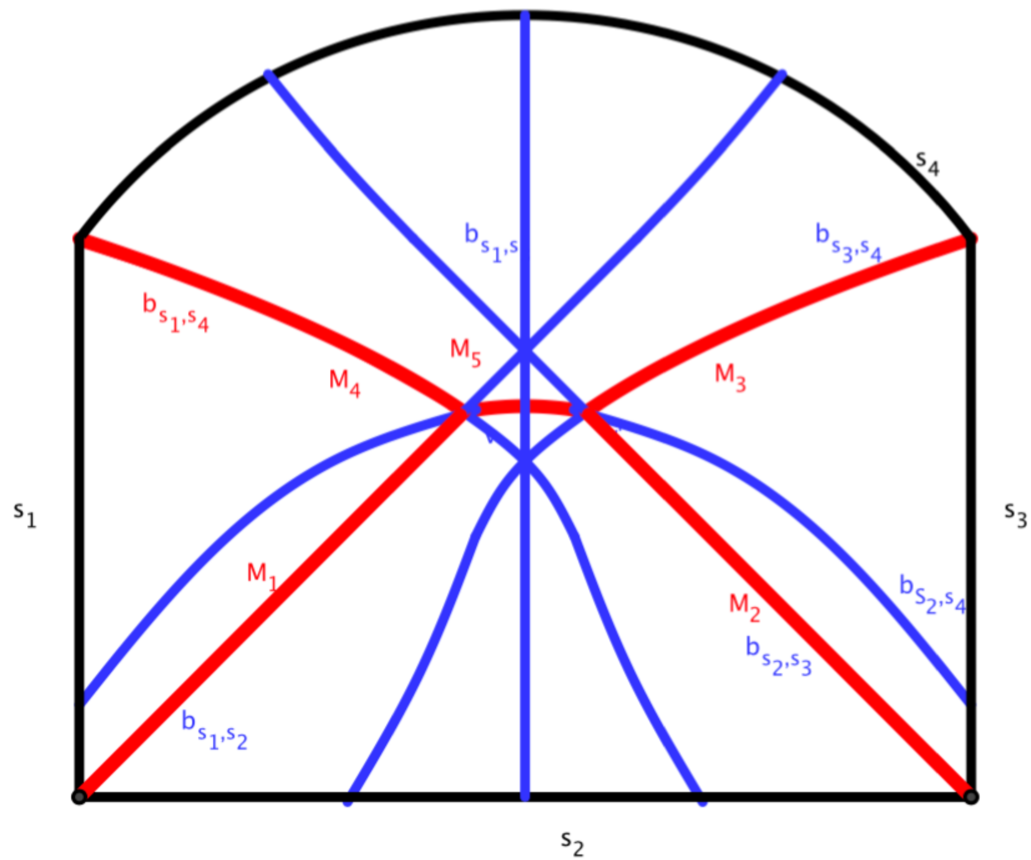

Medial Axis: the background

Medial axis was introduced by Blum (1967) as a concept for efficient shape description. Meanwhile it has proven useful in many scientific areas, and its fast and stable computation is of vital interest.

However, even in the plane, the task of computing the correct medial axis of a given free-form shape is a highly non-trivial one.

There exist two principal problems -apart from stability issues- that need to be addressed when computing a medial axis:

- One of them is determining the combinatorial structure (i.e., the topology) of the medial axis.
- Even when the topology of the medial axis is assumed to be known, the (usually hard) problem of computing its bisectors remains.

O. Aichholzer, W. Aigner, F. Aurenhammer, T. Hackl, B. Jüttler, M. Rabl: *Medial axis computation for planar free--form shapes*. CAD 41, 339-349, 2009.

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Computing the Medial Axis: The Algorithm

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Medial Axis: the algorithm

Let the boundary $\mathcal{C}$ of $\mathcal{D}$ be a finite number of segments and conic arcs.

We introduce a new approach determining the medial axis of $\mathcal{D}$ which is

- topologically correct (no components are missed), and
- geometrically exact (each component is represented exactly).

Medial Axis: the algorithm

Preprocessing step:

- Determining exact representations for the bisector of two parametric curves which are either lines or conics.
- Determining exact representations for the bisector of a point and a parametric curve which is either a lines or a conic.
- Determining all possible "topologies" for the bisector of two parametric curves which are either lines or conics.
- Determining all possible "topologies" for the bisector of a point and a parametric curve which is either a lines or a conic.

Medial Axis: the algorithm

Specialization step, computing the medial axis:

- Analyzing what happens when bisector computations for a concrete domain are applied to segments and (bounded) conic arcs.
- Computing the arrangement of all those bisectors to derive the medial axis of $\mathcal{D}$ by keeping only those curves fulfilling the conditions defining the medial axis.

O. Aichholzer, W. Aigner, F. Aurenhammer, T. Hackl, B. Jüttler, M. Rabl: *Medial axis computation for planar free--form shapes*. CAD 41, 339-349, 2009.

The Equations of the Bisectors

Point and Conic

Point and Conic: equations

The bisector curve of a point and a parametric curve $c(t)=(a(t),b(t))$ is always rational

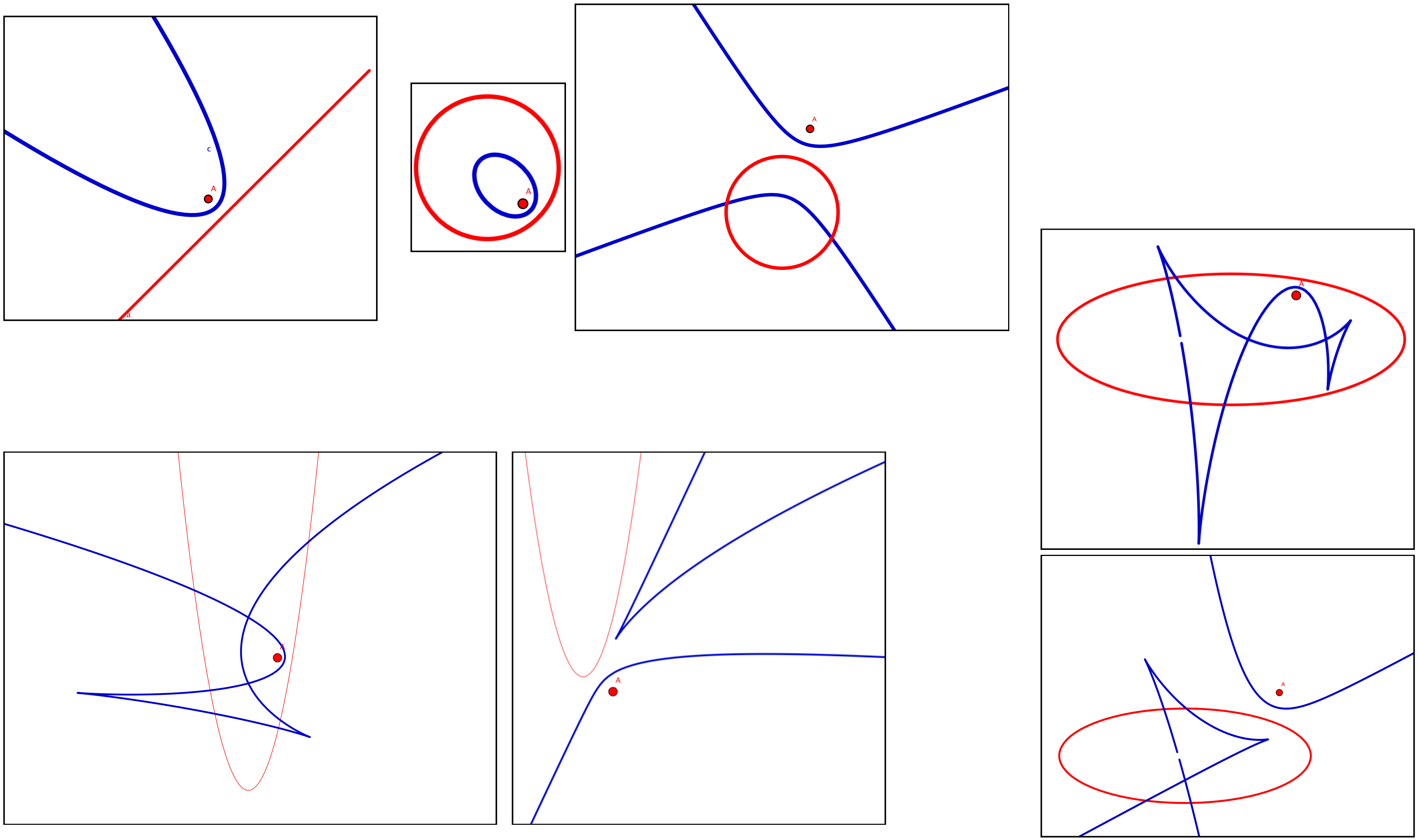
Case	Parametrization	D
point–line	RP	2
point–circle	RP	2
point–ellipse	RP	6
point–parabola	RP	5
point–hyperbola	RP	6

R. T. Farouki, J. K. Johnstone: The bisector of a point and a plane parametric curve. CAGD 11, 117-151, 1994.

The Topologies of the Bisectors

Point and Conic

Point and Conic: topologies



The Equations of the Bisectors

Conic and Conic (including lines)

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Conic and Conic: equations

Case	Parametrization	D
line–line	RP	2
line–circle	RP	4
line-ellipse	NRP	8
line–hyperbola	NRP	8
line–parabola	RP	6
circle–circle	RP	4
circle–ellipse	NRP	12
circle–hyperbola	NRP	12
circle–parabola	RP	10
ellipse–ellipse	NP	28
ellipse–hyperbola	NP	28
ellipse–parabola	NP	22
hyperbola–hyperbola	NP	28
hyperbola–parabola	NP	22
parabola–parabola	NP	17

I. Adamou: *Curvas y Superficies Bisectrices y Diagrama de Voronoi de una familia finita de semirrectas paralelas en R^3*. PhD Thesis, Universidad de Cantabria, 2013.

Conic and Conic: equations

Circle & Ellipse case:

- a parametrization involving radicals

$$\left[\frac{1}{2106} \frac{\left(9t^6 + 53t^4 + 19t^2 + 15\right)\sqrt{\left(t^4 + 7t^2 + 1\right)\left(117t^4 + 118t^2 + 21\right)^2} - 1053t^{12} - 15822t^{10} - 30403t^8 - 19612t^6 - 6523t^4 - 630t^2 + 315}{\left(\left(-\frac{1}{117}t^2 + \frac{1}{351}\right)\sqrt{\left(t^4 + 7t^2 + 1\right)\left(117t^4 + 118t^2 + 21\right)^2} + t^8 + \frac{308}{117}t^6 + \frac{256}{351}t^4 - \frac{16}{351}t^2 + \frac{7}{117}\right)\left(t^2 + 1\right)^2}, \right.$$

$$\left. -\frac{1}{702} \frac{\left(117t^6 + 233t^4 + 55t^2 - \sqrt{\left(t^4 + 7t^2 + 1\right)\left(117t^4 + 118t^2 + 21\right)^2} - 21\right) t \left(57t^4 + 54t^2 + 17\right)}{\left(\left(-\frac{1}{117}t^2 + \frac{1}{351}\right)\sqrt{\left(t^4 + 7t^2 + 1\right)\left(117t^4 + 118t^2 + 21\right)^2} + t^8 + \frac{308}{117}t^6 + \frac{256}{351}t^4 - \frac{16}{351}t^2 + \frac{7}{117}\right)\left(t^2 + 1\right)^2} \right]$$

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Conic and Conic: equations

Circle & Ellipse case:

- a parametrization involving radicals

$$\left[\begin{array}{c}
\frac{1}{2106}\,\frac{\left(9t^6+53t^4+19t^2+15\right)\sqrt{\boxed{\left(t^4+7t^2+1\right)}\left(117t^4+118t^2+21\right)^2}-1053t^{12}-15822t^{10}-30403t^8-19612t^6-6523t^4-630t^2+315}{\left(\left(-\frac{1}{117}t^2+\frac{1}{351}\right)\sqrt{\left(t^4+7t^2+1\right)\left(117t^4+118t^2+21\right)^2}+t^8+\frac{308}{117}t^6+\frac{256}{351}t^4-\frac{16}{351}t^2+\frac{7}{117}\right)\left(t^2+1\right)^2} \\
-\frac{1}{702}\,\frac{\left(117t^6+233t^4+55t^2-\sqrt{\left(t^4+7t^2+1\right)\left(117t^4+118t^2+21\right)^2}-21\right)t\left(57t^4+54t^2+17\right)}{\left(\left(-\frac{1}{117}t^2+\frac{1}{351}\right)\sqrt{\left(t^4+7t^2+1\right)\left(117t^4+118t^2+21\right)^2}+t^8+\frac{308}{117}t^6+\frac{256}{351}t^4-\frac{16}{351}t^2+\frac{7}{117}\right)\left(t^2+1\right)^2}
\end{array}\right]$$

I. Adamou: *Curvas y Superficies Bisectrices y Diagrama de Voronoi de una familia finita de semirrectas paralelas en R^3*. PhD Thesis, Universidad de Cantabria, 2013.

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The Topologies of the Bisectors

Conic and Conic

(including lines)

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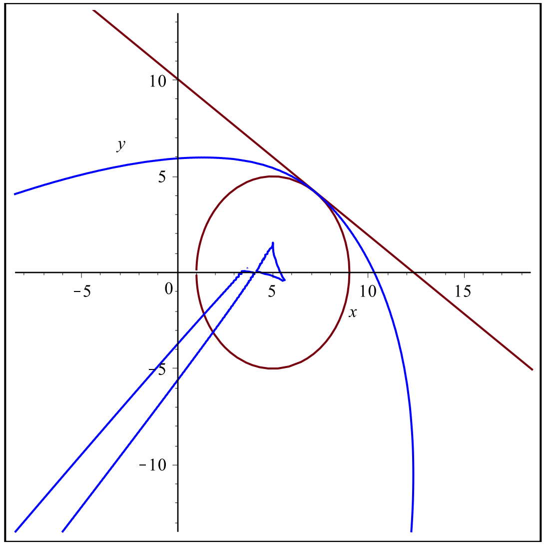
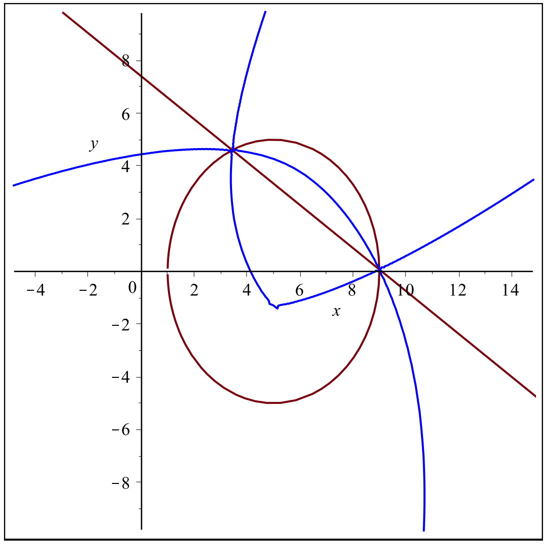
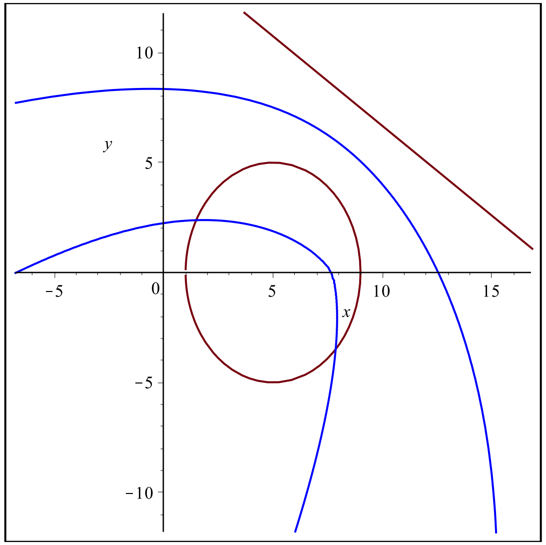
Conic and Conic: topologies

Line & Ellipse case:

- a parametrization involving radicals
- 3 possible topologies







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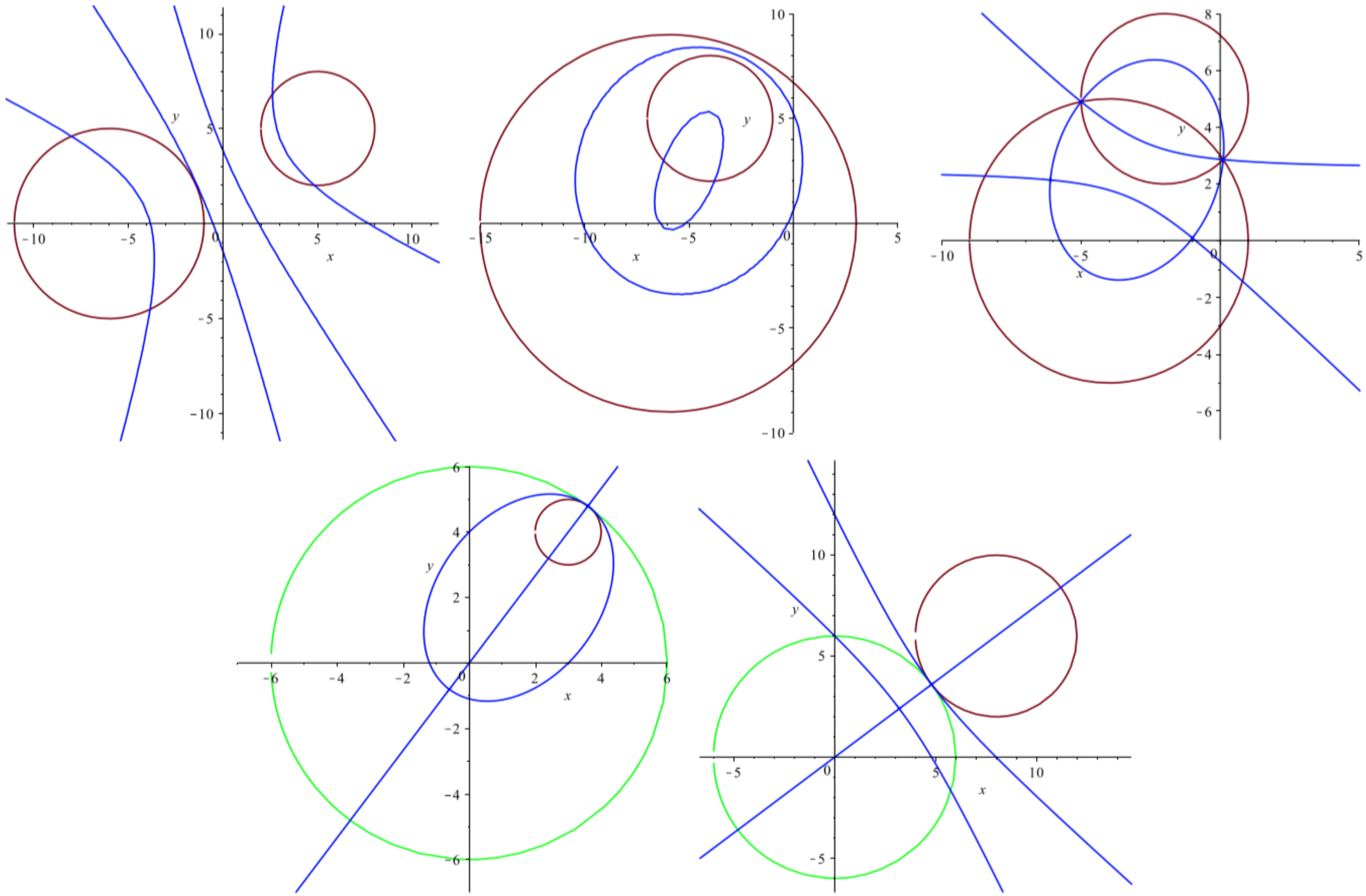
Conic and Conic: topologies

Circle & Circle case



Medial Axis

Bisectors of Curve Segments

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Bisector of two curve segments: trimming

Each segment in the medial axis comes from a point-point, point-curve or curve-curve bisector derived from the points and segments in the boundary of our domain.

Let $s_1(u)$, $(u \in [a_1, b_1])$ and $s_2(t)$, $(t \in [a_2, b_2])$ be two parametric curve segments whose bisector is to be computed. Using

$$\langle P - \mathcal{C}(u_1), \mathcal{C}'(u_1)\rangle = 0 \text{ and } \langle P - \mathcal{C}(u_2), \mathcal{C}'(u_2)\rangle = 0$$

we obtain for P a description $B(u,t)$ that, after replacement in

$$\langle P, 2(\mathcal{C}(u_2) - \mathcal{C}(u_1))\rangle + \|\mathcal{C}(u_1)\|^2 - \|\mathcal{C}(u_2)\|^2 = 0 \ ,$$

produces the following relation for the values of u and t when they generate, as footpoints, a point in the bisector of these two curve segments:

$$h(u,t) = \langle B(u,t), 2(s_1(u) - s_2(t))\rangle + \|s_2(t)\|^2 - \|s_1(u)\|^2 = 0 \ .$$

Bisector of two curve segments: trimming I

Let $s_1(u)$, $(u \in [a_1, b_1])$ and $s_2(t)$, $(t \in [a_2, b_2])$ be two parametric curve segments whose bisector is to be computed. Using

$$\langle P - \mathcal{C}(u_1), \mathcal{C}'(u_1) \rangle = 0 \text{ and } \langle P - \mathcal{C}(u_2), \mathcal{C}'(u_2) \rangle = 0$$

we obtain for P a description $B(u,t)$ that, after replacement in

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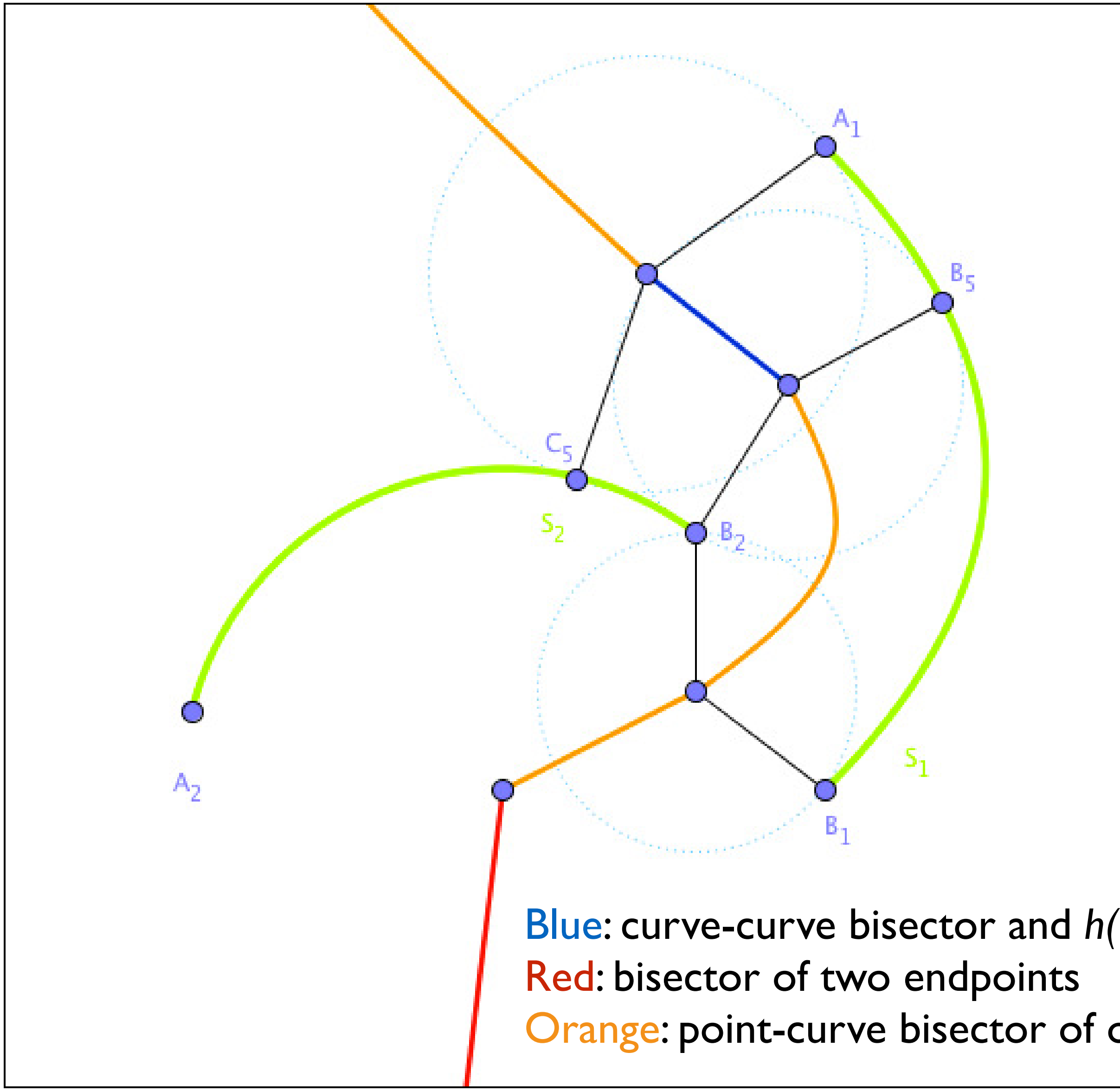
produces the following relation for the values of u and t when they generate, as footpoints, a point in the bisector of these two curve segments:

$$h(u,t) = \langle B(u,t), 2(s_1(u) - s_2(t)) \rangle + \|s_2(t)\|^2 - \|s_1(u)\|^2 = 0 \ .$$

The intersection of $h(u,t) = 0$ with the boundary of $[a_1, b_1] \times [a_2, b_2]$ together with some of the non bounded branches of the involved bisectors produces the searched bisector for the two considered curve segments.

Bisector of two curve segments: trimming II



Blue: curve-curve bisector and $h(u,t)=0$.
Red: bisector of two endpoints
Orange: point-curve bisector of one endpoint and the other curve

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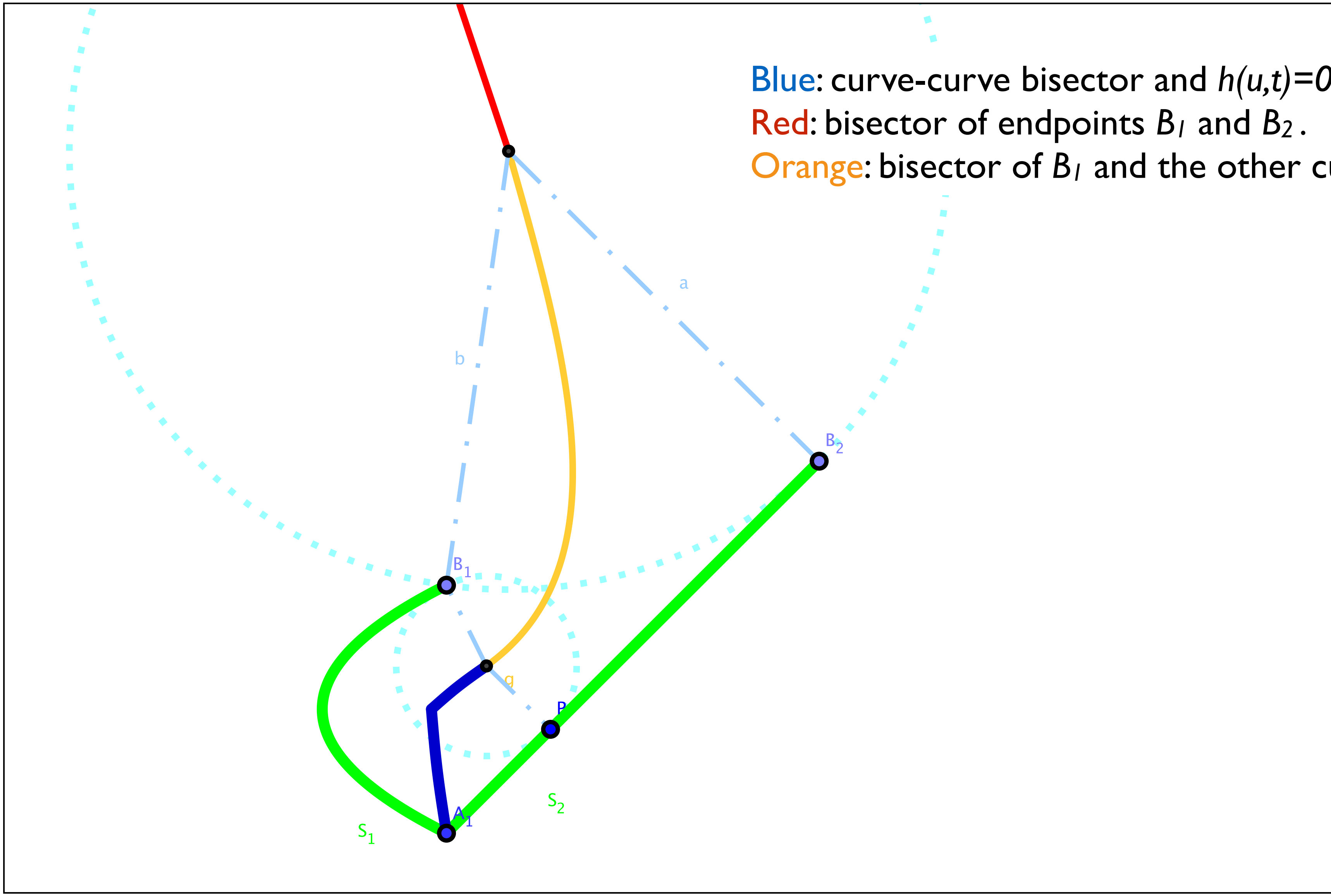
Bisector of two curve segments: trimming



Blue: curve-curve bisector and $h(u,t)=0$.
Red: bisector of endpoints B_1 and B_2 .
Orange: bisector of B_1 and the other curve.

Bisector of two curve segments: trimming

How many components?

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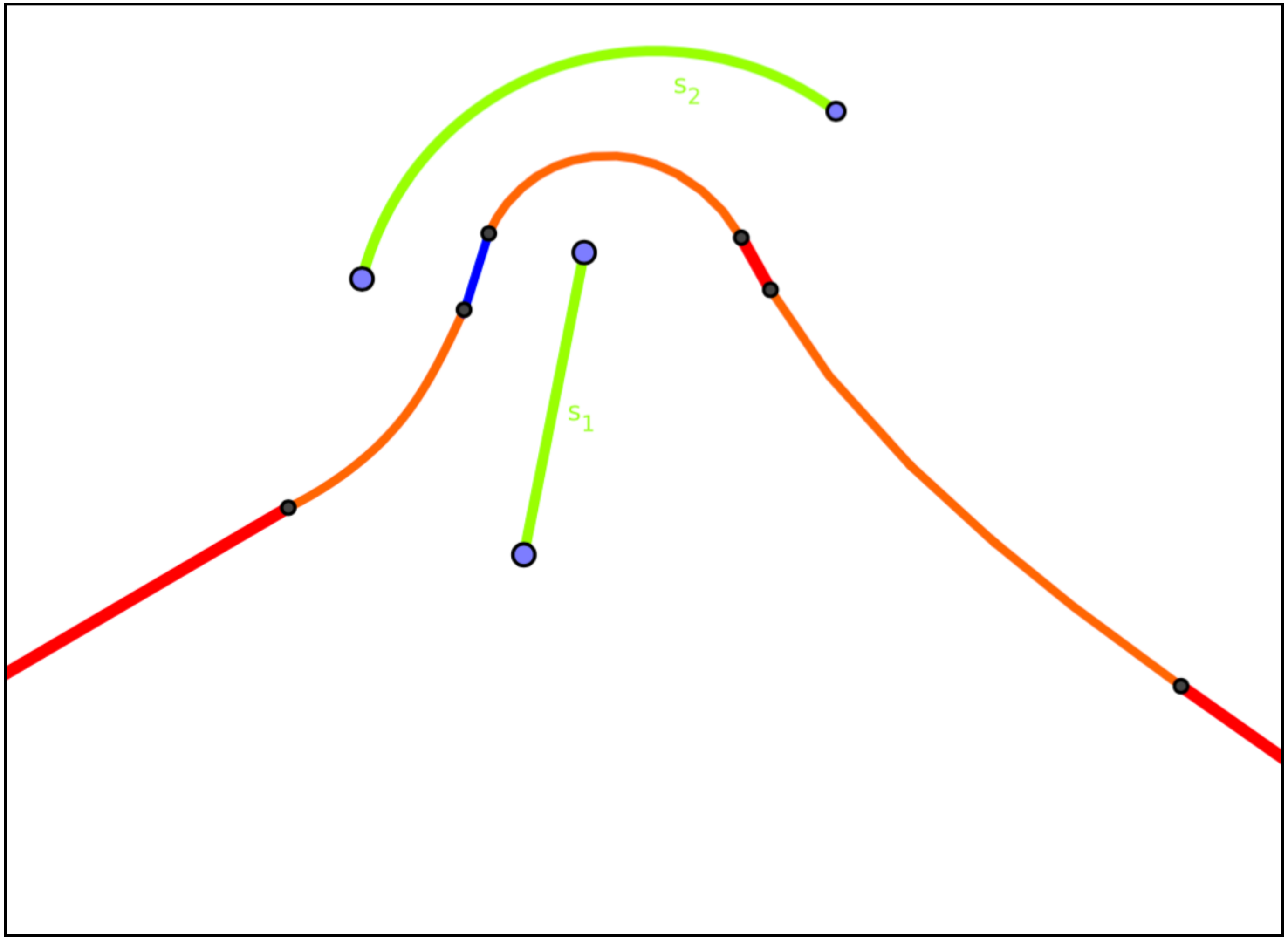
Bisector of two curve segments: trimming

How many components?



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Medial Axis

FINAL COMPUTATION

Medial axis: final computation

Boundary $\mathcal{C}$ of $\mathcal{D}$:

- finitely many bounded segments and conic arcs $\mathcal{C}_i$, $i \in \{1, 2, \ldots, n\}$.

Analyzing the arrangement of the bisectors $\mathcal{S}_{i,j}$ for $\mathcal{C}_i$ and $\mathcal{C}_j$ with $i \neq j$ inside $\mathcal{D}$ produces the medial axis:

- Checking all possible arcs in the arrangement produces the medial axis after keeping only those verifying the conditions in medial axis definition
- It is enough to check one point in each arc in order to select it or to discard it).

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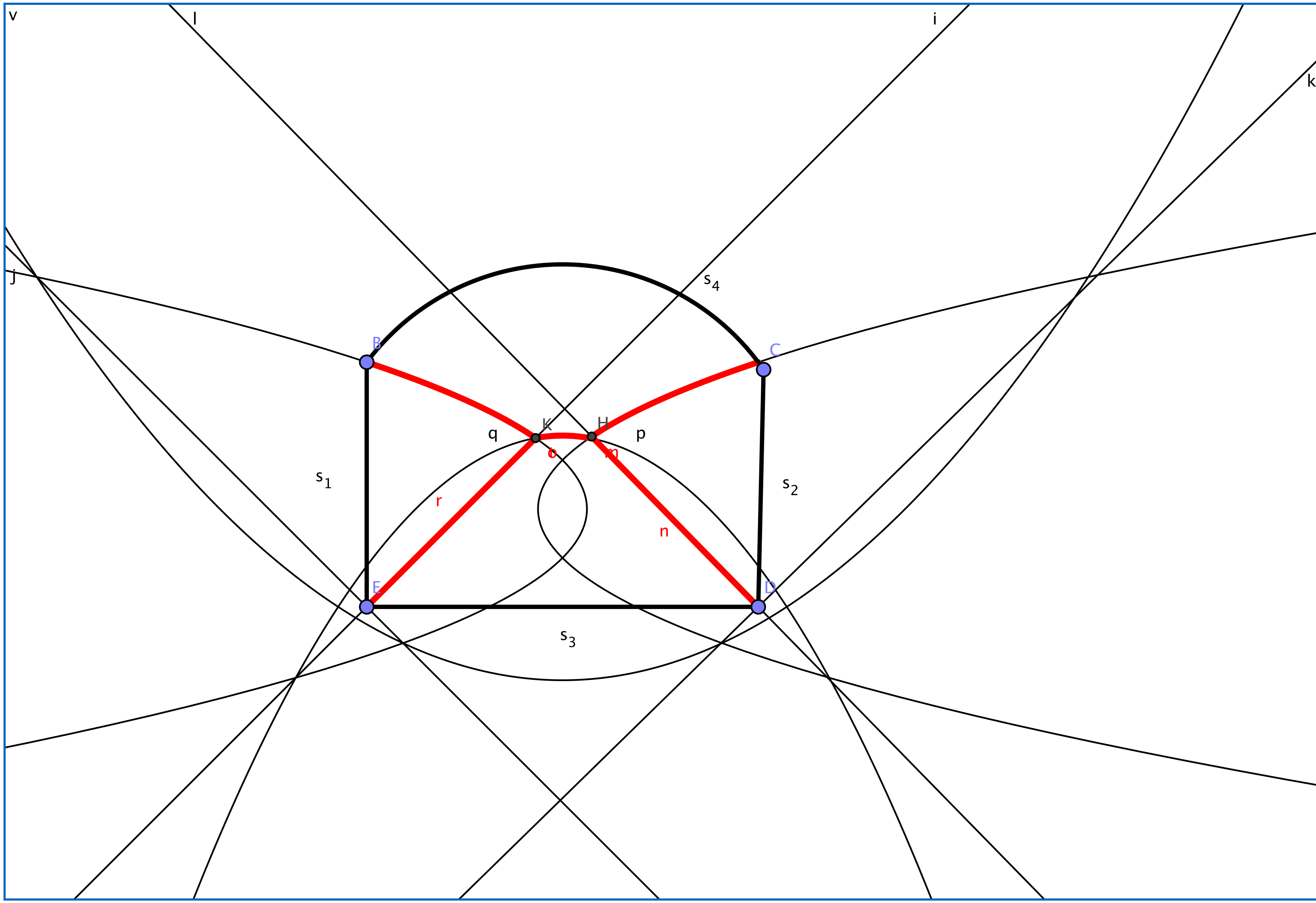
Medial axis: final computation



Medial Axis Computation (Conclusions and ...)

Conclusions and further work

Fully use the the Bentley-Ottmann sweep-line method in order to reduce and simplify the combinatorial “final” burden.

Parametric representations for the bisectors, even those involving radicals, work pretty well. Further work is required when the only available exact representation is the implicit one.

Implicit equations for all involved bisector curves are available in order to be used for answering intersection queries.

Conclusions and further work

Fully use the the Bentley-Ottmann sweep-line method in order to reduce and simplify the combinatorial “final” burden.

Parametric representations for the bisectors, even those involving radicals, work pretty well. Further work is required when the only available exact representation is the implicit one.

Implicit equations for all involved bisector curves are available in order to be used for answering intersection queries.

ellipse–ellipse	NP	28
ellipse–hyperbola	NP	28
ellipse–parabola	NP	22
hyperbola–hyperbola	NP	28
hyperbola–parabola	NP	22
parabola–parabola	NP	17

Conclusions and further work

Apart from stability issues, for exact data representation of the boundary, we provide (when conic segments define the boundary of our planar and closed domain):

- Exact representation for each medial axis component (or curve segment).
- Guaranteed topology or combinatorial description (no components are missed).

GRACIAS
THANKS
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