Computing jumping numbers in higher dimensions

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Introduction

What is algebraic geometry?

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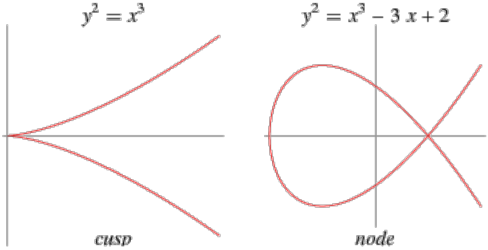
- Consider a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$.
- Study the properties of $\{x \in \mathbb{C}^n \mid f(x) = 0\}$ (=geometry) using algebraic methods.

Introduction



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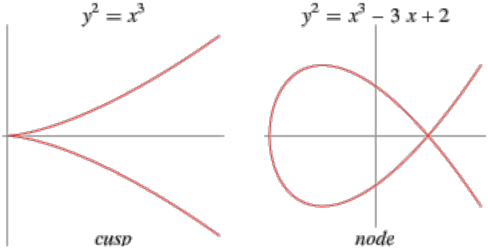

Which is the worst?

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- The jumping numbers of the cusp are $\frac{5}{6}$, 1, $\frac{11}{6}$, 2...
- The cusp is worse!

Setting

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- We study the singularities of D.
- Think of $X = \mathbb{C}^n$, and $D = \{x \in \mathbb{C}^n \mid f(x) = 0\}$ for some $f \in \mathbb{C}[x_1, \ldots, x_n]$.

Definitions and first properties

Definition

A **log resolution** of D is a birational morphism $\pi : Y \to X$ such that

- Y is smooth,
- π is an isomorphism over $X \backslash \mathrm{Sing}(D)$,
- the pullback $\pi^* D$ is a snc divisor, i.e., the irreducible components of $\pi^{-1}(D)$ are smooth and intersect transversally.

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A log resolution ‘makes D smooth’.

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Example

If $\pi : Y \to X$ is a blow-up of a subvariety V of codimension $r \leq 2$ with exceptional divisor E (='replacing V by a hyperplane E') then $K_\pi = (r-1)E$.

Multiplier ideals

Definition

If $\pi : Y \to X$ is a log resolution of a hypersurface $D \subset X$, and $\lambda \in \mathbb{Q}_{\geq 0}$, then the **multiplier ideal** of (X, D) with coefficient λ is

$$\mathcal{J}(\lambda D) = \pi_* \mathcal{O}_Y(K_\pi - \lfloor \lambda \pi^* D \rfloor),$$

where we denote $\lfloor \lambda \pi^* D \rfloor = \sum \lfloor \lambda a_i \rfloor E_i$ if $\pi^* D = \sum a_i E_i$.

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- $\mathcal{J}(\lambda D) = \mathcal{O}_X$ if $\lambda \ll 1$,
- if $\lambda < \lambda'$, then $\mathcal{J}(\lambda D) \supseteq \mathcal{J}(\lambda' D)$,
- if $0 < \varepsilon \ll 1$, then $\mathcal{J}((\lambda + \varepsilon)D) = \mathcal{J}(\lambda D)$.

Jumping numbers

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- if $\lambda \in [\lambda_i, \lambda_{i+1})$ for some i, then $\mathcal{J}(\lambda D) = \mathcal{J}(\lambda_i)$.

The numbers λ_i, $i > 0$, are called the **jumping numbers** of the pair (X, D).

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Hence it suffices to compute the jumping numbers in the interval $(0, 1]$.

Recall: $\mathcal{J}(\lambda D) = \pi_* \mathcal{O}_X(K_\pi - \lfloor \lambda \pi^* D \rfloor)$.
If $\pi : Y \to X$ is a log resolution, and we denote

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If $\pi : Y \to X$ is a log resolution, and we denote

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$$\left\{ \frac{k_i + m}{a_i} \middle| m \in \mathbb{Z}_{>0} \right\}.$$

The numbers in this set are the **candidate jumping numbers**.
The smallest candidate is always a jumping number, it is the **log canonical threshold**, denoted $\mathrm{lct}(D)$.

The algorithm

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- We want to find the biggest divisor D_λ equivalent to $\lfloor \lambda \pi^* D \rfloor - K_\pi$.
- Then the jumping number following to λ is the smallest number λ' such that $\lfloor \lambda' \pi^* D \rfloor - K_\pi$ has a coefficient bigger than the one in D_λ.

π-antieffective divisors

Definition

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Algorithm (Unloading procedure; B., Dachs Cadefau, after Laufer, Enriques, Reguera, AAD. . .)

`Input`: A divisor D on Y.
`Output`: The smallest π-antieffective divisor bigger than D.
While $-D|_E$ is not effective for some E, replace D by $D + E$.

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The algorithm always stops. The result is the π-**antieffective closure** of D.

π-antieffective divisors

Theorem (B., Dachs Cadefau)

If $\tilde{D}$ is the π-antieffective closure of D, then

$$\pi_*\mathcal{O}_Y(-\tilde{D}) = \pi_*\mathcal{O}_Y(-D).$$

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So if λ' is the smallest jumping number bigger than λ, $\lfloor \lambda' \pi^* D \rfloor - K_\pi$ must exceed the π-antieffective closure of $\lfloor \lambda \pi^* D \rfloor - K_\pi$ in at least one coefficient.

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$\Rightarrow$ Lower bound for the next jumping number!

The two-dimensional case

Theorem (Lipman, 1969)

Suppose X is a surface. The mapping $D \mapsto \pi_* \mathcal{O}_Y(-D)$ determines a one-to-one correspondence between

1. *π-antieffective divisors on Y, and*
2. *complete ideal sheaves on X.*

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Theorem (Alberich-Carramiñana, Àlvarez-Montaner, Dachs Cadefau)

If $D_\lambda = \sum a_i^\lambda E_i$ is the π-antieffective closure of $\lfloor \lambda \pi^* D \rfloor - K_\pi$, then the smallest jumping number bigger then λ is

$$\lambda' = \min \left\{ \frac{k_i + a_i^\lambda + 1}{a_i} \right\},$$

where $K_\pi = \sum k_i E_i$ and $\pi^* D = \sum a_i E_i$.

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Algorithm (Alberich-Carramiñana, Àlvarez-Montaner, Dachs Cadefau)

`Input`: A curve C on X.

`Output`: The jumping numbers of (X, C).

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2. Compute $\mathrm{lct}(C)$.
3. Having computed a jumping number λ_j, compute the π-antieffective closure $D_\lambda = \sum a_i^{\lambda_j} E_i$ of $\lfloor \lambda_j \pi^* C \rfloor - K_\pi$ using the unloading procedure. Then the next jumping number is

$$\lambda_{j+1} = \min \left\{ \frac{k_i + a_i^{\lambda_j} + 1}{a_i} \right\}.$$

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$$\lambda_{j+1} = \min \left\{ \frac{k_i + a_i^{\lambda_j} + 1}{a_i} \right\}.$$

4. Repeat until we arrive at 1.

The higher dimensional case

Theorem (B., Dachs Cadefau)

If we run the algorithm of [AAD] until we arrive at 1, then we get a set of so-called **supercandidates**, *containing all the jumping numbers.*

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Remark

No examples are known where not all supercandidates are jumping numbers!

Checking supercandidates

Definition

The *minimal jumping divisor* of a supercandidate λ' is the reduced divisor $G_{\lambda'}$ supported on those E_i where the minimum

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If the minimum jumping divisor of a supercandidate λ' has an irreducible connected component, then λ' is a jumping number.

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Proposition (B., Dachs Cadefau)

If the minimum jumping divisor of a supercandidate λ' has an irreducible connected component, then λ' is a jumping number.

Good news: this happens very often!

Checking supercandidates

Otherwise, there are some more techniques to check whether a supercandidate is a jumping number.

Examples

Example 1

- Take $X = \mathbb{C}^3$ and D given by

$$x(yz - x^4)(x^4 + y^2 - 2yz) + yz^4 - y^5 = 0.$$

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- We find a resolution $\pi : Y \to X$ by six point blow-ups, with

$$K_\pi = 2E_1 + 4E_2 + 8E_3 + 14E_4 + 6E_5 + 6E_6$$

and

$$\pi^* D = D_{aff} + 5E_1 + 9E_2 + 16E_3 + 27E_4 + 11E_5 + 11E_6.$$

Example 1

λ	G_λ
$\frac{5}{9}$	$E_2 + E_4$
$\frac{2}{3}$	$E_2 + E_4$
$\frac{20}{27}$	E_4
$\frac{7}{9}$	$E_2 + E_4$
$\frac{23}{27}$	E_4
$\frac{8}{9}$	$E_2 + E_4$
$\frac{25}{27}$	E_4
$\frac{26}{27}$	E_4
1	$E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + D_{aff}$.

Example 2

- Take $X = \mathbb{C}^3$ and D given by

$$(x^3 + y^3 + z^3)^2 + g(x, y, z) = 0,$$

where $g(x, y, z)$ is a generic homogeneous polynomial of degree 7.

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- E_2 and E_3 are ruled surfaces over C, so their Picard groups are very complicated!

$$\pi^* D = D_{aff} + 6E_1 + 8\sum_{i=1}^{21} E_i^p + 7E_2 + 14E_3,$$

$$K_\pi = 2E_1 + 4\sum_{i=1}^{21} E_i^p + 3E_2 + 6E_3.$$

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and all of them are jumping numbers contributed by E_3.

Implementation

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However, in many examples, our methods are sufficient.

Thank you!