Algebraic invariants of projective monomial curves associated to generalized arithmetic sequences

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Joint work with I. Bermejo and I. García Marco

DEFINITIONS AND NOTATIONS

Let $m_1 < \ldots < m_n$ be a sequence of positive integers. Consider **the projective monomial curve** associated to $m_1 < \ldots < m_n$ $\mathcal{C} \subset \mathbb{P}^n_K$ parametrically defined by

$$x_1 = s^{m_1} t^{m_n - m_1}, \ldots, x_{n-1} = s^{m_{n-1}} t^{m_n - m_{n-1}}, x_n = s^{m_n}, x_{n+1} = t^{m_n}.$$

Consider the k-algebra homomorphism

$$\begin{array}{rcl} \varphi : R & \longrightarrow & K[s,t] \\ x_i & \mapsto & s^{m_i} t^{m_n - m_i}, \text{ for } i \in \{1, \dots, n-1\}, \\ x_n & \mapsto & s^{m_n}, \\ x_{n+1} & \mapsto & t^{m_n}, \end{array}$$

where $R := K[x_1, \dots, x_{n+1}]$ with K an infinite field. Since K is infinite, $\ker(\varphi) \subset R$ is the vanishing ideal $\boldsymbol{I(\mathcal{C})}$ of $\mathcal{C}$, which is binomial, prime and homogeneous. Denote by $\boldsymbol{K[\mathcal{C}]}$ the homogeneous coordinate ring $R/I(\mathcal{C})$ of $\mathcal{C}$.

Consider a minimal graded free resolution of $K[\mathcal{C}]$

$$0 \to \bigoplus_{j=1}^{\beta_p} R(-e_{pj}) \xrightarrow{\phi_p} \cdots \xrightarrow{\phi_2} \bigoplus_{j=1}^{\beta_1} R(-e_{1j}) \xrightarrow{\phi_1} R \to K[\mathcal{C}] \to 0,$$

then,

- the **Castelnuovo-Mumford regularity** of $K[\mathcal{C}]$ is $\mathbf{reg}(K[\mathcal{C}]) := \max\{e_{ij} - i;\ 1 \leq i \leq p,\ 1 \leq j \leq \beta_i\}$,
- $K[\mathcal{C}]$ is **Cohen-Macaulay** if $p = n - 1$;

Whenever $K[\mathcal{C}]$ is Cohen-Macaulay,

- its **Cohen-Macaulay type** is β_p,
- it is **Gorenstein** if its Cohen-Macaulay type is one, and
- $I(\mathcal{C})$ is a **complete intersection** if $\beta_1 = n - 1$.

Moreover

- $K[\mathcal{C}]$ is a **Koszul algebra** if the resolution of the residue class field K over $K[\mathcal{C}]$ is linear.
- $\mathcal{H}_{K[\mathcal{C}]}(t) := \sum_{s\in\mathbb{N}} \dim_K(R_s/I(\mathcal{C}) \cap R_s)\, t^s \in \mathbb{Z}[[t]]$ is the **Hilbert series** of $K[\mathcal{C}]$, where where R_s is the K-vector space generated by all homogeneous polynomials of degree s, $s \in \mathbb{N}$.

IN THIS WORK

we consider $\mathcal{C}$ the projective monomial curve associated to $m_1 < \ldots < m_n$ a **generalized arithmetic sequence**, i.e., there exist $m_1, h, d \in \mathbb{Z}^+$ such that $m_i = hm_1 + (i-1)d$ for all $i \in \{2, \ldots, n\}$.

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In this setting, we characterize both the Cohen-Macaulay and Koszul properties of $K[\mathcal{C}]$. Moreover, we obtain **formulas** for $\mathrm{reg}(K[\mathcal{C}])$ and also for $\mathcal{H}_{K[\mathcal{C}]}(t)$ in terms of the sequence.

MAIN TOOL

- $a, b \in \mathbb{N}^{n+1}$, $x^a >_{rlex} x^b$ if and only if $\deg(x^a) > \deg(x^b)$ or $\deg(x^a) = \deg(x^b)$ and the last non-zero entry of $a - b \in \mathbb{Z}^{n+1}$ is negative.

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- $f \in R$ a polynomial, $\mathrm{in}(f)$ is the greatest monomial of f w.r.t. $>_{\mathrm{rlex}}$. Moreover $\mathrm{in}(I) := \langle \{\mathrm{in}(f) \mid f \in I\} \rangle$ for any $I \subset R$ ideal.

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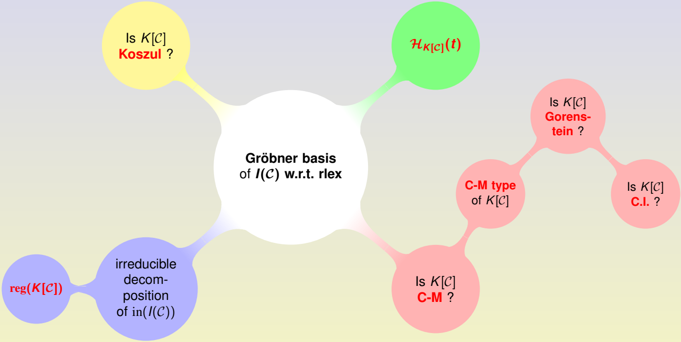
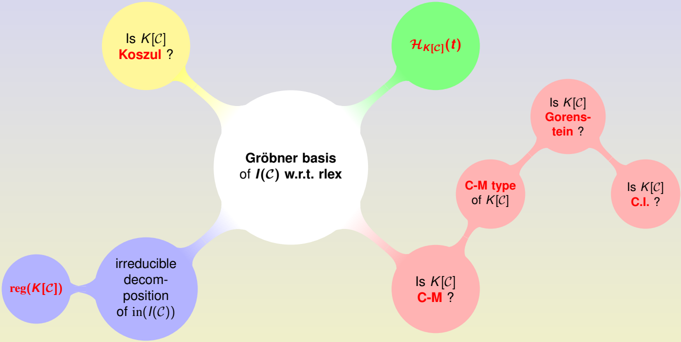
- $f \in R$ a polynomial, $\mathbf{in}(f)$ is the greatest monomial of f w.r.t. $>_{\mathrm{rlex}}$. Moreover $\mathbf{in}(I) := \langle \{\mathrm{in}(f) \mid f \in I\} \rangle$ for any $I \subset R$ ideal.
- $\mathcal{G} = \{g_1, \ldots, g_s\} \subset I$ is a **Gröbner basis** of $I \subset R$ if $\mathrm{in}(I) = \langle \{\mathrm{in}(g_1), \ldots, \mathrm{in}(g_s)\} \rangle$.

STRATEGY



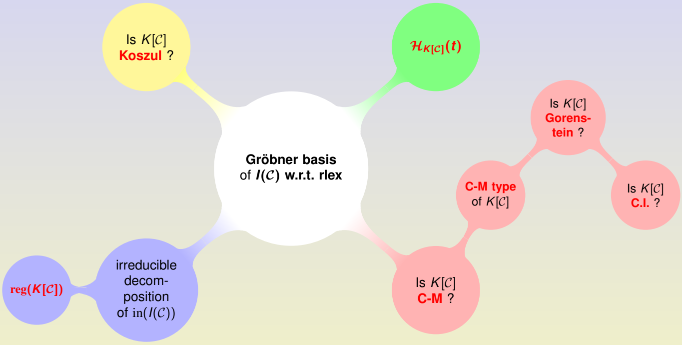
Gröbner basis of $I(\mathcal{C})$ w.r.t. rlex
Is $K[\mathcal{C}]$ Koszul ?
$\mathcal{H}_{K[\mathcal{C}]}(t)$
Is $K[\mathcal{C}]$ Gorens-tein ?
C-M type of $K[\mathcal{C}]$
Is $K[\mathcal{C}]$ C.I. ?
Is $K[\mathcal{C}]$ C-M ?
irreducible decom-position of $\mathrm{in}(I(\mathcal{C}))$
$\mathrm{reg}(K[\mathcal{C}])$

STRATEGY



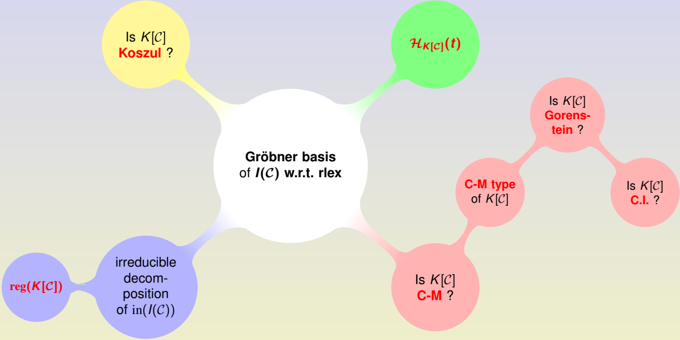
$$\mathcal{H}_{K[\mathcal{C}]}(t) = \mathcal{H}_{R/\mathrm{in}(I(\mathcal{C}))}(t)$$

STRATEGY



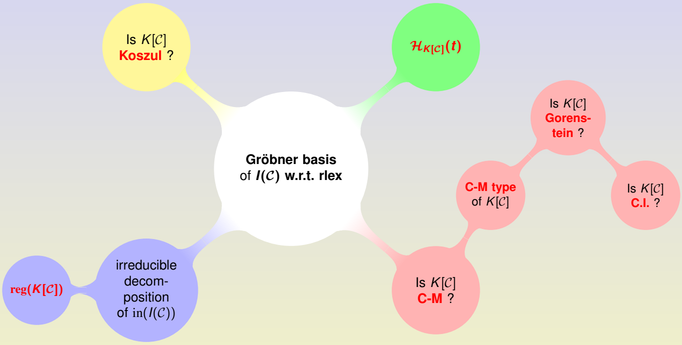
$K[\mathcal{C}]$ is C-M $\iff$ none of the minimal generators of $\mathrm{in}(I(\mathcal{C}))$ is divisible by x_n, x_{n+1}.

STRATEGY



$I(\mathcal{C})$ possesses a quadratic Gröbner basis $\Rightarrow$ $K[\mathcal{C}]$ is Koszul $\Rightarrow$ $I(\mathcal{C})$ is generated by quadrics.

STRATEGY



$$
\begin{aligned}
\mathrm{reg}(K[\mathcal{C}]) &= \mathrm{reg}(R/\mathrm{in}(I(\mathcal{C}))) \\
&= \max\{\mathrm{reg}(R/\mathfrak{q}_i) \mid \mathrm{in}(I(\mathcal{C})) = \cap_i \mathfrak{q}_i\}
\end{aligned}
$$

ARITHMETIC SEQUENCES

Let $\boldsymbol{m_1} < \cdots < \boldsymbol{m_n}$ be an **arithmetic sequence** of relatively prime positive integers, i.e., there exist $\boldsymbol{d}, \boldsymbol{m_1} \in \mathbb{Z}^+$ such that $m_i = m_1 + (i-1)d$ for all $i \in \{1, \ldots, n\}$, where $\gcd\{m_1, d\} = 1$.

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Theorem

$$\begin{aligned} \mathcal{G} \;=\; & \{\boldsymbol{x_i x_j} - x_{i-1}x_{j+1} \mid 2 \leq i \leq j \leq n-1\} \cup \\ & \{\boldsymbol{x_1^{\alpha} x_i} - x_{n+1-i}x_n^q x_{n+1}^d \mid 1 \leq i \leq k\}. \end{aligned}$$

where

$\boldsymbol{q} := \lfloor \frac{m_1 - 1}{n-1} \rfloor$,

$\alpha := q + d$ y

$\boldsymbol{k} := n - m_1 + \lfloor \frac{m_1 - 1}{n-1} \rfloor (n-1)$,

is a minimal Gröbner basis of $I(\mathcal{C})$ w.r.t. rlex.

Corollary

- $\mathrm{in}(I(\mathcal{C})) = \langle\{\boldsymbol{x_i x_j};\ 2 \leq i \leq j \leq n-1\}\rangle + \langle\{\boldsymbol{x_1^{\alpha} x_i};\ 1 \leq i \leq k\}\rangle.$

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- The C-M type of $K[\mathcal{C}]$ is $m_1 - 1 - \lfloor \frac{m_1-2}{n-1} \rfloor (n-1)$.

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- The C-M type of $K[\mathcal{C}]$ is $m_1 - 1 - \lfloor \frac{m_1-2}{n-1} \rfloor (n-1)$.
- $K[\mathcal{C}]$ is Gorenstein $\Longleftrightarrow m_1 \equiv 2 \pmod{n-1}$
- $I(\mathcal{C})$ is a C.I. $\Longleftrightarrow n = 3$ and m_1 is even.

- $\operatorname{reg}(K[\mathcal{C}]) = \left\lceil \frac{m_1-1}{n-1} \right\rceil + d.$

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- $\mathcal{H}_{K[\mathcal{C}]}(t) = \frac{1+(n-1)(t+\cdots+t^{\alpha})+(n-1-k)t^{\alpha+1}}{(1-t)^2}$.

Example

We consider $m_1 < \cdots < m_5$ with $m_1 = 10$, $m_2 = 13$, $m_3 = 16$, $m_4 = 19$ and $m_5 = 22$. Then:

- the C-M type $K[\mathcal{C}]$ is $10 - 1 - \lfloor \frac{10-2}{4} \rfloor 4 = 1$,
- $K[\mathcal{C}]$ is Gorenstein, and
- $I(\mathcal{C})$ is not a complete intersection.

Moreover, since $\alpha = \lfloor \frac{10-1}{4} \rfloor + 3 = 5$,

- $\operatorname{reg}(K[\mathcal{C}]) = \left\lceil \frac{10-1}{5-1} \right\rceil + 3 = 6$, and
- $\mathcal{H}_{K[\mathcal{C}]}(t) = \frac{1+\sum_{i=1}^{5} 4\, t^i + t^6}{(1-t)^2}$.

GENERALIZED ARITHMETIC SEQUENCES

Let $m_1 < \cdots < m_n$ be a sequence of positive integers containing a generalized arithmetic sequence $m_{i_1} < \cdots < m_{i_l} = m_n$ with $l \geq 3$, i.e., there exist $\boldsymbol{m_{i_1}}$, $\boldsymbol{h}$ and $\boldsymbol{d}$ such that $m_{i_j} = hm_{i_1} + (j-1)d$ for all $j \in \{2, \ldots, l\}$

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Theorem

If $h > 1$ and $hm_{i_1} \notin \{m_1, \ldots, m_n\}$, then $K[\mathcal{C}]$ is non Cohen Macaulay.

Let $m_1 < \cdots < m_n$ a generalized arithmetic sequence.

Corollary

$K[\mathcal{C}]$ is Cohen-Macaulay $\Longleftrightarrow h = 1$ (arithmetic sequence)

Theorem

$K[\mathcal{C}]$ is a Koszul algebra $\Longleftrightarrow h = 1$, $d = 1$ and $n > m_1$.

GENERALIZED ARITHMETIC SEQUENCES such that *h* divides *d*

Consider $\mathcal{C}'$ the projective monomial curve associated to $m_2 < \cdots < m_n$.

Why do we consider that h divides d?

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Why do we consider that h divides d?

Proposition

$$\mathrm{in}(I(\mathcal{C}')) = \mathrm{in}(I(\mathcal{C})) \cap K[x_2, \ldots, x_{n+1}].$$

Theorem ($h > 1$)

$$\begin{aligned}\mathcal{G} \;:=\; \mathcal{G}_1 \;&\cup\; \{x_1^h x_i - x_2 x_{i-1} x_{n+1}^{h-1} \mid 3 \le i \le n\} \\ &\cup\; \{x_1^{jh} x_2^{\beta_j} - x_{\sigma_j} x_n^{\lambda_j} x_{n+1}^{j(h-1)+(d/h)} \mid 1 \le j \le \delta/h\},\end{aligned}$$

where

$\mathcal{G}_1 \subset K[x_2, \ldots, x_n, x_{n+1}]$ *is a minimal Gröbner basis of* $I(\mathcal{C}')$,

$\delta := \left(\lfloor \frac{m_1-1}{n-1} \rfloor + 1\right) h + d$,

$\beta_{\delta/h-j} := j + \lfloor (j+r-2)/(n-2) \rfloor, \; \forall j \in \{1, \ldots, \delta/h - 1\}$,

con $r = m_1 - \lfloor \frac{m_1-1}{n-1} \rfloor (n-1)$,

$\sigma_{\delta/h-j} \in \{3, \ldots, n\} \mid \sigma_{\delta/h-j} \equiv r + j - 1 \pmod{n-2}$, y

$\lambda_{\delta/h-j} := p + \lfloor (j+r-2)/(n-2) \rfloor, \; \forall j \in \{1, \ldots, \delta/h - 1\}$,

is a Gröbner basis of $I(\mathcal{C})$ *w.r.t rlex.*

Corollary

$$\begin{aligned}\mathrm{in}(\mathbf{I}(\mathcal{C})) &= \langle\{\mathrm{in}(g) \mid g \in \mathcal{G}_1\}\rangle \\ &+ \langle\{\boldsymbol{x_1^h x_i};\ 3 \le i \le n\}\rangle \\ &+ \langle\{\boldsymbol{x_1^{jh} x_2^{\beta_j}};\ 1 \le j \le \delta/h\}\rangle\end{aligned}$$

- $\operatorname{reg}(K[\mathcal{C}]) = \begin{cases} \left(\lfloor \frac{m_1-1}{n-1} \rfloor + 1\right) h + d - 1, & \text{if } n-1 \nmid m_1, \\ \left(\lfloor \frac{m_1-1}{n-1} \rfloor + 1\right) h + d, & \text{if } n-1 \mid m_1. \end{cases}$.

- $\operatorname{reg}(K[\mathcal{C}]) = \begin{cases} \left(\lfloor \frac{m_1-1}{n-1} \rfloor + 1\right) h + d - 1, & \text{if } n-1 \not| m_1, \\ \left(\lfloor \frac{m_1-1}{n-1} \rfloor + 1\right) h + d, & \text{if } n-1 | m_1. \end{cases}$.
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- $\mathrm{reg}(K[\mathcal{C}])$ is attained at the last step of the resolution of $K[\mathcal{C}]$.
- $\mathcal{H}_{K[\mathcal{C}]}(t) =$
  $(1 + \cdots + t^{h-1})\mathcal{H}_{K[\mathcal{C}']}(t) + \frac{\sum_{j=h}^{\delta-1} t^j - \left(\sum_{j=0}^{h-1} t^j\right)\left(\sum_{i=1}^{(\delta/h)-1} t^{ih+\beta_i}\right)}{(1-t)^2}$.

Consider the sequence $7 < 30 < 39 < 48 < 57 < 66$. Notice that the sequence $30 < 39 < 48 < 57 < 66$ defines the same projective monomial curve that the sequence $10 < 13 < 16 < 19 < 22$. We observe that $h = 3$ divides $d = 9$. Then

- $\mathrm{reg}(K[\mathcal{C}]) = \left(\left\lfloor \frac{7-1}{6-1} \right\rfloor + 1\right) 3 + 9 - 1 = 14$, since $5 \not| 7$.
- We compute $r = 7 - \left\lfloor \frac{7-1}{6-1} \right\rfloor 5 = 2$, $\beta_1 = 4 + \lfloor (4+2-2)/4 \rfloor = 5$, $\beta_2 = 3$, $\beta_3 = 2$ y $\beta_4 = 1$, and in the previous example we computed $\mathcal{H}_{K[\mathcal{C}']}(t)$, so that

$$\mathcal{H}_{K[\mathcal{C}]}(t) = (1+t+t^2)\left(\frac{1+\sum_{i=1}^{5} 4\, t^i + t^6}{(1-t)^2}\right) + \frac{\sum_{j=3}^{14} t^j - (1+t+t^2)(t^8+t^9+t^{11}+t^{13})}{(1-t)^2}.$$