Noether resolutions in dimension 2

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Framework

$R := K[x_1, \ldots, x_n]$ a polynomial ring over an infinite field K.

A vector $\omega = (\omega_1, \ldots, \omega_n) \in (\mathbb{Z}^+)^n$ induces a grading in R by assigning $\deg_\omega(x_i) := \omega_i \in \mathbb{Z}^+$. Then, $R = \bigoplus_{s \in \mathbb{N}} R_s$.

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Let $I \subset R$ be a ω-homogeneous ideal.

The ideal $I = (f_1, f_2) \subset K[x_1, x_2, x_3]$ with

- $f_1 := x_1^2 - 2x_2x_3 + x_3^3$
- $f_2 := x_2^2 + 3x_3^4$

is ω-homogeneous for $\omega = (3, 4, 2) \subset (\mathbb{Z}^+)^3$.

Framework

A common aim is to try to describe a **minimal ω-graded free resolution of R/I as R-module**

$$0 \rightarrow \bigoplus_{j=1}^{\beta_p} R(-e_{pj}) \xrightarrow{\phi_p} \cdots \xrightarrow{\phi_1} \bigoplus_{j=1}^{\beta_0} R(-e_{0j}) \xrightarrow{\phi_0} R/I \rightarrow 0$$

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The **projective dimension** of R/I as R-module is $pd_R(R/I) = p$.

We say that R/I is **Cohen-Macaulay** if $\mathrm{pd}_R(R/I) = n - \dim(R/I)$.

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The ω**-weighted Hilbert series** of R/I is

$$HS_{R/I}(t) \quad := \quad \sum_{s\in\mathbb{N}}[\dim_K(R_s) - \dim_K(I \cap R_s))]\, t^s$$

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If $\omega = (1, \ldots, 1)$, the **Castelnuovo-Mumford regularity** of R/I is

$$\operatorname{reg}(R/I) := \max\{e_{ij} - i;\ 0 \le i \le p,\ 1 \le j \le \beta_i\}.$$

Today's framework

Let $d := \dim(R/I)$ and set $A := K[x_{n-d+1}, \ldots, x_n]$.

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Hence, it makes sense to study its minimal ω-graded resolution as A-module (the **Noether resolution**).

$$0 \longrightarrow \bigoplus_{j=1}^{\gamma_p} A(-s_{pj}) \xrightarrow{\psi_p} \cdots \xrightarrow{\psi_1} \bigoplus_{j=1}^{\gamma_0} A(-s_{0j}) \xrightarrow{\psi_0} R/I \longrightarrow 0$$

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The **projective dimension** of R/I as A-module is $pd_A(R/I) := p$.

In general, $0 \leq p < d$.

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3. *If $\omega = (1,\ldots,1)$, then*

   $$\mathrm{reg}(R/I) = \max\{s_{ij} - i;\ 0 \le i \le \mathrm{pd}_A(R/I),\ 1 \le j \le \gamma_i\}.$$

The tool: Gröbner bases

We consider the ω-weighted degree reverse lexicographic order $>_\omega$, defined as follows:

For $\alpha = (\alpha_1, \ldots, \alpha_n), \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, we have that $x^\alpha >_\omega x^\beta$ iff

- $\deg_\omega(x^\alpha) > \deg_\omega(x^\beta)$, or
- $\deg_\omega(x^\alpha) = \deg_\omega(x^\beta)$ and the last nonzero entry of $\alpha - \beta \in \mathbb{Z}^n$ is negative.

Notation:

- For $J \subset R$ ideal, $\mathrm{in}\,(J)$ denotes its initial ideal (w.r.t. $>_\omega$).

First step of the Noether resolution

Let $\mathcal{B}_0$ be **the set of monomials NOT belonging** to $\mathrm{in}\,(I) + (x_{n-d+1}, \ldots, x_n)$

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The set

$$\{x^\alpha + I \mid x^\alpha \in \mathcal{B}_0\} \subset R/I$$

is a minimal set of generators of R/I as A-module.

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Indeed, the first step of the Noether resolution is

$$\begin{array}{rrcl} \psi_0 : & \oplus_{v \in \mathcal{B}_0} A(-\deg_\omega(v)) & \longrightarrow & R/I \\ & e_v & \mapsto & v + I. \end{array}$$

Cohen-Macaulay criterion

Proposition

R/I is **Cohen-Macaulay** *if and only if $x_{n-d+1}, \ldots, x_n$ do not divide any minimal generator of $\mathrm{in}\,(I)$.*

This result (slightly) generalizes [Bermejo & Gimenez (2001)].

Whenever R/I is Cohen-Macaulay, the result in the previous slide provides the whole Noether resolution.

R/I is 2-dimensional

We work under the following hypotheses:

- $d = \dim(R/I) = 2$,
- $A = K[x_{n-1}, x_n]$ is a Noether normalization of R/I, and
- x_n is a nonzero divisor of R/I (or I saturated).

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Consider the evaluation map

$$\begin{array}{rccl} \chi : & R & \longrightarrow & R \\ & x_i & \mapsto & 1 \text{ for } i \in \{n-1, n\}. \end{array}$$

Theorem

Let $J := \chi(\mathrm{in}(I)) \cdot R$ and set $\mathcal{B}_1 := \mathcal{B}_0 \cap J$.

The shifts of the second step of the Noether resolution are

$$\deg_\omega(ux_{n-1}^{\delta_u}), \text{ where } u \in \mathcal{B}_1 \text{ and } \delta_u := \min\{\delta \mid ux_{n-1}^{\delta} \in \mathrm{in}(I)\}.$$

Corollary

$$HS_{R/I}(t) = \frac{\sum_{v \in \mathcal{B}_0} t^{\deg_\omega(v)} - \sum_{u \in \mathcal{B}_1} t^{\deg_\omega(u) + \delta_u \omega_{n-1}}}{(1 - t^{\omega_{n-1}})(1 - t^{\omega_n})}$$

Corollary (Bermejo-Gimenez (1999))

If $\omega = (1, \ldots, 1)$*, then:*

$$\operatorname{reg}(R/I) = \max\left(\{\deg(v) \mid v \in \mathcal{B}_0\} \cup \{\deg(u) + \delta_u - 1 \mid u \in \mathcal{B}_1\}\right)$$

A worked out example

Let $I = I(\Gamma)$ be the ideal of **the surface** $\Gamma \subset \mathbb{A}^4_K$ defined as

$$\Gamma := \{(s^3 + s^2 t, t^4 + st^3, s^2, t^2) \in \mathbb{A}^4_K \mid s, t \in K\}$$

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If $\mathrm{ch}(K) = 0$, then $\mathcal{G} = \{g_1, g_2, g_3, g_4\}$ is a Gröbner basis of I w.r.t. $>_\omega$ with $\omega = (3, 4, 2, 2)$, where

- $g_1 := 2x_2x_3^2 - x_1^2x_4 + x_3^3x_4 - x_3^2x_4^2$
- $g_2 := x_1^4 - 2x_1^2x_3^3 + x_3^6 - 2x_1^2x_3^2x_4 - 2x_3^5x_4 + x_3^4x_4^2$
- $g_3 := x_2^2 - 2x_2x_4^2 - x_3x_4^3 + x_4^4$
- $g_4 := 2x_1^2x_2 - x_1^2x_3x_4 + x_3^4x_4 - 3x_1^2x_4^2 - 2x_3^3x_4^2 + x_3^2x_4^3.$

Hence, $\mathrm{in}\,(I) = (x_2x_3^2, x_1^4, x_2^2, x_1^2x_2)$

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Hence, $\mathrm{in}\,(I) = (x_2x_3^2, x_1^4, x_2^2, x_1^2x_2) \Longrightarrow R/I$ is **not C-M**.

$$\mathrm{in}(I) = (x_2 x_3^2, x_1^4, x_2^2, x_1^2 x_2) \Rightarrow \mathrm{in}(I) + (x_3, x_4) = (x_1^4, x_2^2, x_1^2 x_2, x_3, x_4)$$

- $\mathcal{B}_0 = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ with
  $u_1 := 1,\ u_2 := x_1,\ u_3 := x_2,\ u_4 := x_1^2,\ u_5 := x_1 x_2,\ u_6 := x_1^3.$

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The **first step of the Noether resolution is:**

$$\psi_0 : \begin{array}{c} A \oplus A(-3) \oplus A(-4) \oplus \\ A(-6) \oplus A(-7) \oplus A(-9) \end{array} \to R/I \to 0,$$

induced by $e_i \mapsto u_i + I$.

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Now we have that:

- $J = \chi(\mathrm{in}(I)) = (x_1^4, x_2)$, and
- $\mathcal{B}_1 = \mathcal{B}_0 \cap J = \{x_2\} = \{u_3\}$.

We compute $\delta_3 = \min\{\delta \mid u_3 x_3^\delta \in \mathrm{in}\,(I)\} = 2$, and observe that $\deg_\omega(u_3 x_3^2) = 8$.

Hence, the **Noether resolution** is given by:

$$0 \to A(-8) \xrightarrow{\psi_1} \begin{array}{c} A \oplus A(-3) \oplus A(-4) \oplus \\ \oplus A(-6) \oplus A(-7) \oplus A(-9) \end{array} \to R/I \to 0,$$

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Moreover, the remainder of the division of $x_3^2 u_3$ by $\mathcal{G}$ is:

$$-x_4 u_4 + (x_3^3 x_4 - x_3^2 x_4^2) u_1 .$$

Thus, ψ_1 is given by the matrix

$$\begin{pmatrix} -x_3^3 x_4 + x_3^2 x_4^2 \\ 0 \\ x_3^2 \\ x_4 \\ 0 \\ 0 \end{pmatrix}$$

Since the Noether resolution is:

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Then, the ω-weighted Hilbert series of R/I is

$$HS_{R/I}(t) = \frac{1 + t^3 + t^4 + t^6 + t^7 - t^8 + t^9}{(1 - t^2)^2}.$$

Simplicial semigroup rings

For $\mathcal{S} = \langle a_1, \ldots, a_n \rangle \subset \mathbb{N}^d$ a finitely generated semigroup, we consider the semigroup ring $K[\mathcal{S}] := K[t^s \mid s \in \mathcal{S}] \subset K[t_1, \ldots, t_d]$.

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We have that $K[\mathcal{S}] \simeq R/I_{\mathcal{S}}$ with $I_{\mathcal{S}} = \mathrm{Ker}(\varphi)$ and

$$\begin{array}{rccl} \varphi : & R & \to & K[t_1, \ldots, t_d] \\ & x_i & \mapsto & t^{a_i} = t_1^{a_{i1}} \cdots t_d^{a_{id}} \textit{ for all } i \in \{1, \ldots, n\}. \end{array}$$

Moreover, $I_{\mathcal{S}}$ is $\mathcal{S}$-graded, i.e., multigraded with respect to the grading $\deg_{\mathcal{S}}(x_i) = a_i \in \mathbb{N}^m$ for all $i \in \{1, \ldots, n\}$.

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We also assume that $A = K[x_{n-d+1}, \ldots, x_n] \hookrightarrow R/I_{\mathcal{S}}$ is a Noether normalization (this is equivalent to $\mathcal{S}$ simplicial semigroup).

In this setting we may consider a **$\mathcal{S}$-graded Noether resolution** of $K[\mathcal{S}]$, i.e., a minimal multigraded free resolution of $K[\mathcal{S}]$ as A-module:

$$0 \longrightarrow \oplus_{s\in\mathcal{S}_p} A\cdot s \xrightarrow{\psi_p} \cdots \xrightarrow{\psi_1} \oplus_{s\in\mathcal{S}_0} A\cdot s \xrightarrow{\psi_0} K[\mathcal{S}] \longrightarrow 0,$$

where $\mathcal{S}_i \subset \mathcal{S}$ for all $i \in \{0,\ldots,p\}$ and $A\cdot s$ denotes the shifting of A by $s\in\mathcal{S}$.

First step of the multigraded Noether resolution

The first step of this resolution corresponds to a minimal set of generators of $K[\mathcal{S}]$ as A-module and is given by the following well known result.

Proposition

$$\mathcal{S}_0 = \{s \in \mathcal{S} \mid s - a_i \notin \mathcal{S} \text{ for all } i \in \{n-d+1, \ldots, n\}\}.$$

Moreover,

$$\begin{array}{rccc} \psi_0 : & \oplus_{s \in \mathcal{S}_0} A \cdot s & \longrightarrow & K[\mathcal{S}] \\ & e_s & \mapsto & t^s \end{array}$$

Cohen-Macaulay criterion

From the previous result we derive the following one, which can be seen to be equivalent to [Goto-Suzuki-Watanabe (1976)] and [Stanley (1978)].

Proposition

$$K[\mathcal{S}] \text{ is } \textbf{Cohen-Macaulay} \iff |\mathcal{S}_0| = [\mathbb{Z}\mathcal{S} : \mathbb{Z}\mathcal{S}'],$$

where $\mathbb{Z}\mathcal{S}$ *is the group generated by* $a_1, \ldots, a_n$ *and* $\mathbb{Z}\mathcal{S}'$ *is the group generated by* $a_{n-d+1}, \ldots, a_n$.

$K[\mathcal{S}]$ is 2-dimensional

From now on suppose that $K[\mathcal{S}]$ is a 2-dimensional semigroup ring. The second step of the Noether resolution is given by the following result.

Theorem

$$\mathcal{S}_1 = \{s \in \mathcal{S} \mid s - a_{n-1}, s - a_n \in \mathcal{S} \text{ and } s - a_n - a_{n-1} \notin \mathcal{S}\}.$$

An application

Let $\mathcal{C} \subset \mathbb{P}^n_K$ be a projective monomial curve:

$$x_1 = s^{m_1} t^{m_n - m_1}, \ldots, x_{n-1} = s^{m_{n-1}} t^{m_n - m_{n-1}}, x_n = s^{m_n}, x_{n+1} = t^{m_n}.$$

We set $\mathcal{S} := \langle a_1, \ldots, a_{n+1} \rangle \subset \mathbb{N}^2$ with $a_i := (m_i, m_n - m_i)$ for $i \in \{1, \ldots, n-1\}$, $a_n := (m_n, 0)$ and $a_{n+1} := (0, m_n)$, then

$$K[\mathcal{S}] \simeq K[x_1, \ldots, x_{n+1}]/I(\mathcal{C}).$$

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$$K[\mathcal{S}] \simeq K[x_1, \dots, x_{n+1}]/I(\mathcal{C}).$$

For $r \in \{2, \dots, n-1\}$ one may consider the r-th canonical projection of $\mathcal{C}$, i.e., the curve $\mathcal{C}_r := \pi_r(\mathcal{C})$ where

$$\pi_r(p_1 : \cdots : p_{n+1}) = (p_1 : \cdots : p_{r-1} : p_{r+1} : \cdots : p_{n+1})$$

We have that $K[\mathcal{S}_r] \simeq R/I(\mathcal{C}_r)$, where

$$\mathcal{S}_r = \langle a_1, \dots, a_{r-1}, a_{r+1}, \dots, a_{n+1} \rangle.$$

Arithmetic sequences

Theorem (Li-Patil-Roberts (2012))

If $m_1 < \cdots < m_n$ is an arithmetic sequence, i.e., there exist $e, m_1 \in \mathbb{Z}^+$ such that $m_i = m_1 + (i-1)\,e$ for all $i \in \{1, \ldots, n\}$, then:

$$\mathcal{S}_0 = \left\{ \left(\left\lceil \frac{j}{n-1} \right\rceil m_n - je,\, je \right) \mid j \in \{0, \ldots, m_n - 1\} \right\}$$

In particular, $K[\mathcal{S}]$ is **Cohen-Macaulay**, and this result provides the shifts of the only step of the multigraded Noether resolution.

Now, we study the Noether resolution of the canonical projections of $K[\mathcal{S}]$.

Theorem

Let $q := \lceil (m_1 - 1)/(n-1) \rceil$ *and* $\ell := m_1 - q(n-1)$*. If we set* $s_\lambda := \left(\left\lceil \frac{\lambda}{n-1} \right\rceil m_n - \lambda e, \lambda e \right) \in \mathbb{N}^2$ *for all* $\lambda \in \{0, \ldots, m_n - 1\}$*, then the* **multigraded Noether resolution of** $K[\mathcal{S}_r]$ *is given by:*

- *For* $m_1 \geq 2$*, then*

$$0 \to \left(\oplus_{\lambda=0,\, \lambda \notin \Lambda_1}^{m_n - 1} A \cdot s_\lambda \right) \oplus \left(\oplus_{\lambda \in \Lambda_1} A \cdot (s_\lambda + a_n) \right) \to K[\mathcal{S}_2] \to 0,$$

  where $\Lambda_1 := \{\mu(n-1) - 1 \,|\, 1 \leq \mu \leq q + e + \epsilon\}$*, and* $\epsilon = 1$ *if* $\ell = n-1$*, or* $\epsilon = 0$ *otherwise.*

- *For* $r \in \{3, \ldots, n-2\}$ *and* $r \leq m_1$*, then*

$$0 \to \left(\oplus_{\lambda=0,\, \lambda \neq n-r}^{m_n - 1} A \cdot s_\lambda \right) \oplus A \cdot (a_r + a_n) \to K[\mathcal{S}_r] \to 0$$

- *For* $r = n-1 \leq m_1$*, then*

$$0 \to \left(\oplus_{\lambda=0,\, \lambda \neq 1}^{m_n - 1} A \cdot s_\lambda \right) \oplus A \cdot (a_{n-1} + (q+\epsilon) a_n + e a_{n+1}) \to K[\mathcal{S}_{n-1}] \to 0,$$

  where $\epsilon = 1$ *if* $\ell = n-1$*, or* $\epsilon = 0$ *otherwise.*

Theorem (Continuation)

- *For* $m_1 = 1$*, then*

$$0 \to \oplus_{\lambda\in\Lambda_2} A\cdot(s_\lambda + a_n + ea_{n+1}) \to \begin{array}{c} \oplus_{\lambda=0,\lambda\notin\Lambda_2}^{m_n-1} A\cdot s_\lambda \\ \oplus \\ \oplus_{\lambda\in\Lambda_2} A\cdot(s_\lambda + a_n) \\ \oplus \\ \oplus_{\lambda\in\Lambda_2} A\cdot(s_\lambda + ea_{n+1}) \end{array} \to K[\mathcal{S}_2] \to 0,$$

where $\Lambda_2 := \{\mu(n-1) - 1 \mid 1 \le \mu \le e\}$.

- *For* $r \in \{3, \ldots, n-2\}$ *and* $r > m_1$*, then*

$$0 \to A\cdot(a_r + a_n + ea_{n+1}) \to \begin{array}{c} \left(\oplus_{\lambda=0,\ \lambda\neq n-r}^{m_n-1} A\cdot s_\lambda\right) \\ \oplus \\ A\cdot(a_r + a_n) \oplus A\cdot(a_r + ea_{n+1}) \end{array} \to K[\mathcal{S}_r] \to 0.$$

- *For* $r = n-1 > m_1$*, then*

$$0 \to \left(\oplus_{\lambda=0,\ \lambda\neq 1}^{m_n-1} A\cdot s_\lambda\right) \oplus A\cdot(a_{n-1} + ea_{n+1}) \to K[\mathcal{S}_{n-1}] \to 0.$$

Corollary

$K[\mathcal{S}_r]$ *is* **Cohen-Macaulay** $\Longleftrightarrow r \leq m_1$ *or* $r = n-1$.

Corollary

$$\operatorname{reg}(K[\mathcal{S}_r]) = \begin{cases} \left\lceil \frac{m_n - 1}{n-1} \right\rceil + 1 & \text{if } r \in \{2, n-1\} \text{ and } r \leq m_1, \\ 2e & \text{if } r = 2 \text{ and } m_1 = 1, \text{ and} \\ \left\lceil \frac{m_n - 1}{n-1} \right\rceil & \text{if } r \in \{3, \ldots, n-2\}, \text{ or} \\ & \text{if } r = n-1 \text{ and } m_1 < r \end{cases}$$

Consider the projective curve given parametrically by:

$$x_1 = st^6, x_2 = s^5t^2, x_4 = s^7, x_5 = t^7.$$

This corresponds to $\mathcal{C}_2$, where $\mathcal{C}$ is associated to the arithmetic sequence $1 < 3 < 5 < 7$ (i.e., with $m_1 = 1$, $d = 2$ and $n = 4$).

The **multigraded Noether resolution** of $K[\mathcal{S}_2]$ is

$$0 \to \begin{array}{c} A \cdot (10,18) \\ \oplus \\ A \cdot (11,24) \end{array} \to \begin{array}{c} A \oplus A \cdot (1,6) \oplus A \cdot (5,2) \oplus \\ A \cdot (2,12) \oplus A \cdot (6,8) \oplus \\ A \cdot (10,4) \oplus A \cdot (3,18) \oplus \\ A \cdot (11,10) \oplus A \cdot (4,24) \end{array} \to K[\mathcal{S}_2] \to 0.$$

If we consider the standard grading on R, we get:

$$0 \to A(-4) \oplus A(-5) \to \begin{array}{c} A \oplus A(-1)^2 \oplus A(-2)^3 \\ A(-3)^2 \oplus A(-4) \end{array} \to K[\mathcal{S}_2] \to 0,$$

and the following expression for the **Hilbert series** of $K[\mathcal{S}_2]$:

$$HS_{K[\mathcal{S}_2]}(t) = \frac{1 + 2t + 3t^2 + 2t^3 - t^5}{(1-t)^2}.$$

We also have that $\mathrm{reg}(K[\mathcal{S}_2]) = 4$.