The condition number of polynomials and its relationship with a set of points on the sphere

Ujué Etayo

Departamento Matesco
Facultad de Ciencias
Universidad de Cantabria

XV Encuentro de Álgebra Computacional y Aplicaciones
Universidad de la Rioja

June 22th - 24th, 2016

Introduction

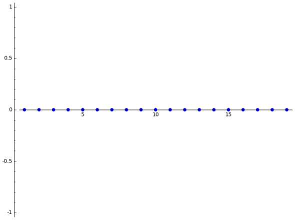
Finding solutions to polynomial equations

Definition

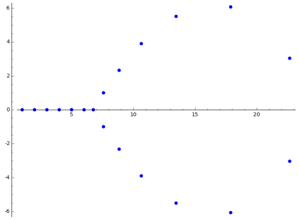
The Wilkinson's polynomial is defined by:

$$p_W(x) = \prod_{i=1}^{20} (x - i)$$

Roots of $p_W(x)$



Roots of $p_W(x) + 0.001x^{18}$



Index

1. The Bombieri-Weyl norm
2. Condition number
   - *The solution variety*
   - *Main idea and definition*
   - *Shub-Smale normalization*
   - *Open problem*
3. The Armentano-Beltrán-Shub formula
   - *Logarithmic energy*
4. Our work
   - *An interesting hypotesis*
   - *First interesting case: degree 5*

We are going to work in $\mathbb{C}[x]$.

The Bombieri-Weyl norm

Definition

Let $f = z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$ be a polynomial of $\mathbb{C}[z]$, then we can define the *Bombieri-Weyl norm* of f as follows.

$$||f||^2 = \sum_{j=0}^{n} \binom{n}{j}^{-1} |a_j|^2$$

The Bombieri-Weyl norm

To point out: if we have a polynomial $f(z) = a_0 + a_1 z + a_2 z^2 + ... + a_{n-2} z^{n-2} + a_{n-1} z^{n-1} + a_n z^n$, then BW norm is given by $||f||^2 = |a_0|^2 + \frac{|a_1|^2}{n} + \frac{2|a_2|^2}{n(n-1)} + ... + \frac{2|a_{n-2}|^2}{n(n-1)} + \frac{|a_{n-1}|^2}{n} + |a_n|^2.$

The Bombieri-Weyl norm

To point out: if we have a polynomial $f(z) = a_0 + a_1 z + a_2 z^2 + ... + a_{n-2} z^{n-2} + a_{n-1} z^{n-1} + a_n z^n$, then BW norm is given by $||f||^2 = |a_0|^2 + \frac{|a_1|^2}{n} + \frac{2|a_2|^2}{n(n-1)} + ... + \frac{2|a_{n-2}|^2}{n(n-1)} + \frac{|a_{n-1}|^2}{n} + |a_n|^2$.

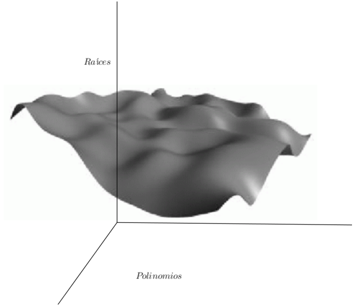
1. The BW norm weights more the coeffients of the extremes.

The Bombieri-Weyl norm

To point out: if we have a polynomial
$f(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_{n-2} z^{n-2} + a_{n-1} z^{n-1} + a_n z^n$,
then BW norm is given by
$||f||^2 = |a_0|^2 + \frac{|a_1|^2}{n} + \frac{2|a_2|^2}{n(n-1)} + \dots + \frac{2|a_{n-2}|^2}{n(n-1)} + \frac{|a_{n-1}|^2}{n} + |a_n|^2.$

1. The BW norm weights more the coeffients of the extremes.
2. Example:
   $f(x) = x^5 - 2x^4 - x + 2 = (x-1)(x+1)(x-2)(x^2+1)$
   $||f||^2 = \binom{5}{0}^{-1}|2|^2 + \binom{5}{1}^{-1}|-1|^2 + \binom{5}{4}^{-1}|-2|^2 + \binom{5}{5}^{-1}|1|^2$
   $||f|| = \sqrt{6}$

Condition number
The solution variety



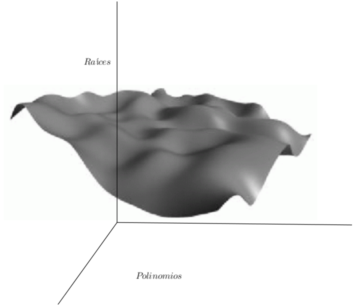
$$E = \mathscr{P}_d(\mathbb{C}) \times \mathbb{C}$$

$$E = \{(f, z) : f \in \mathscr{P}_d(\mathbb{C}), z \in \mathbb{C}\}$$

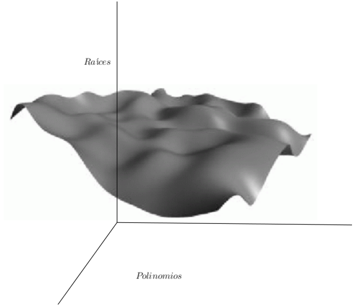
$$V = \{(f, z) : f(z) = 0\}$$

Riemmanian manifold

Condition number
Main idea and definition



$$\text{cond}^f(z) = ||D\Pi_{Pol}(f,z)^{-1}||_{op}$$



$$\text{cond}^f(z) = ||D\Pi_{Pol}(f,z)^{-1}||_{op}$$

$$\mu(f,z) = \sqrt{n}\,\text{cond}^f(z) = \frac{\sqrt{n}(1+|z|^2)^{\frac{n-2}{2}}}{|f'(z)|}||f||$$

As we can find in [Shub and Smale, 1993].

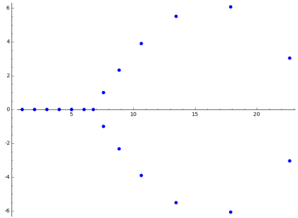
Example:

$$f(x) = x^5 - 2x^4 - x + 2 = (x-1)(x+1)(x-2)(x^2+1)$$
$$f(x)' = 5x^4 - 8x^3 - 1$$
$$||f|| = \sqrt{6}$$
$$\mu(f,1) = \frac{\sqrt{5}(1+|1|^2)^{\frac{3}{2}}}{|f'(1)|}||f|| = \sqrt{15}$$

Condition number
Wilkinson's polynomial

$$p_W(x) = \prod_{i=1}^{20}(x-i)$$

Roots of $p_W(x) + 0.001x^{18}$



$$\mu(p_W,1) \approx 40976, \quad \mu(p_W,2) \approx 10^9, \quad \mu(p_W,5) \approx 10^{17}$$

$$\mu(p_W,10) \approx 10^{23}, \quad \mu(p_W,15) \approx 10^{25}, \quad \mu(p_W,20) \approx 10^{23}$$

Problem

Find explicitly a family $\{f_n\}_{n\in\mathbb{N}}$ with $\mu(f_n) \leq n$.

Proposed in [Shub and Smale, 1993].

An application of the problem:

Complexity of *path followings* starting at $(g,z) \leq \text{cte}.n^{\frac{5}{2}}\mu^2_{\max}(g,z)$

proved in [Bürgisser and Cucker, 2013].

The Armentano-Beltrán-Shub formula

$$\mathscr{E}_0(\omega_n) = \underbrace{\frac{1}{2}\sum_{i=1}^{n} \ln(\mu(f, z_i))}_{\mathscr{M}} + \underbrace{\frac{n}{2} \ln \left(\frac{\prod_{i=1}^{n} \sqrt{1+|z_i|^2}}{||f||} \right)}_{\mathscr{N}} - \frac{n}{4}\ln(n)$$

[Armentano et al., 2011], [Beltrán, 2015].

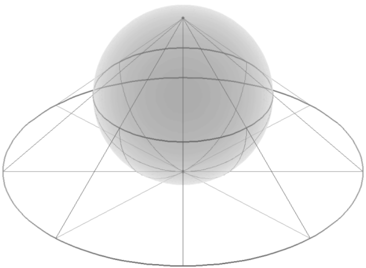
- $\mathscr{E}_0(\omega_n)$: logarithmic energy of a subset of points ω_n on the Riemann sphere *(Potential theory)*.
- $\mu(f, z_i)$: SS condition number of the polynomial f in its root z_i *(Numerical stability)*.
- $\mathscr{N}$: classical measure *(Number theory).*

Definition

The logarithmic energy of a collection of points $\omega_n = \{x_1, ..., x_n\}$ in the complex projective space

$$\mathcal{E}_0(\omega_n) = \sum_{i,j=1, i<j}^{n} \ln\left(\frac{1}{||x_i - x_j||}\right)$$

The Armentano-Beltrán-Shub formula
Variables



Stereographic projection

$$\pi : \mathbb{S}\backslash(0,0,1) \longrightarrow \mathbb{C}$$
$$(a,b,c) \mapsto \frac{c}{a-ib}$$
$$(0,0,0) \mapsto 0$$

$$x_i \mapsto z_i, \quad f = \prod_{i=1}^{n}(z - z_i)$$

The Armentano-Beltrán-Shub formula

$$\mathcal{E}_0(\omega_n) = \underbrace{\frac{1}{2}\sum_{i=1}^{n}\ln(\mu(f,z_i))}_{\mathcal{M}} + \underbrace{\frac{n}{2}\ln\left(\frac{\prod_{i=1}^{n}\sqrt{1+|z_i|^2}}{||f||}\right)}_{\mathcal{N}} - \frac{n}{4}\ln(n)$$

The Armentano-Beltrán-Shub formula

$$\mathcal{E}_0(\omega_n) = \underbrace{\frac{1}{2}\sum_{i=1}^{n} \ln(\mu(f,z_i))}_{\mathcal{M}} + \underbrace{\frac{n}{2}\ln\left(\frac{\prod_{i=1}^{n}\sqrt{1+|z_i|^2}}{||f||}\right)}_{\mathcal{N}} - \frac{n}{4}\ln(n)$$

Hypothesis: $1, 2$ and 3 are equivalent.

1. Minimize $\mathcal{E}_0(\omega_n)$.
2. Minimize $\sum_{i=1}^{n} \ln(\mu(f,z_i))$.
3. Maximize $\frac{\prod_{i=1}^{n}\sqrt{1+|z_i|^2}}{||f||}$.

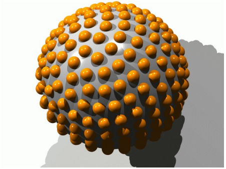
The Armentano-Beltrán-Shub formula

Minimal logarithmic energy

$$\operatorname{argmin} \sum_{i,j=1,i<j}^{n} \ln\left(\frac{1}{\|x_i - x_j\|}\right)$$ *Whyte's problem*

equivalently

$$\operatorname{argmax} \prod_{i,j=1,i<j}^{n} \|x_i - x_j\|$$ *Elliptic Fekete points*

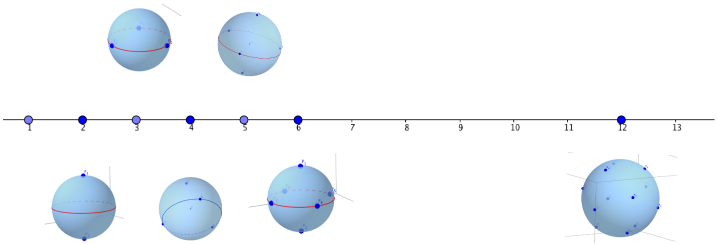


Smale's 7th problem Find $\omega_n = \{x_1, ..., x_n\}$ such that:

$$\mathcal{E}_0(X) - m_n \leq c \ln(n)$$

where c is a constant.

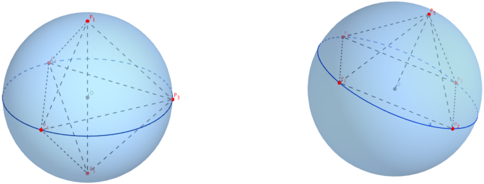
Schedule of the solved minimal logarithmic energy problem: 2 – 6 and 12 points.



See Brauchart and Grabner [2015].

The Armentano-Beltrán-Shub formula

First interesting case: degree 5



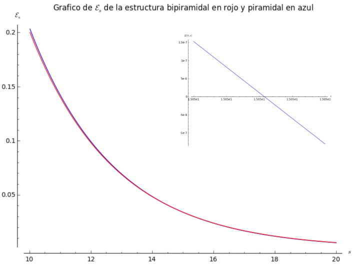
(a) Bipyramidal structure (b) Pyramidal structure

The Armentano-Beltrán-Shub formula

First interesting case: degree 5

Logarithmic energy $\mathcal{E}_0$ is lower in the bipyramidal structure.

Bipyramidal structure	**Pyramidal structure**
$\mathcal{E}_0^{\Diamond}(\omega_5) = 2.511$	$\mathcal{E}_0^{\triangle}(\omega_5) = 2.520$

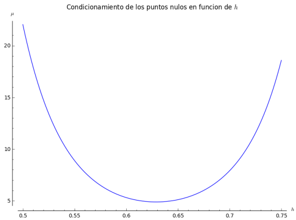


The Armentano-Beltrán-Shub formula

First interesting case: degree 5

$\mathscr{M} = \prod_{i=1}^{n} \mu(f, z_i)$ is lower in the bipyramidal structure.

Bipyramidal structure	**Pyramidal structure**
$\mathscr{M}^{\Diamond} = 4.\widehat{740}$	$\mathscr{M}^{\triangle} \approx 4.897$

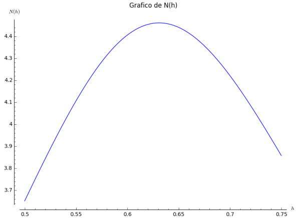


The Armentano-Beltrán-Shub formula

First interesting case: degree 5

$$\mathscr{N} = \frac{\prod_{i=1}^{n} \sqrt{1+|z_i|^2}}{||f||}$$ is greater in the bipyramidal structure.

Bipyramidal structure	**Pyramidal structure**
$\mathscr{N}^{\Diamond} \approx 4.472$	$\mathscr{N}^{\triangle} \approx 4.459$



Conclusions

- We have found a polynomial with small condition number and the procedure we employed was new.
- We have tested our hypothesis for the first interesting case, that is, the polynomials of degree five.

Michael Shub and Steve Smale. Complexity of bezout's theorem: III. condition number and packing. *J. Complexity*, 9(1):4–14, 1993.

Peter Bürgisser and Felipe Cucker. *Condition: The Geometry of Numerical Algorithms*. Springer Publishing Company, Incorporated, 2013. ISBN 3642388957, 9783642388958.

Diego Armentano, Carlos Beltrán, and Michael Shub. Minimizing the discrete logarithmic energy on the sphere: the role of random polynomials. *Trans. Am. Math. Soc.*, 363(6): 2955–2965, 2011.

Carlos Beltrán. A facility location formulation for stable polynomials and elliptic fekete points. *Foundations of Computational Mathematics*, 15(1):125–157, 2015.

Johann S. Brauchart and Peter J. Grabner. Distributing many points on spheres: Minimal energy and designs. *Journal of Complexity*, 31(3):293 – 326, 2015. ISSN 0885-064X. doi: http://dx.doi.org/10.1016/j.jco.2015.02.003. URL http://www.sciencedirect.com/science/article/pii/S0885064X15000205. Oberwolfach 2013.

Peter D Dragnev, DA Legg, and DW Townsend. Discrete logarithmic energy on the sphere. *Pacific journal of mathematics*, 207(2):345–358, 2002.

Thank you very much!