FACULTADE DE MATEMÁTICAS

The universal enveloping algebra of an n-Lie algebra

Xabier García Martínez

Universidade de Santiago de Compostela

EACA 2016

Joint work with Rustam Turdibaev and Tim Van der Linden

Logroño; June 22-24, 2016

European FEDER support included

Definition

Let $\mathbb{K}$ be a field. A $\mathbb{K}$-Lie algebra is a $\mathbb{K}$-vector space L with a bilinear operation $[-,-]\colon L \times L \to L$ satisfying

$$[x, x] = 0,$$

$$\big[[y_1, y_2], x\big] = \big[[y_1, x], y_2\big] + \big[y_1, [y_2, x]\big].$$

Lie algebras

Definition

Let $\mathbb{K}$ be a field. A $\mathbb{K}$-Lie algebra is a $\mathbb{K}$-vector space L with a bilinear operation $[-,-]\colon L \times L \to L$ satisfying

$$[x,x] = 0,$$

$$\big[[y_1,y_2],x\big] = \big[[y_1,x],y_2\big] + \big[y_1,[y_2,x]\big].$$

Example

Associative algebras.

Representations

Definition

A Lie algebra representation over L is a $\mathbb{K}$-vector space M with a bilinear map $L \times M \to M, (x, m) \mapsto xm$ such that

$$[x, y]m = x(ym) - y(xm)$$

Representations

Definition

A Lie algebra representation over L is a $\mathbb{K}$-vector space M with a bilinear map $L \times M \to M, (x, m) \mapsto xm$ such that

$$[x, y]m = x(ym) - y(xm)$$

Definition

A Lie algebra representation over L is a $\mathbb{K}$-vector space M with a Lie algebra homomorphism

$$L \to \mathrm{Der}_{\mathbb{K}}(M)$$

Representations

Definition

A Lie algebra representation over L is a $\mathbb{K}$-vector space M with a bilinear map $L \times M \to M, (x, m) \mapsto xm$ such that

$$[x, y]m = x(ym) - y(xm)$$

Definition

A Lie algebra representation over L is a $\mathbb{K}$-vector space M with a Lie algebra homomorphism

$$L \to \mathrm{Der}_{\mathbb{K}}(M)$$

Definition

A Lie algebra representation is a $\mathbb{K}$-vector space M with a Lie algebra structure on $L \oplus M$ such that

- M is an ideal,
- M is abelian,
- L is a subalgebra.

Lie algebras case

Definition

Let L be a $\mathbb{K}$-Lie algebra. The universal enveloping algebra $\mathrm{U}(L)$ is the tensor algebra $\mathrm{T}(L)$ quotient by the ideal generated by the elements

$$x \otimes y - y \otimes x - [x, y].$$

Lie algebras case

Definition

Let L be a $\mathbb{K}$-Lie algebra. The universal enveloping algebra $\mathrm{U}(L)$ is the tensor algebra $\mathrm{T}(L)$ quotient by the ideal generated by the elements

$$x \otimes y - y \otimes x - [x, y].$$

Properties

(U1) The category of representations over L is isomorphic to the category of modules over $\mathrm{U}(L)$.

(U2) The universal enveloping functor $\mathrm{U}\colon \mathrm{Lie}_{\mathbb{K}} \to \mathrm{Alg}_{\mathbb{K}}$ is left adjoint of the liezation functor $(-)_{\mathrm{Lie}}\colon \mathrm{Alg}_{\mathbb{K}} \to \mathrm{Lie}_{\mathbb{K}}$.

(U3) The inclusion $i\colon L \to \mathrm{U}(L)$ is injective.

Definition

An n-Lie algebra or Filippov algebra L is a $\mathbb{K}$-vector space with a multilinear map $[-,\dots,-]\colon L^{\times n} \to L$ satisfying

(i) $[x_1,\dots,x_i,\dots,x_j,\dots,x_n] = 0$ if $x_i = x_j$ for some i,j,

(ii) $\big[[x_1,\dots,x_n],y_1,\dots,y_{n-1}\big] = \sum_{i=1}^{n} \big[x_1,\dots,x_{i-1},[x_i,y_1,\dots,y_n],x_{i+1},\dots,x_n\big].$

n-Lie algebras

Definition

An n-Lie algebra or Filippov algebra L is a $\mathbb{K}$-vector space with a multilinear map $[-,\dots,-]\colon L^{\times n} \to L$ satisfying

(i) $[x_1,\dots,x_i,\dots,x_j,\dots,x_n] = 0$ if $x_i = x_j$ for some i,j,

(ii) $\big[[x_1,\dots,x_n],y_1,\dots,y_{n-1}\big] = \sum_{i=1}^{n}\big[x_1,\dots,x_{i-1},[x_i,y_1,\dots,y_n],x_{i+1},\dots,x_n\big]$.

Definition

An n-Lie algebra representation is a $\mathbb{K}$-vector space M with an n-Lie algebra structure on $L \oplus M$ such that

- M is an ideal,
- M is abelian,
- L is a subalgebra.

Attempts

First attempt

Let be the functor $F\colon n\text{-Lie}_{\mathbb{K}} \to \text{Lie}_{\mathbb{K}}$, $L \mapsto \Lambda^{n-1}L$, with bracket

$$[x_1\wedge\cdots\wedge x_{n-1}, y_1\wedge\cdots\wedge y_{n-1}] = \sum_{i=1}^{n-1}(-1)^{i+1}\left[x_1,\ldots,\widehat{x_i},\ldots,x_{n-1},[x_i,y_1,\ldots,y_{n-1}]\right].$$

Consider the universal enveloping algebra of $F(L)$ in the Lie algebras sense.

Attempts

First attempt

Let be the functor $F\colon n\text{-Lie}_{\mathbb{K}} \to \text{Lie}_{\mathbb{K}}$, $L \mapsto \Lambda^{n-1}L$, with bracket

$$[x_1\wedge\cdots\wedge x_{n-1}, y_1\wedge\cdots\wedge y_{n-1}] = \sum_{i=1}^{n-1}(-1)^{i+1}\big[x_1,\dots,\widehat{x_i},\dots,x_{n-1},[x_i,y_1,\dots,y_{n-1}]\big].$$

Consider the universal enveloping algebra of $F(L)$ in the Lie algebras sense.

Second attempt

Let be the functor $\text{G}\colon n\text{-Lie}_{\mathbb{K}} \to \text{Lie}_{\mathbb{K}}$, $L \mapsto \Lambda^{n-1}L$, with bracket

$$[x_1\wedge\cdots\wedge x_{n-1}, y_1\wedge\cdots\wedge y_{n-1}] = \frac{1}{2}\Big(\sum_{i=1}^{n-1}(-1)^{i+1}\big[x_1,\dots,\widehat{x_i},\dots,x_{n-1},[x_i,y_1,\dots,y_{n-1}]\big]$$

$$-\sum_{i=1}^{n-1}(-1)^{i+1}\big[y_1,\dots,\widehat{y_i},\dots,y_{n-1},[y_i,x_1,\dots,x_{n-1}]\big]\Big).$$

Necessary and sufficient conditions

Proposition

Let L be an n-Lie algebra with abelian $\mathrm{InnDer}(L)$*. Then* $\mathrm{G}(L)$ *is a Lie algebra. In particular, if L is simple,* $\mathrm{G}(L)$ *is a Lie algebra.*

Proposition

Let $\mathbb{K}$ *be a field and let L be a 3-Lie algebra over* $\mathbb{K}$ *such that* $\mathrm{InnDer}(L)$ *is not abelian. If* $\dim Z(L) \geq 2$ *then* $\mathrm{G}(L)$ *is not a Lie algebra.*

Definition

Let L an n-Lie algebra, the universal enveloping algebra $U(L)$ is the tensor algebra $T(\Lambda^{n-1}L)$ quotient by

$$(x_1 \wedge \cdots \wedge x_{n-1})(y_1 \wedge \cdots \wedge y_{n-1}) - (y_1 \wedge \cdots \wedge y_{n-1})(x_1 \wedge \cdots \wedge x_{n-1}) = \sum_{i=1}^{n-1} y_1 \wedge \cdots \wedge [x_1, \ldots, x_{n-1}, y_i] \wedge \cdots \wedge y_{n-1},$$

and

$$[x_1, \ldots, x_n] \wedge y_2 \wedge \cdots \wedge y_{n-1} = \sum_{i=1}^{n} (-1)^{n-i} (x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_n)(x_i \wedge y_2 \wedge \cdots \wedge y_{n-1}).$$

Definition

Let L an n-Lie algebra, the universal enveloping algebra $\mathrm{U}(L)$ is the tensor algebra $\mathrm{T}(\Lambda^{n-1}L)$ quotient by

$$(x_1 \wedge \cdots \wedge x_{n-1})(y_1 \wedge \cdots \wedge y_{n-1}) - (y_1 \wedge \cdots \wedge y_{n-1})(x_1 \wedge \cdots \wedge x_{n-1}) \\ = \sum_{i=1}^{n-1} y_1 \wedge \cdots \wedge [x_1, \ldots, x_{n-1}, y_i] \wedge \cdots \wedge y_{n-1},$$

and

$$[x_1, \ldots, x_n] \wedge y_2 \wedge \cdots \wedge y_{n-1} \\ = \sum_{i=1}^{n} (-1)^{n-i} (x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_n)(x_i \wedge y_2 \wedge \cdots \wedge y_{n-1}).$$

Justification

$L\text{-Mod}_{\mathbb{K}} \cong \mathrm{Mod}_{\mathrm{End}(L_1)}$, where L_1 is the free object on one generator in the category of n-Lie representations.

Non-existence of adjoint

Theorem

The functor $\mathrm{U}\colon n\text{-}\mathrm{Lie}_{\mathbb{K}} \to \mathrm{Alg}_{\mathbb{K}}$ *has a right adjoint if and only if* $n = 2$.

Non-existence of adjoint

Theorem

The functor $\mathrm{U}\colon n\text{-}\mathrm{Lie}_{\mathbb{K}} \to \mathrm{Alg}_{\mathbb{K}}$ *has a right adjoint if and only if* $n = 2$*.*

Theorem

For $n > 2$ *there is no functor* $F\colon n\text{-}\mathrm{Lie}_{\mathbb{K}} \to \mathrm{Alg}_{\mathbb{K}}$ *with a right adjoint* $G\colon \mathrm{Alg}_{\mathbb{K}} \to n\text{-}\mathrm{Lie}_{\mathbb{K}}$ *such that there is an equivalence of categories between* $L\text{-}\mathrm{Mod}_{\mathbb{K}}$ *and* $\mathrm{Mod}_{F(L)}$ *for all* L*.*

Non-existence of adjoint

Theorem

The functor $\mathrm{U}\colon n\text{-}\mathrm{Lie}_{\mathbb{K}} \to \mathrm{Alg}_{\mathbb{K}}$ *has a right adjoint if and only if* $n = 2$.

Theorem

For $n > 2$ *there is no functor* $F\colon n\text{-}\mathrm{Lie}_{\mathbb{K}} \to \mathrm{Alg}_{\mathbb{K}}$ *with a right adjoint* $G\colon \mathrm{Alg}_{\mathbb{K}} \to n\text{-}\mathrm{Lie}_{\mathbb{K}}$ *such that there is an equivalence of categories between* $L\text{-}\mathrm{Mod}_{\mathbb{K}}$ *and* $\mathrm{Mod}_{F(L)}$ *for all* L.

Proposition

If L *is a simple n-Lie algebra, then* $\mathrm{U}(L)$ *coincides with the algebras generated in the first and second attempts.*