Automatic Discovery in GeoGebra

First Steps

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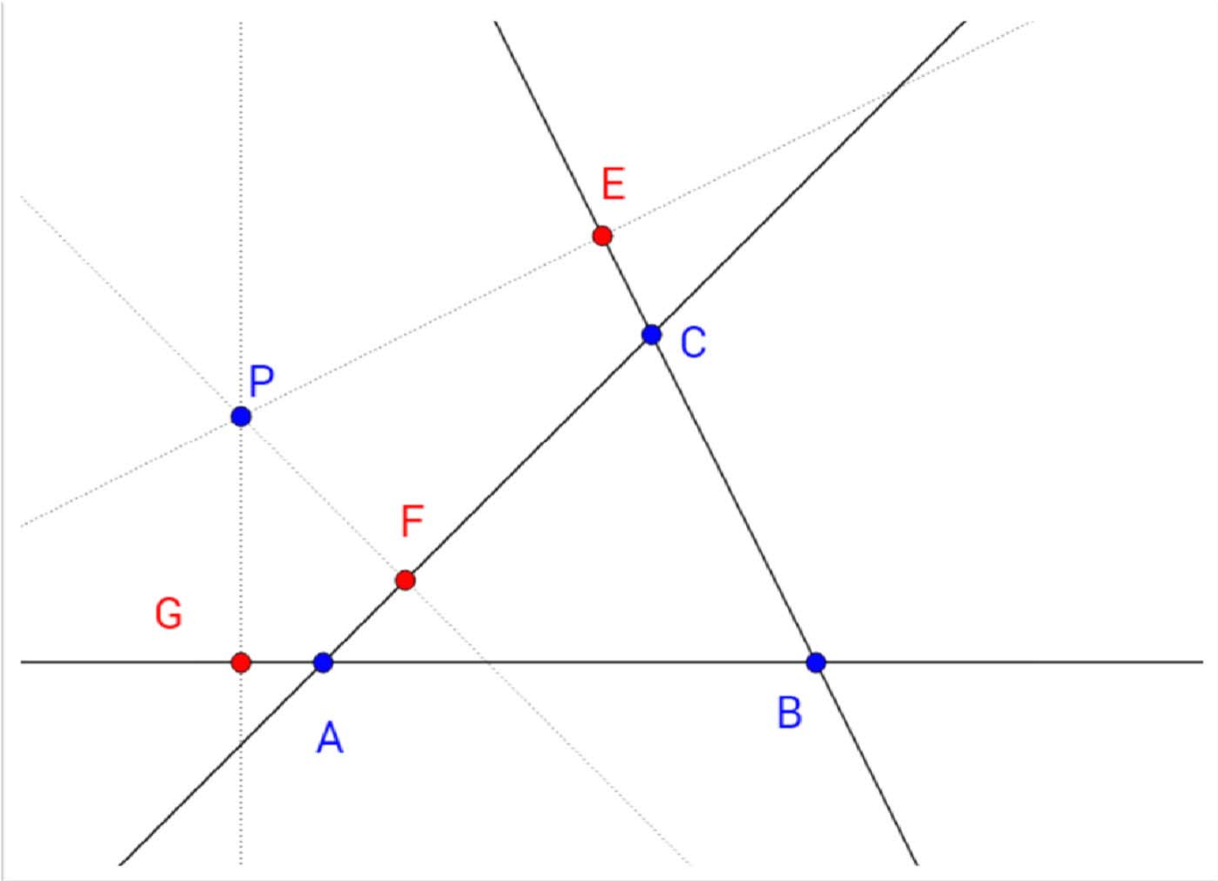
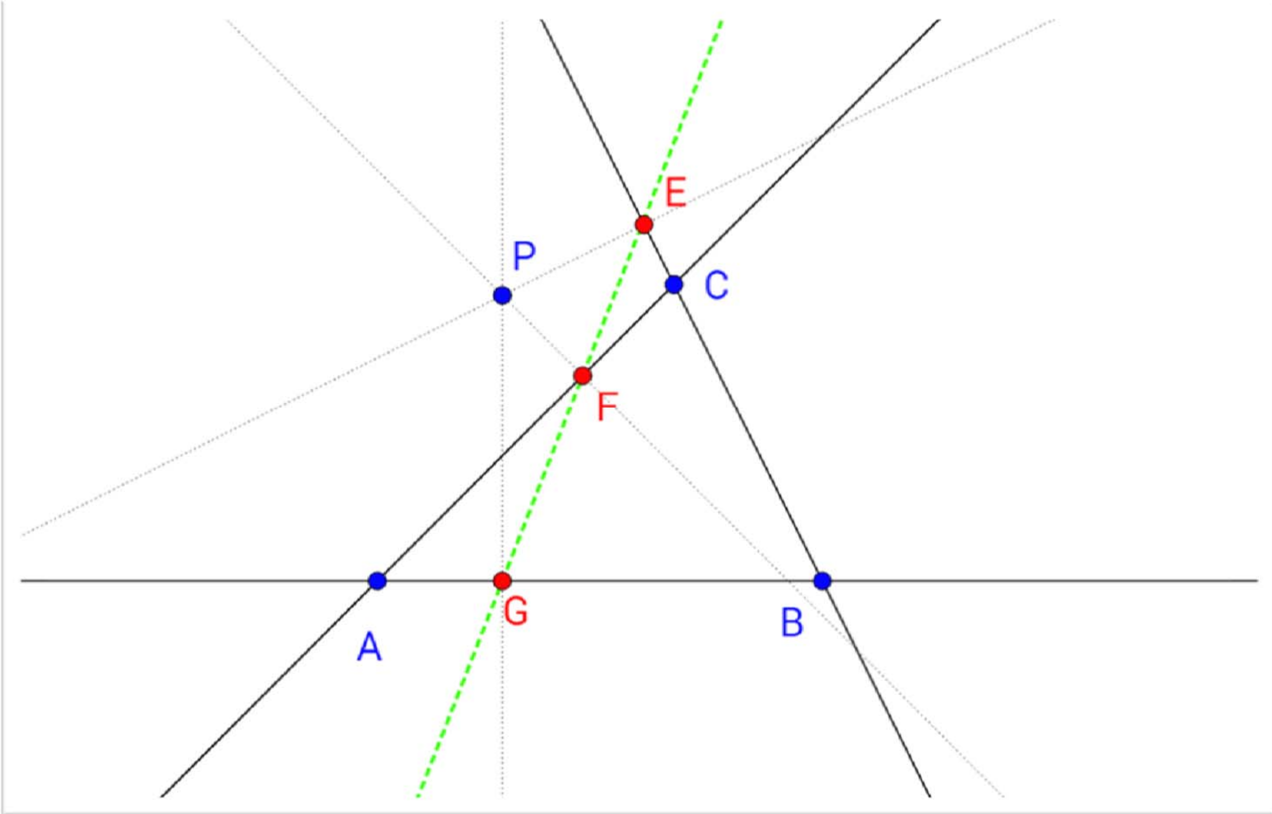
Automatic Proving vs. Automatic Discovering

- Automatic Proving:
  - establishing if some statement is true

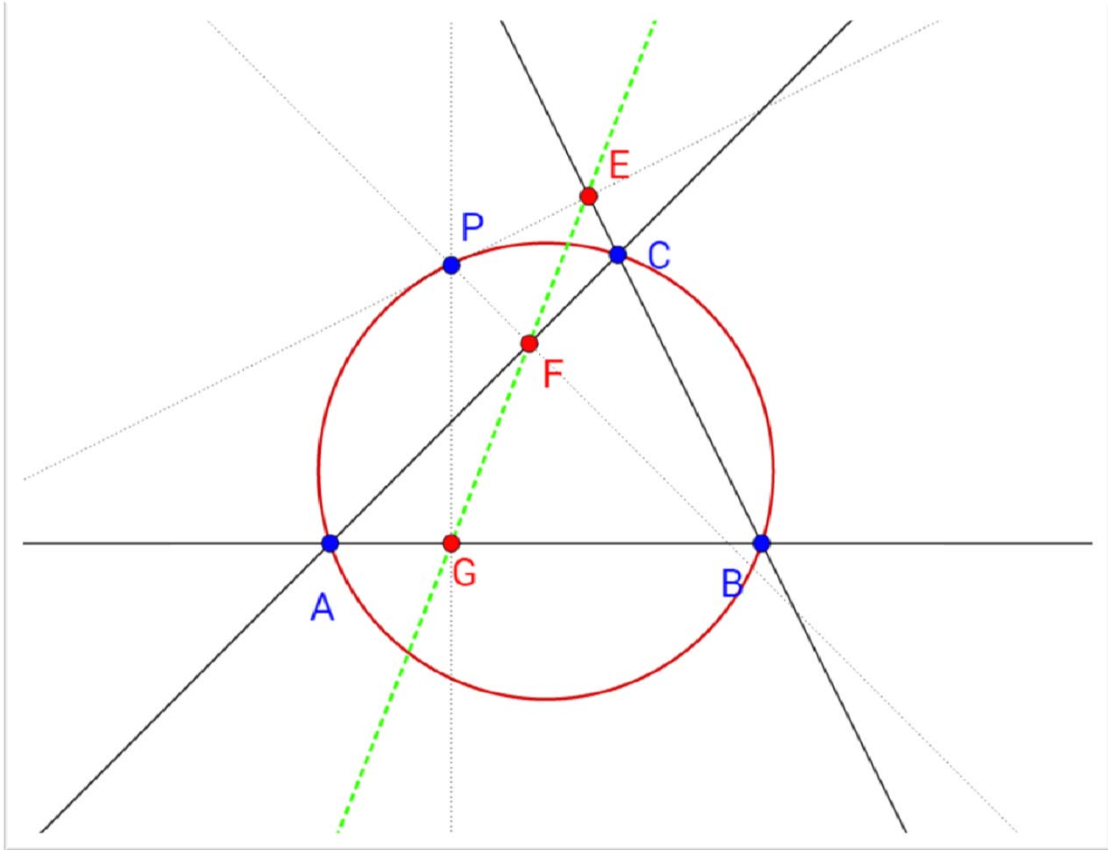
- Automatic Discovery:
  - establishing when some statement is true



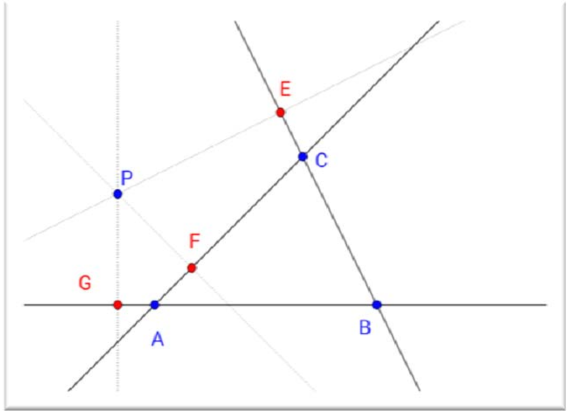
- E, F and G not aligned in general



- When are E, F and G aligned?
  - i.e. for which positions of P?

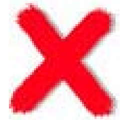


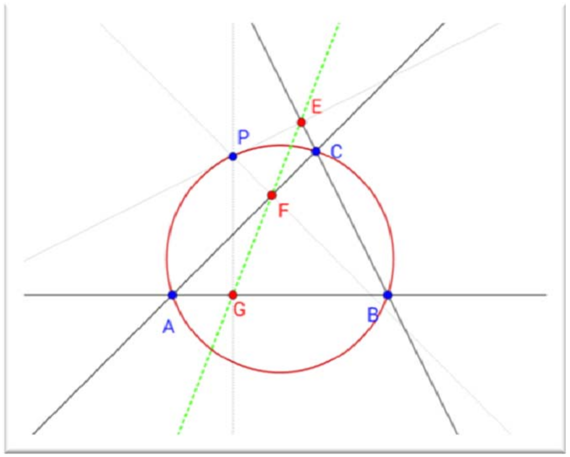
- E, F and G are aligned if and only if P is on circle through A, B and C
  - Wallace-Simson theorem



Theorem:

If E, F and G are the orthogonal projections of P onto the sides of triangle ABC, then E, F and G are aligned.





Theorem:

If E, F and G are the orthogonal projections of P onto the sides of triangle ABC **and P is on the circumcircle of ABC**, then E, F and G are aligned.

Automatic Proving in elementary geometry

- Algorithms, using computer algebra methods, for confirming (or refuting) the truth of some given geometric statement
  - Translate hypotheses and theses into systems of polynomial equations

$$\left.\begin{matrix} H \to S_H \\ T \to S_T \end{matrix}\right\} \to \left[H \Rightarrow T \right] \sim \left[S_H \subseteq S_T \right]$$

  - Geometric statements become set inclusion statements
    - Elucidated by some computer algebra tools
- Initiated by Wu in the 1980's
  - Other authors: Chou, Kapur,…

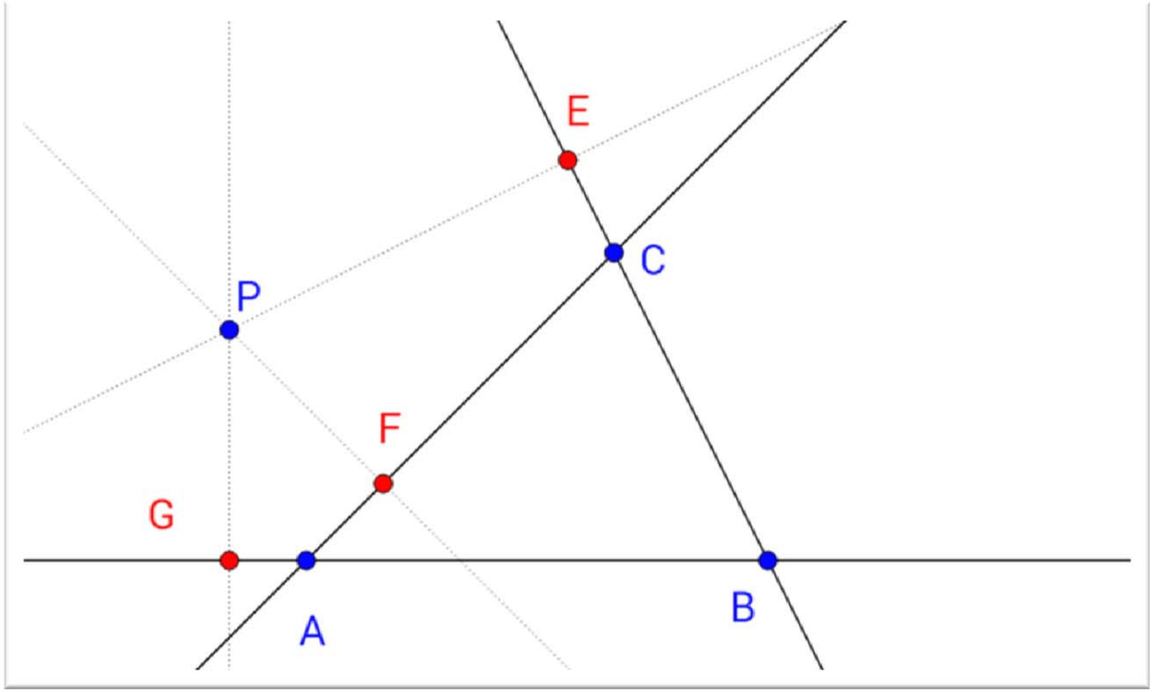
Automatic Discovery in elementary geometry

- Considers a statement $H \Rightarrow T$ that is false in most relevant cases.
- It aims to automatically produce additional hypotheses H_0 for the (new) statement $(H \wedge H_0) \Rightarrow T$ to be true.

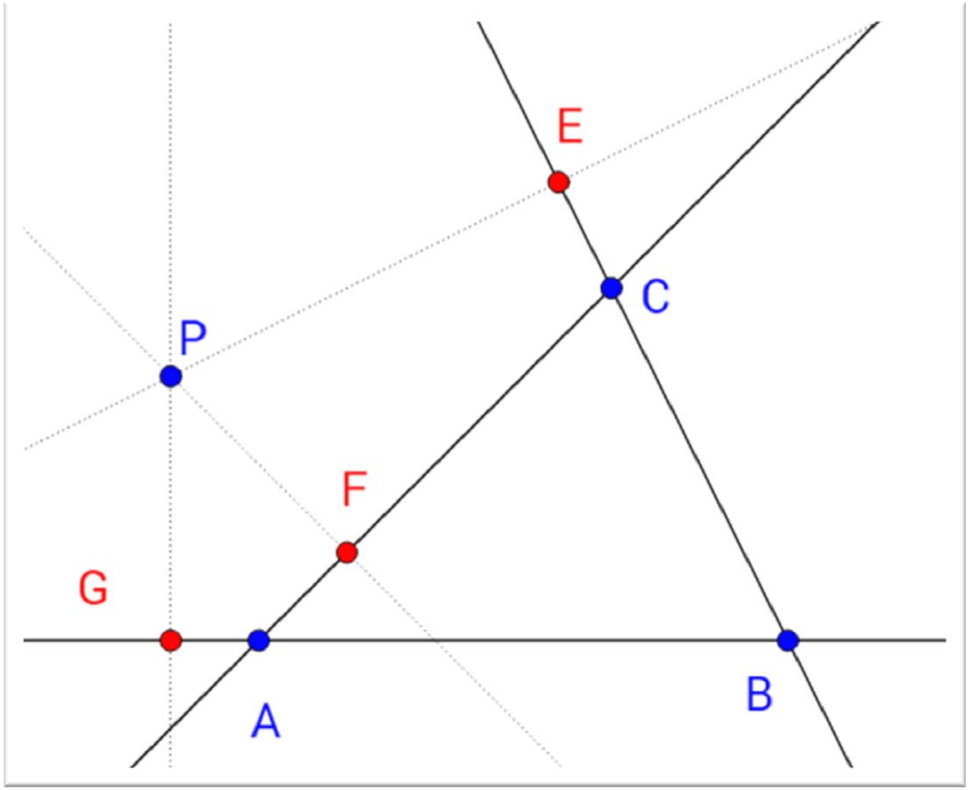
$$\text{we have: } \boxed{H \Rightarrow T \text{ false}}$$

$$\text{we want: } \boxed{(H \wedge H_0) \Rightarrow T \text{ true}}$$

- Complementary hypotheses in terms of the free variables for the construction.
- Proposed in
  - T. Recio, M.P. Vélez: Automatic discovery of theorems in elementary geometry, Journal of Automated Reasoning 23: pp. 63-82, 1999



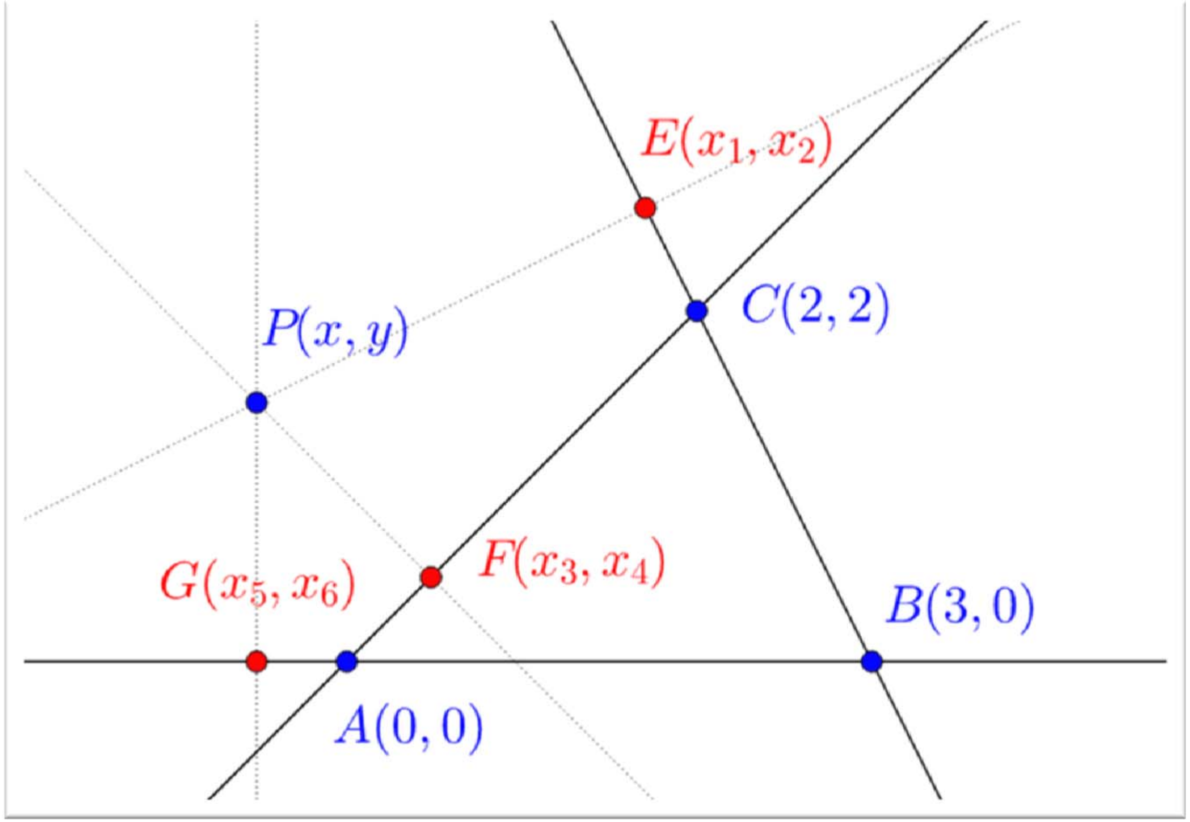
- E, F and G not aligned in general
- When are E, F and G aligned?
  - for which positions of P?



$$\begin{cases} \text{Line}(P,E) \perp \text{Line}(C,B) \\ E \in \text{Line}(C,B) \\ \text{Line}(P,F) \perp \text{Line}(A,C) \\ F \in \text{Line}(A,C) \\ \text{Line}(P,G) \perp \text{Line}(A,B) \\ G \in \text{Line}(A,B) \end{cases}$$

- Assign coordinates:

$$A(0,0) \;\; B(3,0) \;\; C(2,2) \;\; P(x,y) \;\; E(x_1,x_2) \;\; F(x_3,x_4) \;\; G(x_5,x_6)$$



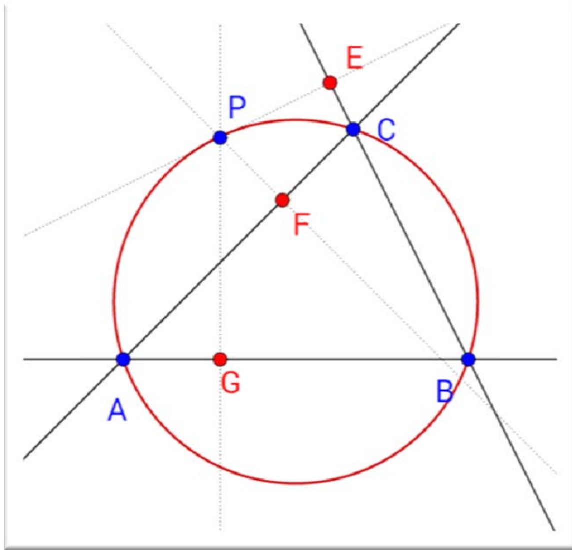
$$\begin{cases} x - y - x_1 + 2x_2 = 0 \\ -2x_1 - x_2 + 2 = 0 \\ x + y - x_3 - x_4 = 0 \\ x_3 - x_4 = 0 \\ x - x_5 = 0 \\ x_6 = 0 \end{cases}$$

$$\begin{cases} x - y - x_1 + 2x_2 = 0 \\ -2x_1 - x_2 + 2 = 0 \\ x + y - x_3 - x_4 = 0 \\ x_3 - x_4 = 0 \\ x - x_5 = 0 \\ x_6 = 0 \end{cases}$$

E, F, G collinear

$$+ \quad (x_5 - x_1)\cdot(x_4 - x_2) - (x_3 - x_1)\cdot(x_6 - x_2) = 0$$

$$\Rightarrow \quad \boxed{x^2 + y^2 - 3x - y = 0} \quad \Rightarrow$$



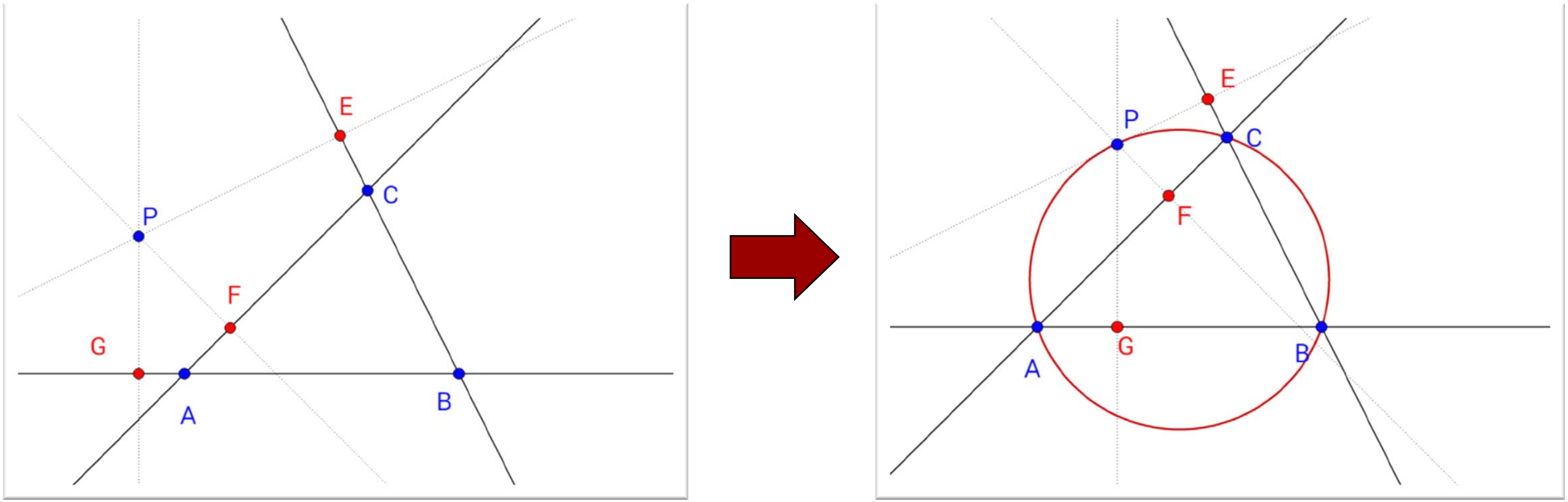
"Solving for x and y"

(Elimination theory – Gröbner bases)

- Discovery over one free point P in the plane
  - (In general) Results in a curve
  - Locus of positions of P such that the extra condition is satisfied
    - e.g. E, F and G collinear in the example
  - Locus set defined implicitly by a condition on the "locus point"

- Implicit Locus = locus obtained from "discovery"
  - <u>Can not be constructed</u>
    - Only "discovered"

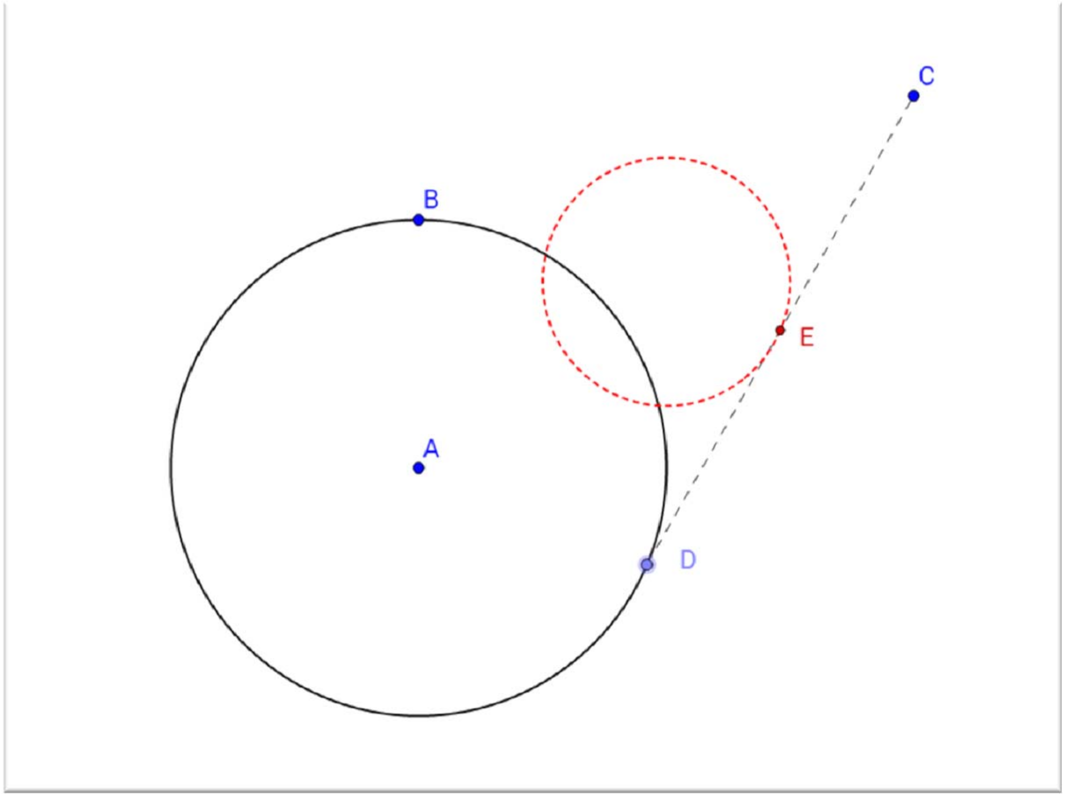
▪ Example of implicit locus:



▪ Locus of points P such that its projections are aligned

- Standard loci in Dynamic Geometry
  - “tracer-mover”
  - Defined by the positions of a tracer point that depends on a mover point running along a 1-dimensional set
  - Can be constructed

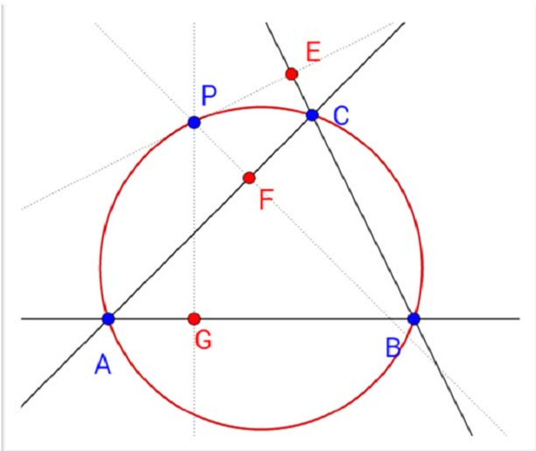
- Example of “tracer-mover” locus:



Circle with center A through B
C point in the plane
D point on the black circle
E = midpoint(D,C)
E *traces* the locus (red circle) as D *moves* (along black circle)

- Computation of loci in GeoGebra
  - LocusEquation[<Locus Point>,<Moving Point>]
  - Command in GeoGebra that computes equation of locus
    - Only for tracer-mover loci
    - Based on previous collaboration (2010)

- Discovery in GeoGebra
  - Collaboration with GeoGebra developing team
  - Generalization of LocusEquation[<Locus Point>,<Moving Point>]
  - LocusEquation[<Boolean Expression>,<Free Point>]
    - Boolean Expression = extra condition (thesis)
    - Free Point = point over which we “discover”
      - For which positions of P is the extra condition satisfied?

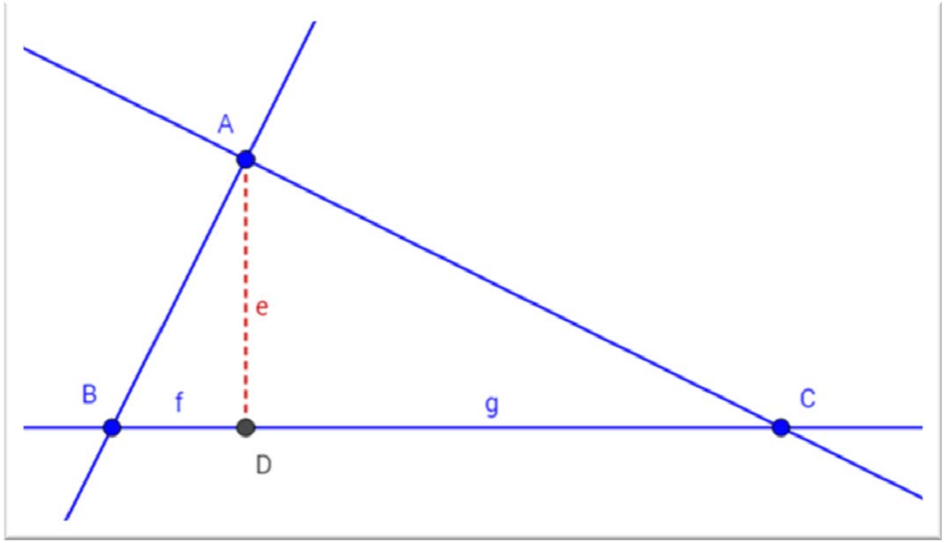


LocusEquation[AreCollinear[E,F,G], P]

Example of discovery in GeoGebra

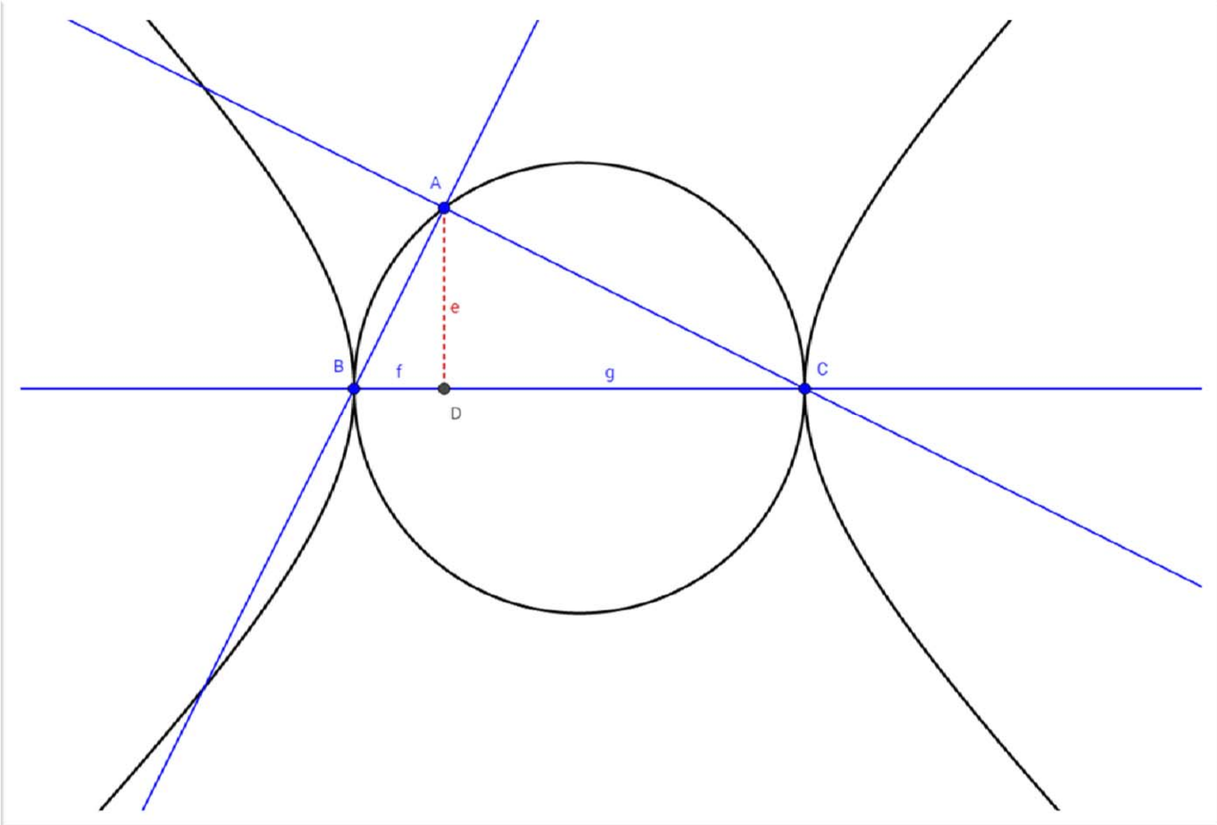
- Right triangle altitude theorem

$$\left.\begin{array}{l} ABC \text{ right triangle} \\ D = \text{Projection of } A \text{ onto } BC \\ e = \text{Distance}(A, D) \\ f = \text{Distance}(B, D) \\ g = \text{Distance}(C, D) \end{array}\right\} \Rightarrow e^2 = f \cdot g$$



- True for any non-right triangles?
- When is $\text{Distance}(A, D)^2 = \text{Distance}(B, D) \cdot \text{Distance}(C, D)$?
  - For which positions of A?

- LocusEquation[e*e == f*g, A]



- Locus = circle + hyperbola

Example of discovery in GeoGebra

- Orthic triangle

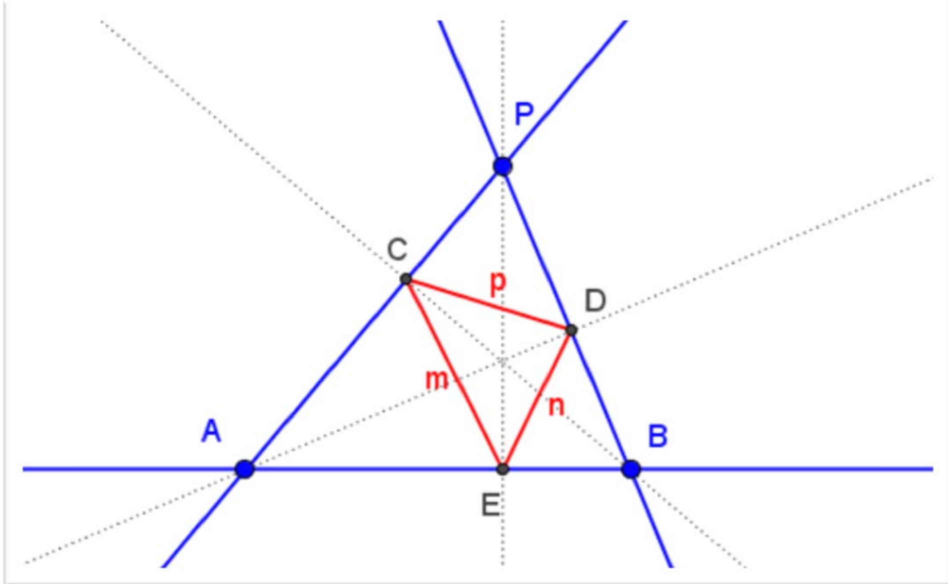
ABP triangle

$C =$ Projection of B onto AP

$D =$ Projection of A onto BP

$E =$ Projection of P onto BA

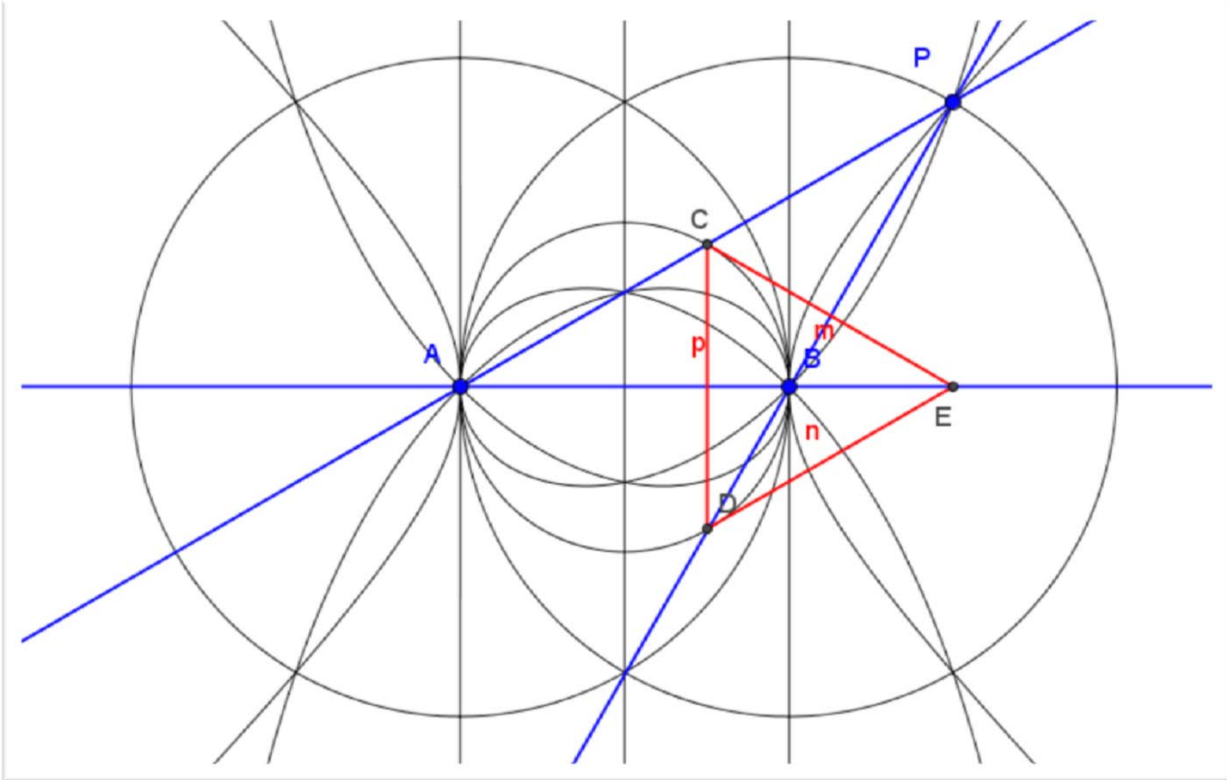
$CDE =$ Orthic triangle of ABP



- When is the orthic triangle equilateral?
- When is $m = n = p$?
  - For which positions of P?

LocusEquation[m == n, P], LocusEquation[m == p, P],

LocusEquation[n == p, P]



Locus = six intersection points

Example of discovery in GeoGebra

- Variation of Simson-Wallace Theorem

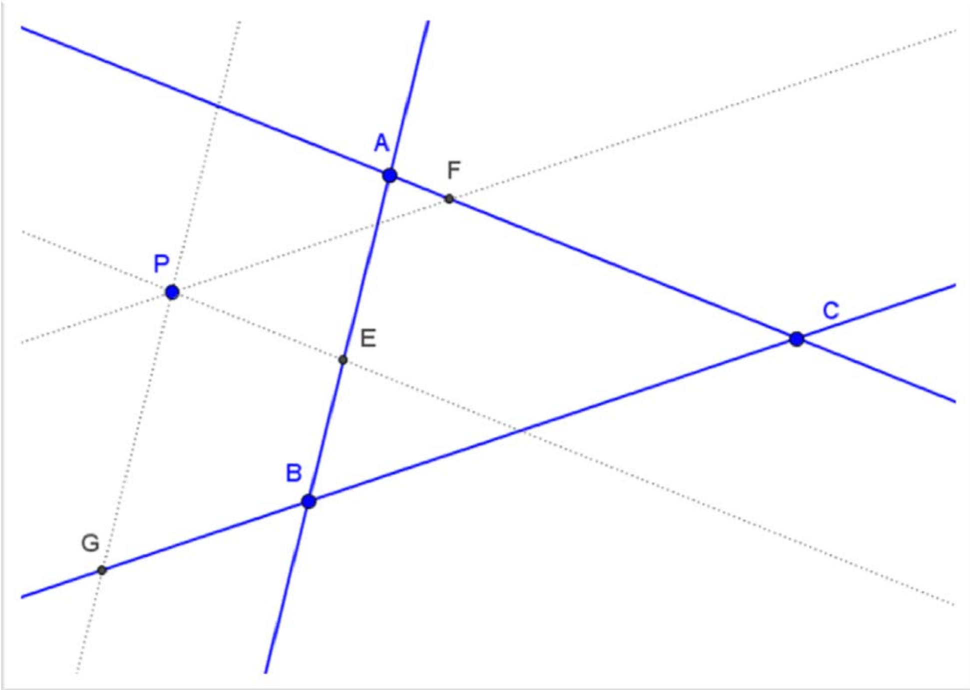
ABC triangle

P point in the plane

$E = $ Parallel projection of P onto AB

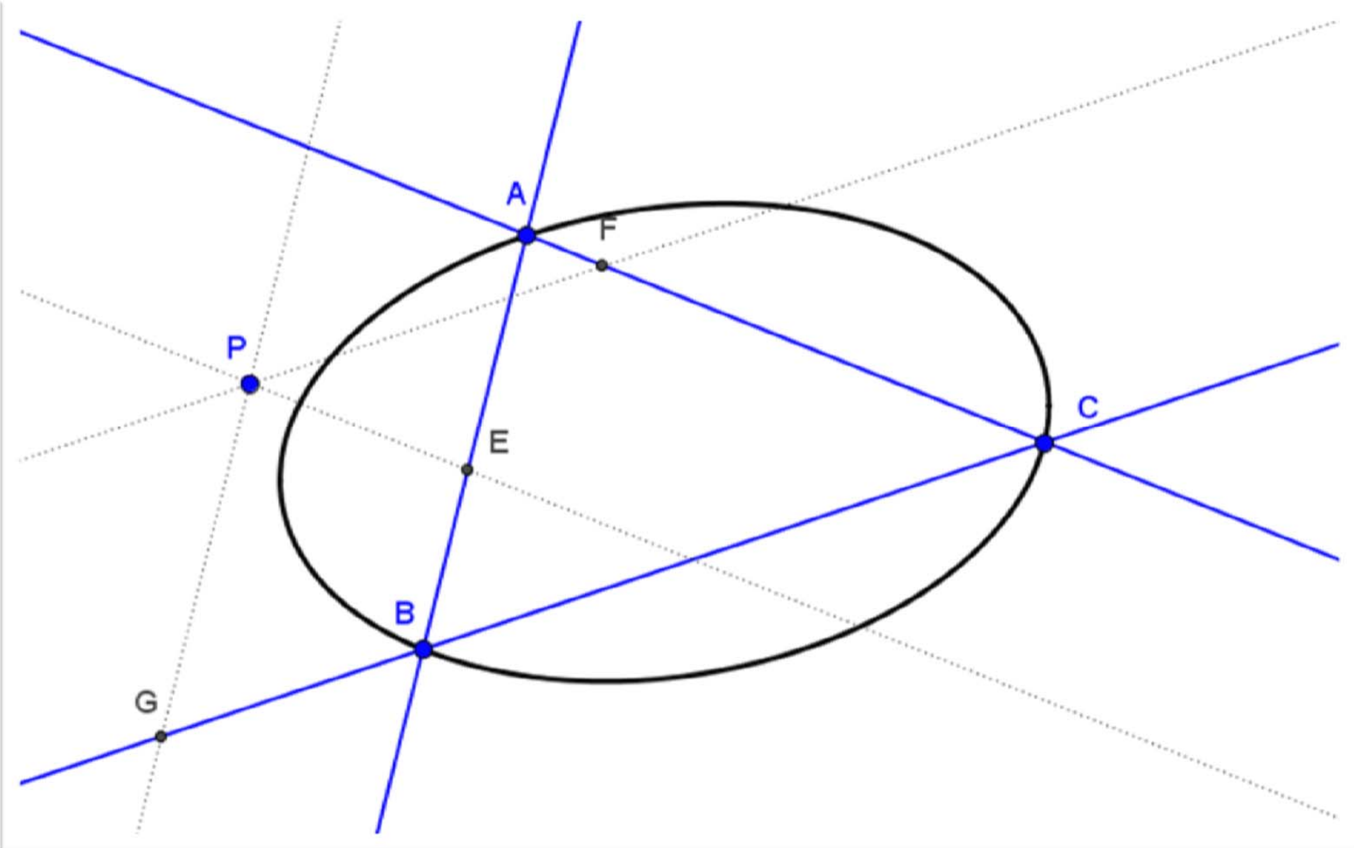
$F = $ Parallel projection of P onto AC

$G = $ Parallel projection of P onto BC



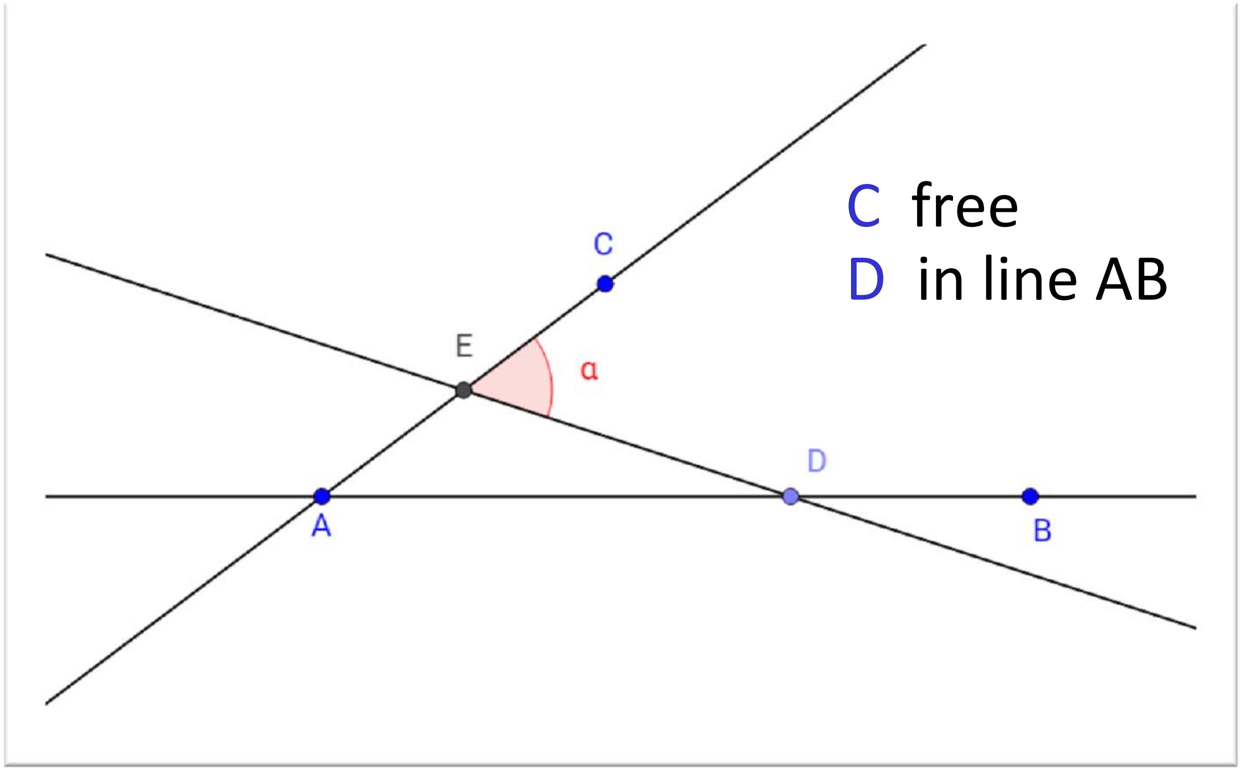
- When are E,F and G aligned?
  - For which positions of P?

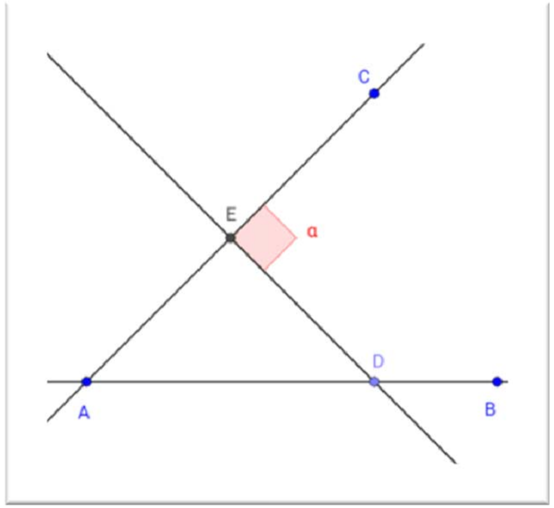
- LocusEquation[AreCollinear[E,F,G], P]

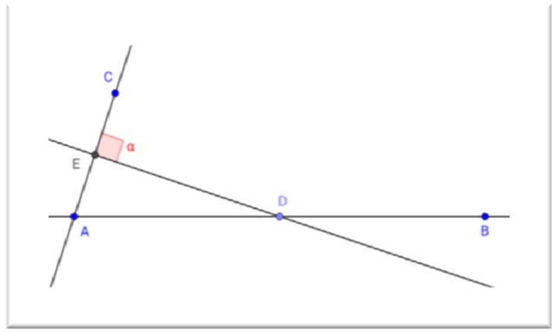


- Locus = ellipse

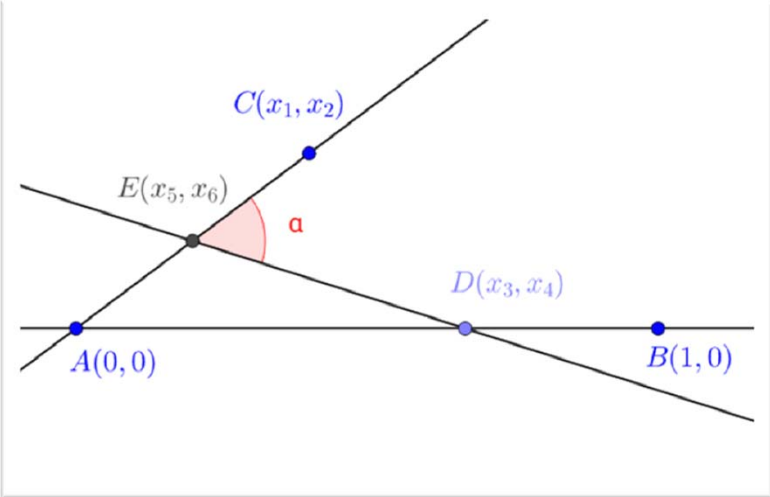
▪ Discovery over several points







- When is α a right angle?
  - for which positions of C and D?



$$\begin{cases} D \in \text{Line}(A, B) \\ E = \text{Midpoint}(A, C) \end{cases}$$

+

$$\{\text{Line}(A, C) \perp \text{Line}(E, D)$$

$$\begin{bmatrix} A(0,0) \\ B(1,0) \\ C(x_1, x_2) \\ D(x_3, x_4) \\ E(x_5, x_6) \end{bmatrix}$$

$$\begin{cases} x_4 = 0 \\ x_5 - \dfrac{x_1}{2} = 0 \\ x_6 - \dfrac{x_2}{2} = 0 \\ \boxed{x_1 \cdot (x_3 - x_5) + x_2 \cdot (x_4 - x_6) = 0} \end{cases}$$

solving for x_1, x_2, x_3, x_4

$$x_1^2 + x_2^2 - x_1 x_3 = 0$$

Not direct graphic interpretation

Not implemented in GeoGebra

Conclusion

- Dynamic Geometry + Discovery helps…

". . . exploring and modeling the more creative human-like thought processes of inductively exploring and manipulating diagrams to discover new insights about geometry".

- Johnson, L. E.: Automated Elementary Geometry Theorem Discovery via Inductive Diagram Manipulation.

  Master Thesis. MIT. (2015).

Thank you