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On the Stability of a Class of Permanent Rotations of a Heavy Asymmetric Gyrostat

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Abstract—We consider the motion of an asymmetric gyrostat under the attraction of a uniform Newtonian field. It is supposed that the center of mass lies along one of the principal axes of inertia, while a rotor spins around a different axis of inertia. For this problem, we obtain the possible permanent rotations, that is, the equilibria of the system. The Lyapunov stability of these permanent rotations is analyzed by means of the Energy – Casimir method and necessary and sufficient conditions are derived, proving that there exist permanent stable rotations when the gyrostat is oriented in any direction of the space. The geometry of the gyrostat and the value of the gyrostatic momentum are relevant in order to get stable permanent rotations. Moreover, it seems that the necessary conditions are also sufficient, but this fact can only be proved partially.

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Dedicated to Professor Markeev on his 75th birthday

1. INTRODUCTION

A gyrostat is made of a rigid body, a platform, and other bodies, the rotors, which move in such a way that their motions do not alter the mass distribution of the gyrostat (see Fig. 1). Since the rotor may be spinning along any axis, it is in general assumed that there are three rotors, each one along the three axes of the reference frame, usually the principal axes of inertia. This model is important in the field of mechanical systems because the rotors allow the spin vector to be stabilized. In particular, in spacecraft dynamics where the rotors, or spinning wheels, are used to control the attitude [23].

One of the key questions is the analysis of the stability of the permanent rotations or, equivalently, the equilibria of the system. When no external forces act upon the gyrostat, we have an extension of the problem of the motion of a free rigid body. This is an integrable problem and the stability and bifurcations of the permanent rotations have been studied in detail by many authors from different points of view, especially in the context of spacecraft attitude dynamics [5, 13, 15, 17, 18, 21]. These studies help in the analysis of the motion of the gyrostat under small perturbations, giving rise to chaotic behaviors which can be controlled by the action of the rotors [4, 24, 28, 29].

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The problem becomes more difficult when we consider more realistic approximations, as it is the motion of the gyrostat under the action of external torques, in particular, a uniform gravity field. When the rotors are at rest, we are left with the classical problem of the dynamics of a heavy rigid body or a heavy top. For this special case, there are many results concerning the stability of particular solutions under some specific assumptions, such as the stability of permanent rotations [26, 38], pendulum-like motions [6–8], planar motions [9, 10] or regular precessions [31, 32] (see also the book of Leimanis [30]). In the general case, when the action of the rotors is considered, some particular cases have been studied. For instance, the motion of a heavy symmetric gyrostat with a variable gyrostatic torque, with two rotors along orthogonal axes [37] or the families of isoconic motions of a heavy nonautonomous gyrostat with a fixed point [40]. On the other hand, other authors have considered the stability problem of particular solutions, especially those interesting for their practical applications [14, 20, 36, 39].

In this paper we consider the attitude of a gyrostat with a fixed point under a uniform Newtonian gravity field. In particular, we will focus on the stability of permanent or Staude's rotations, that is, rotations around the vertical axis, along which the gravity force acts. For this case, both necessary and sufficient conditions of stability have been obtained by means of different methods. In this sense, by building appropriate Lyapunov functions, Rumiantsev [35] and Anchev [1, 2] gave sufficient conditions of stability. In the particular case in which the center of mass lies on one principal axis and the gyrostatic momentum is directed along the same axis, Kovalev [27] derived sufficient conditions that matched those of Rumiantsev. He also applied KAM theory to study the stability when the associated quadratic form of the perturbed Hamiltonian is not sign definite, but the necessary conditions are satisfied. These results can also be obtained and improved by using the Energy – Casimir method [33, 34], provided the problem is a Lie – Poison system. Indeed, the Energy – Casimir method has been successfully used in rigid body dynamics [11, 22] and more recently applied to study the stability of permanent rotations of a heavy gyrostat in the same situation considered by Kovalev [12, 25]. The results obtained in [25] complemented those given by de Bustos et al. [12], adding new sufficient conditions for all the permanent rotations and proving also that, in some configurations of the moments of inertia, they are also necessary conditions. This case is very general as a wide class of stable permanent rotations can be achieved, with the gyrostat revolving around the vertical axis and oriented any which way. However, the action of a different rotor can also produce stable rotations, but losing some freedom in the parameters defining such stable rotations. This is the aim of this paper, where we will consider the stability of Staude's permanent rotations produced by the action of a spinning rotor that is not aligned with the principal axis where the center of mass lies.

This case presents rich dynamics and also some important differences with respect to the case previously studied in [12, 25, 27]. Indeed, the number of equilibria, or permanent rotations, is different, as are the stability conditions, the proof of the results being more complex. However, the most interesting fact is that, with this configuration, it is also possible to obtain stable permanent rotations in such a way that the gyrostat is oriented in any direction of the space. Even more, in the limiting case in which the center of mass coincides with the fixed point, we recover the classical stability conditions for a gyrostat in free motion.

2. EQUATIONS OF MOTION

Let us consider an asymmetric heavy gyrostat in a uniform gravity field. The gyrostat is composed of a rigid asymmetric platform and three axisymmetric wheels or rotors that can be in a relative rotation motion with respect to the platform. The rotation axis of each of the rotors coincides with one of the principal axes of the platform. We assume that the relative motion of the rotors does not modify the mass distribution of the gyrostat. We also consider that the whole gyrostat rotates with a fixed point O. Note that this fixed point O may not be located at the center of mass G of the gyrostat.

In order to study the permanent rotations of the gyrostat, we use two orthonormal reference frames centered at the fixed point O (see Fig. 1). On the one hand, the space or inertial reference frame $\mathcal{F}\{O, X, Y, Z\}$ fixed in the space, with the direction of the Z axis opposite to the acceleration $\boldsymbol{g}$ of the gravity field. On the other hand, the body frame $\mathcal{B}\{O, x, y, z\}$ fixed with the gyrostat, so that

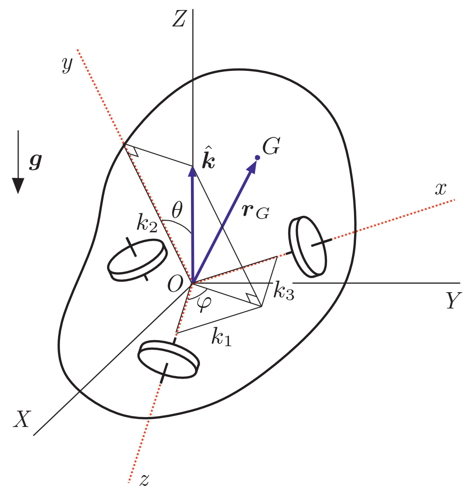

Fig. 1. Asymmetric gyrostat and reference frames.

the axes coincide with the principal axes of inertia of the gyrostat. The relative attitude between these two reference frames results from three consecutive rotations involving three angles such us the Euler angles [30]. Note that, as we study the permanent rotations of the gyrostat, we only need two of these angles to define the orientation of the rotating gyrostat in the inertial fixed frame $\mathcal{F}$.

Furthermore, the inertia tensor $\mathbb{I}$ of the gyrostat is determined by its mass distribution. In the body frame $\mathcal{B}$ (we recall that it was chosen to be the principal axis of inertia), the expression of this inertia tensor is diagonal, $\mathbb{I} = \mathrm{diag}(I_1, I_2, I_3)$ with $I_1 \neq I_2 \neq I_3$, as we assume an asymmetric gyrostat.

Let $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ denote the angular momentum of the gyrostat considered as a rigid body, $\boldsymbol{l} = (l_1, l_2, l_3)$ the angular momentum of the rotors, and $\hat{\boldsymbol{k}} = (k_1, k_2, k_3)$ the unitary vector in the direction of the fixed Z axis, the three vectors being expressed in the body frame $\mathcal{B}$. The total angular momentum of the gyrostat is $\boldsymbol{H} = \boldsymbol{\pi} + \boldsymbol{l}$ (for details see, e. g., [15, 17]). Besides, let (x_0, y_0, z_0) denote the coordinates of the center of mass G in the same frame.

In addition, if we denote by $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ the angular velocity of the gyrostat expressed in the body frame, we have the following relation:

$$\boldsymbol{\pi} = \mathbb{I}\boldsymbol{\omega}. \tag{2.1}$$

Because of the gravity field, the gyrostat is under the action of a gravitational torque $\boldsymbol{N}$ about the fixed point O, given by

$$\boldsymbol{N} = \boldsymbol{r}_G \times m\boldsymbol{g} = -mg\, \boldsymbol{r}_G \times \hat{\boldsymbol{k}},$$

where $\boldsymbol{r}_G$ is the position vector of the center of mass G of the gyrostat and m the mass of the gyrostat. Under all these assumptions, and by the angular momentum theorem about the fixed point O,

$$\frac{d\boldsymbol{H}}{dt} = \boldsymbol{N},$$

whereas the evolution on the space frame of the vector $\hat{\boldsymbol{k}}$ is given by

$$\frac{d\hat{\boldsymbol{k}}}{dt} = -\boldsymbol{\omega} \times \hat{\boldsymbol{k}} = -\left(\frac{\pi_1}{I_1}, \frac{\pi_2}{I_2}, \frac{\pi_3}{I_3}\right) \times (k_1, k_2, k_3).$$

In this way [25], the complete set of equations that governs the rotation dynamics of the gyrostat can be written as

$$\begin{aligned}
\frac{d\pi_1}{dt} &= \left(\frac{I_2 - I_3}{I_2 I_3}\right)\pi_2\pi_3 + \frac{l_2\pi_3}{I_3} - \frac{l_3\pi_2}{I_2} + (z_0k_2 - y_0k_3)mg,\\
\frac{d\pi_2}{dt} &= \left(\frac{I_3 - I_1}{I_1 I_3}\right)\pi_1\pi_3 + \frac{l_3\pi_1}{I_1} - \frac{l_1\pi_3}{I_3} + (x_0k_3 - z_0k_1)mg,\\
\frac{d\pi_3}{dt} &= \left(\frac{I_1 - I_2}{I_1 I_2}\right)\pi_1\pi_2 + \frac{l_1\pi_2}{I_2} - \frac{l_2\pi_1}{I_1} + (y_0k_1 - x_0k_2)mg,\\
\frac{dk_1}{dt} &= \frac{k_2\pi_3}{I_3} - \frac{k_3\pi_2}{I_2},\\
\frac{dk_2}{dt} &= \frac{k_3\pi_1}{I_1} - \frac{k_1\pi_3}{I_3},\\
\frac{dk_3}{dt} &= \frac{k_1\pi_2}{I_2} - \frac{k_2\pi_1}{I_1}.
\end{aligned} \tag{2.2}$$

The first three equations in (2.2) are the well-known Euler equations, while the last three equations are the Poisson equations.

On the other hand, the system can be regarded as a Lie–Poisson system, so that the equations of motion (2.2) can be derived from the following Hamiltonian function [25]:

$$\mathcal{H} = \frac{1}{2}\left(\frac{\pi_1^2}{I_1} + \frac{\pi_2^2}{I_2} + \frac{\pi_3^2}{I_3}\right) + (x_0k_1 + y_0k_2 + z_0k_3)mg. \tag{2.3}$$

We are interested in the stability of the permanent rotations of the gyrostat. These permanent rotations correspond to the equilibrium solutions of the equations of motion (2.2). Therefore, as the problem at hand is a Lie–Poisson system, we can study the stability of the permanent rotations by means of the so called Energy–Casimir method [3, 22, 33, 34]. In this sense, the system has two conserved quantities that can be used as Casimir functions in the stability analysis

$$C_1 \equiv k_1^2 + k_2^2 + k_3^2 = 1, \tag{2.4}$$

$$C_2 \equiv (\boldsymbol{\pi} + \boldsymbol{l}) \cdot \hat{\boldsymbol{k}} = (\pi_1 + l_1)k_1 + (\pi_2 + l_2)k_2 + (\pi_3 + l_3)k_3, \tag{2.5}$$

the last one, the projection of the total angular momentum of the gyrostat, $\boldsymbol{\pi} + \boldsymbol{l}$, on the Z axis of the fixed inertial frame $\mathcal{F}$.

3. EQUILIBRIUM SOLUTIONS

Hereinafter, we will consider the following situation for the gyrostat. The center of mass, G, is located at the z axis of the body frame $\mathcal{B}$, that is, $x_0 = y_0 = 0$. Besides, the direction of the gyrostatic momentum $\boldsymbol{l}$ of the rotors is parallel to the x axis of the body frame $\mathcal{B}$, that is, $l_2 = l_3 = 0$. Taking into account all these premises, the equations of motion (2.2) reduce to

$$\begin{aligned}
\frac{d\pi_1}{dt} &= \left(\frac{I_2 - I_3}{I_2 I_3}\right)\pi_2\pi_3 + mgz_0k_2,\\
\frac{d\pi_2}{dt} &= \left(\frac{I_3 - I_1}{I_1 I_3}\right)\pi_1\pi_3 - \frac{l_1\pi_3}{I_3} - mgz_0k_1,\\
\frac{d\pi_3}{dt} &= \left(\frac{I_1 - I_2}{I_1 I_2}\right)\pi_1\pi_2 + \frac{l_1\pi_2}{I_2},\\
\frac{dk_1}{dt} &= \frac{k_2\pi_3}{I_3} - \frac{k_3\pi_2}{I_2},\\
\frac{dk_2}{dt} &= \frac{k_3\pi_1}{I_1} - \frac{k_1\pi_3}{I_3},\\
\frac{dk_3}{dt} &= \frac{k_1\pi_2}{I_2} - \frac{k_2\pi_1}{I_1}.
\end{aligned} \tag{3.1}$$

Under these assumptions, we will focus on Staude's permanent rotations, when the angular velocity is aligned with the Z axis of the inertial frame, that is, $\boldsymbol{\omega} = \omega \hat{\boldsymbol{k}}$, ω being the norm of the angular velocity vector. In this way, we have the following result.

Theorem 1. *There are two families of equilibrium points. The first is given by points of the form*

$$E_0 \equiv (I_1 \omega \sin\varphi, 0, I_3\omega\cos\varphi, \sin\varphi, 0, \cos\varphi),$$

where $\varphi \in [0, 2\pi)$ *and* $\omega \in \mathbb{R}$ *such that*

$$(I_1 - I_3)\omega^2 \sin\varphi\cos\varphi + \omega l_1 \cos\varphi + gmz_0 \sin\varphi = 0. \tag{3.2}$$

The second is defined by points of the form

$$E_1 \equiv (I_1\omega\sin\varphi\sin\theta, I_2\omega\cos\theta, I_3\omega\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta, \cos\varphi\sin\theta),$$

where $\varphi \in [0, 2\pi)$, $\theta \in [0, \pi/2) \cup (\pi/2, \pi]$ *and* $\omega \in \mathbb{R}$ *such that*

$$l_1 + (I_1 - I_2)\omega\sin\varphi\sin\theta = 0, \qquad (I_2 - I_3)\omega^2\cos\varphi\sin\theta + gmz_0 = 0. \tag{3.3}$$

It is worth noting here that the angles φ and θ give us the orientation of the gyrostat with respect to the fixed frame $\mathcal{F}$ (see Fig. 1). Also note that the equilibrium solutions correspond to rotations of the gyrostat around the vertical axis in the inertial frame $\mathcal{F}$.

Proof. Taking into account that $\hat{\boldsymbol{k}}$ is a unitary vector, we introduce angles $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$ in such a way that

$$k_1 = \sin\varphi\sin\theta, \quad k_2 = \cos\theta, \quad k_3 = \cos\varphi\sin\theta. \tag{3.4}$$

Analogously, for the components of the angular momentum, from (2.1), we have

$$\pi_1 = \omega I_1 \sin\varphi\sin\theta, \quad \pi_2 = \omega I_2\cos\theta, \quad \pi_3 = \omega I_3 \cos\varphi\sin\theta. \tag{3.5}$$

It is easy to verify that, under this parameterization, the last three equations of system (3.1) vanish. Thus, to obtain the equilibrium solutions, we have to see the conditions under which the three first equations vanish. We rewrite them as

$$\begin{cases} (gmz_0 + (I_2 - I_3)\omega^2\cos\varphi\sin\theta)\cos\theta = 0, \\[2ex] (gmz_0\sin\varphi + (l_1 + (I_1 - I_3)\omega\sin\varphi\sin\theta)\omega\cos\varphi)\sin\theta = 0, \\[2ex] (l_1 + (I_1 - I_2)\omega\sin\varphi\sin\theta)\omega\cos\theta = 0. \end{cases} \tag{3.6}$$

Discarding the case $\omega = 0$, we divide the solutions into two groups: those satisfying $\cos\theta = 0$ and those which do not.

If $\cos\theta = 0$, the first and third equations of (3.6) vanish. Taking into account that $\theta \in [0, \pi]$, $\sin\theta = 1$, and then the second equation vanishes if

$$(I_1 - I_3)\omega^2\sin\varphi\cos\varphi + \omega l_1\cos\varphi + gmz_0\sin\varphi = 0.$$

If $\cos\theta \neq 0$, from the third equation of (3.6) we see that equilibrium solutions satisfy

$$l_1 + (I_1 - I_2)\omega\sin\varphi\sin\theta = 0.$$

By eliminating l_1, the first two equations become

$$\begin{cases} (gmz_0 + (I_2 - I_3)\omega^2\cos\varphi\sin\theta)\cos\theta = 0, \\[2ex] (gmz_0 + (I_2 - I_3)\omega^2\cos\varphi\sin\theta)\sin\varphi\sin\theta = 0, \end{cases}$$

which vanish at the same time if the condition

$$gmz_0 + (I_2 - I_3)\omega^2\cos\varphi\sin\theta = 0$$

holds. □

Remark 1. Note that the first family is a limiting case of the second when $\theta = \pi/2$. However, the two conditions

$$l_1 + (I_1 - I_2)\omega \sin\varphi \sin\theta = 0, \qquad gmz_0 + (I_2 - I_3)\omega^2 \cos\varphi \sin\theta = 0$$

do not need to be satisfied at the same time, but only a linear combination of them.

Remark 2. If $\omega = 0$, there are also equilibrium solutions where the angular momentum is equal to zero. One of these solutions takes place when the fixed point is precisely the center of mass, that is, $z_0 = 0$. The other one occurs if $\theta = \pi/2$ and $\varphi = 0$ or $\varphi = \pi$. In this case, the gyrostat is upright or hanging from the fixed point and the center of mass along the direction of the gravity field.

4. STABILITY ANALYSIS

In this section we will focus on the Lyapunov stability analysis of the equilibrium solutions given in Theorem 1. Taking into account that we are considering a Poisson system, to establish sufficient stability conditions we can use the classical Energy–Casimir method [3, 22] and a generalized version given by Ortega and Ratiu [34], which reads:

Theorem 2 (Generalized Energy – Casimir method). *Let $(M, \{.,.\}, h)$ be a Poisson system, and $m \in M$ be an equilibrium of the Hamiltonian vector field X_h. If there is a set of conserved quantities $C_1, \ldots, C_n \in C^\infty(M)$ for which*

$$\mathbf{d}(h + C_1 + \cdots + C_n)(m) = 0,$$

and

$$\mathbf{d}^2(h + C_1 + \cdots + C_n)(m)\Big|_{W\times W}$$

is definite for W defined by

$$W = \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m),$$

then m is stable. If $W = \{0\}$, m is always stable.

This result is more convenient, as the definiteness of the Hessian, given by the second derivative of the augmented Hamiltonian $h + C_1 + \cdots + C_n$, is checked in the reduced space $W \times W$ and not in the whole space $T_mM \times T_mM$.

In this case, $M = \mathbb{R}^6$ and h, C_1 and C_2 are given, respectively, by (2.3), (2.4) and (2.5). Besides, it is well known that each equilibrium point of the problem is a critical point of the augmented Hamiltonian function

$$H_A = \frac{1}{2}\left(\frac{\pi_1^2}{I_1} + \frac{\pi_2^2}{I_2} + \frac{\pi_3^2}{I_3}\right) + mgz_0k_3 + \Big((\pi_1 + l_1)k_1 + \pi_2k_2 + \pi_3k_3\Big)\lambda + (k_1^2 + k_2^2 + k_3^2)\mu,$$

λ and μ being parameters to be determined with this condition. On the one hand, permanent rotations E_0 and E_1 are critical points of H_A if they satisfy

$$\mathbf{d}(H_A)(E_0) = 0 \Rightarrow \lambda = -\omega, \quad \mu = \tfrac{1}{2}(I_3\omega^2 - gmz_0 \sec\varphi),$$

$$\mathbf{d}(H_A)(E_1) = 0 \Rightarrow \lambda = -\omega, \quad \mu = \tfrac{1}{2}I_2\omega^2.$$

On the other hand, let us now determine the space

$$W = \ker \mathbf{d}C_1 \cap \ker \mathbf{d}C_2.$$

Taking into account the parameterization (3.4), (3.5), we obtain

$$\begin{aligned}
\mathbf{d}C_1 &= 2\sin\varphi\sin\theta dk_1 + 2\cos\theta dk_2 + 2\cos\varphi\sin\theta dk_3,\\
\mathbf{d}C_2 &= \sin\varphi\sin\theta d\pi_1 + \cos\theta d\pi_2 + \cos\varphi\sin\theta d\pi_3\\
&\quad + (l_1 + \omega I_1\sin\varphi\sin\theta)dk_1 + \omega I_2\cos\theta dk_2 + \omega I_3\cos\varphi\sin\theta dk_3.
\end{aligned}$$

Solving the system $\mathbf{d}C_1 = \mathbf{d}C_2 = 0$, we get

$$\pi_3 = -\big(l_1 k_1 + ((I_2 - I_3)k_2\omega + \pi_2)\cos\theta\big)\csc\theta\sec\varphi - \big((I_1 - I_3)k_1\omega + \pi_1\big)\tan\varphi,$$

$$k_3 = -k_1\tan\varphi - k_2\cot\theta\sec\varphi.$$

Thus, when $\cos\varphi \neq 0$ and $\sin\theta \neq 0$, W is generated by the following four vectors:

$$\begin{cases} \mathbf{v}_1 = \cos\varphi\sin\theta\,\hat{\mathbf{e}}_1 - \sin\varphi\sin\theta\,\hat{\mathbf{e}}_3, \\ \mathbf{v}_2 = \cos\varphi\sin\theta\,\hat{\mathbf{e}}_2 - \cos\theta\,\hat{\mathbf{e}}_3, \\ \mathbf{v}_3 = \big(-l_1 - (I_1 - I_3)\omega\sin\varphi\sin\theta\big)\,\hat{\mathbf{e}}_3 + \cos\varphi\sin\theta\,\hat{\mathbf{e}}_4 - \sin\varphi\sin\theta\,\hat{\mathbf{e}}_6, \\ \mathbf{v}_4 = (I_3 - I_2)\omega\cos\theta\,\hat{\mathbf{e}}_3 + \cos\varphi\sin\theta\,\hat{\mathbf{e}}_5 - \cos\theta\,\hat{\mathbf{e}}_6, \end{cases}$$

$\{\hat{\mathbf{e}}_i\}_{1\leqslant i\leqslant 6}$ being the canonic basis of $\mathbb{R}^6$.

Let us now consider a vector $\mathbf{v}$ in W, which is expressed as

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4,$$

where $x_i \in \mathbb{R}$, $i = 1, \ldots, 4$. The quadratic form in the variables x_i given by

$$\mathbf{v}^T \cdot \mathrm{Hess}(H_A) \cdot \mathbf{v}$$

leads us to $\mathrm{Hess}(H_A)\Big|_{W\times W}$. In this way we obtain

$$\mathrm{Hess}(H_A)\Big|_{W\times W} = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{12} & h_{22} & h_{23} & h_{24} \\ h_{13} & h_{23} & h_{33} & h_{34} \\ h_{14} & h_{24} & h_{34} & h_{44} \end{bmatrix}, \tag{4.1}$$

where

$$h_{11} = \frac{1}{I_1 I_3}(I_1\sin^2\varphi + I_3\cos^2\varphi)\sin^2\theta,$$

$$h_{12} = \frac{1}{2I_3}\sin\varphi\sin 2\theta,$$

$$h_{13} = \frac{1}{I_3}\Big[l_1\sin\varphi + I_3\lambda\cos^2\varphi\sin\theta + \big[I_3\lambda + (I_1 - I_3)\omega\big]\sin^2\varphi\sin\theta\Big]\sin\theta,$$

$$h_{14} = \frac{I_3\lambda + (I_2 - I_3)\omega}{2I_3}\sin\varphi\sin 2\theta,$$

$$h_{22} = \frac{1}{I_2 I_3}(I_2\cos^2\theta + I_3\cos^2\varphi\sin^2\theta),$$

$$h_{23} = \frac{1}{I_3}\Big[l_1 + \big[I_3\lambda + (I_1 - I_3)\omega\big]\sin\varphi\sin\theta\Big]\cos\theta,$$

$$h_{24} = \frac{1}{I_3}\Big[\big[I_3\lambda + (I_2 - I_3)\omega\big]\cos^2\theta + I_3\lambda\cos^2\varphi\sin^2\theta\Big],$$

$$h_{33} = \frac{1}{I_3}[l_1 + (I_1 - I_3)\omega\sin\varphi\sin\theta]^2 + 2\big[l_1 + (I_1 - I_3)\omega\sin\varphi\sin\theta\big]\lambda\sin\varphi\sin\theta + \mu\sin^2\theta,$$

$$h_{34} = \frac{\cos\theta}{I_3}\Big[(I_2 - I_3)[l_1 + (I_1 - I_3)\omega\sin\varphi\sin\theta]\omega + [l_1 + (I_1 + I_2 - 2I_3)\omega\sin\varphi\sin\theta]I_3\lambda + 2I_3\mu\sin\varphi\sin\theta\Big],$$

$$h_{44} = \frac{1}{I_3}\Big[(I_2 - I_3)^2\omega^2\cos^2\theta + 2(I_2 - I_3)I_3\lambda\omega\cos^2\theta + 2(\cos^2\theta + \cos^2\varphi\sin^2\theta)I_3\mu\Big].$$

4.1. Stability of Family E_1

Once we have introduced the tools we need to analyze the stability, to begin with, we state the first stability result, concerning the equilibrium solutions E_1.

Theorem 3. *The equilibrium E_1 is stable if the following conditions are satisfied:*

$$I_2 - I_1 > 0, \qquad (I_2 - I_3)\big(I_2 + 3(I_2 - I_3)\cos^2\varphi\sin^2\theta\big) > 0.$$

Proof. The quadratic form

$$\mathbf{d}^2(h + C_1 + C_2)(E_1)\Big|_{W\times W}$$

is positive definite if $\mathrm{Hess}(H_A)\Big|_{W\times W}$ satisfies Sylvester's criterion. Taking into account that, for E_1,

$$\lambda = -\omega, \quad \mu = \frac{I_2\omega^2}{2}, \quad l_1 + (I_1 - I_2)\omega\sin\varphi\sin\theta = 0,$$

we obtain for the leading principal minors:

$$D_1 = \frac{(I_3\cos^2\varphi + I_1\sin^2\varphi)\sin^2\theta}{I_1I_3},$$

$$D_2 = \left(\frac{\cos^2\theta}{I_1I_3} + \frac{\cos^2\varphi\sin^2\theta}{I_1I_2} + \frac{\sin^2\varphi\sin^2\theta}{I_2I_3}\right)\cos^2\varphi\sin^2\theta,$$

$$D_3 = \big((2I_1 + I_2 - 3I_3)(I_2 - I_3)\cos^2\varphi\sin^2\varphi\sin^2\theta + (I_1 - I_3)I_2\sin^2\varphi$$
$$+(I_2 - I_1)(I_2 + (I_3 - I_2)\sin^2\theta)\big)\frac{\omega^2\cos^2\varphi\sin^4\theta}{I_1I_2I_3},$$

$$D_4 = (I_2 - I_1)(I_2 - I_3)\big(I_2 + 3(I_2 - I_3)\cos^2\varphi\sin^2\theta\big)\frac{\omega^4\cos^4\varphi\cos^2\theta\sin^4\theta}{I_1I_2I_3}.$$

On the one hand, it is clear that D_1 and D_2 are always positive, provided we suppose

$$\cos\varphi\sin\theta \neq 0.$$

On the other hand, D_3 and D_4 are positive if

$$\Delta_1 \equiv (2I_1 + I_2 - 3I_3)(I_2 - I_3)\cos^2\varphi\sin^2\varphi\sin^2\theta$$
$$+ I_2(I_1 - I_3)\sin^2\varphi + (I_2 - I_1)\big(I_2 + (I_3 - I_2)\sin^2\theta\big) > 0,$$

and

$$(I_2 - I_1)(I_2 - I_3)\Delta_2 > 0,$$

where

$$\Delta_2 = I_2 + 3(I_2 - I_3)\cos^2\varphi\sin^2\theta.$$

We will divide our analysis into two cases: $I_2 - I_3 > 0$ and $I_2 - I_3 < 0$.

Case $I_2 - I_3 > 0$.

In this case, $\Delta_2 > 0$ and, to obtain sufficient stability conditions, it is necessary that $I_2 - I_1 > 0$. We are going to show that then also $D_3 > 0$. To this end, let us introduce the change of variables

$$\sin^2\theta = x, \quad \sin^2\varphi = y, \quad x, y \in [0, 1]. \tag{4.2}$$

Then Δ_1 can be written as

$$\Delta_1(x, y) = (I_2 - I_1)(I_2 - I_2x + I_3x) + I_2(I_1 - I_3)y$$
$$+ (2I_1 + I_2 - 3I_3)(I_2 - I_3)(1 - y)xy,$$

which is a linear function in x. If, for every y, $\Delta_1(0,y) > 0$ and $\Delta_1(1,y) > 0$, we can ensure that $\Delta_1(x,y) > 0$ for every $x, y \in [0,1]$.

On the one hand, we have

$$\Delta_1(0,y) = \big((I_2 - I_1) + (I_1 - I_3)y\big)I_2 > 0,$$

provided that $I_2 - I_1 > 0$ and $I_2 - I_3 > 0$. On the other hand,

$$\Delta_1(1,y) = (I_2 - I_1)I_3 + (I_1 - I_3)I_2 y + (2I_1 + I_2 - 3I_3)(I_2 - I_3)(1-y)y.$$

Here, we distinguish two situations, depending on the sign of the coefficient of $(1-y)y$. If this coefficient is positive, the second-degree polynomial in y has a maximum and, taking into account that

$$\Delta_1(1,0) = (I_2 - I_1)I_3 > 0, \quad \Delta_1(1,1) = (I_2 - I_3)I_1 > 0, \tag{4.3}$$

it follows that, in this case, $\Delta_1(1,y) > 0$ for every $y \in [0,1]$. If, on the contrary, the coefficient of $(1-y)y$ is negative, we obtain immediately that

$$I_1 - I_3 < 0. \tag{4.4}$$

Moreover, there is a minimum at the point

$$y_m = \frac{(I_3 - I_1)I_2 + (2I_1 + I_2 - 3I_3)(I_3 - I_2)}{2(2I_1 + I_2 - 3I_3)(I_3 - I_2)}.$$

However, $y_m > 1$. Indeed, $y_m > 1$ implies

$$(I_3 - I_1)I_2 - (2I_1 + I_2 - 3I_3)(I_3 - I_2) = I_2^2 + (I_2 - 2I_3)I_1 - 3I_2I_3 + 3I_3^2 > 0.$$

Now, if $(I_2 - 2I_3) > 0$, it follows

$$I_2^2 + (I_2 - 2I_3)I_1 - 3I_2I_3 + 3I_3^2 > I_2^2 - 3I_2I_3 + 3I_3^2 > 0,$$

because the quadratic form $I_2^2 - 3I_2I_3 + 3I_3^2$ has a negative discriminant. If $(I_2 - 2I_3) < 0$, from (4.4) we obtain

$$I_2^2 + (I_2 - 2I_3)I_1 - 3I_2I_3 + 3I_3^2 > I_2^2 + (I_2 - 2I_3)I_3 - 3I_2I_3 + 3I_3^2 = (I_2 - I_3)^2 \geqslant 0.$$

Thus, regardless of the sign of $(I_2 - 2I_3)$, $y_m > 1$ and, by (4.3), $\Delta_1(1,y) > 0$ for every $y \in [0,1]$.

Case $I_2 - I_3 < 0$.

To analyze the sign of D_3 and D_4 we observe that

$$\Delta_1(I_1 = I_2) \equiv (I_2 - I_3)\Delta_2 \sin^2\varphi.$$

This allows us introduce a real parameter ε in such a way that $I_1 = I_2 + \varepsilon$. By doing so, we obtain

$$\Delta_1 = (I_2 - I_3)\Delta_2 \sin^2\varphi + \varepsilon\Delta_3, \tag{4.5}$$

where

$$\Delta_3 = -I_2 + (I_2 - I_3)\sin^2\theta + (I_2 + 2(I_2 - I_3)\cos^2\varphi \sin^2\theta)\sin^2\varphi < 0.$$

Indeed, by introducing the change of variables (4.2), Δ_3 becomes

$$\Delta_3 = -(1-y)I_2 + (I_2 - I_3)(1 + 2y - 2y^2)x,$$

which is a linear function in x. However,

$$\Delta_3(x = 0) = -(1-y)I_2 < 0,$$

whereas the slope is also negative, provided

$$I_2 - I_3 < 0, \quad 1 + 2y - 2y^2 > 0, \quad y \in [0,1].$$

Thus, $\Delta_3 < 0$.

Now we note that $D_4 > 0$ if $(I_2 - I_1)\Delta_2 < 0$. There are two different situations. On the one hand, if $\Delta_2 < 0$, then $I_2 - I_1 > 0$. In this case, it follows from (4.5) that $\Delta_1 > 0$ as $\varepsilon < 0$, and thus $D_3 > 0$. On the other hand, if $\Delta_2 > 0$, a similar reasoning yields $\Delta_1 < 0$ and, consequently, $D_3 < 0$. $\square$

Remark 3. It can be proved that, in the cases $\cos\varphi = 0$ and $\sin\theta = 0$, the above theorem is also valid, and the stability conditions are those obtained in the limit, that is, I_2 is the largest moment of inertia.

It is worth noting that the sufficient conditions given in Theorem 3 match the necessary conditions in the case $I_2 > I_1$. Indeed, we have the following result:

Theorem 4. *A necessary condition for the equilibrium E_1 to be stable is*

$$(I_2 - I_1)(I_2 - I_3)[I_2 + 3(I_2 - I_3)\cos^2\varphi \sin^2\theta] > 0.$$

Proof. It is known that spectral stability is a necessary condition for stability, that is to say, all the eigenvalues of the Jacobian matrix associated to the equilibrium must have zero real part. Making use of (3.3), the Jacobian matrix at E_1 reads

$$\begin{bmatrix} 0 & \frac{(I_2-I_3)\omega c_1 s_2}{I_2} & \frac{(I_2-I_3)\omega c_2}{I_3} & 0 & (I_3-I_2)\omega^2 c_1 s_2 & 0 \\ \frac{(I_3-I_1)\omega c_1 s_2}{I_1} & 0 & \frac{(I_3-I_2)\omega s_1 s_2}{I_3} & (I_2-I_3)\omega^2 c_1 s_2 & 0 & 0 \\ \frac{(I_1-I_2)\omega c_2}{I_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{c_1 s_2}{I_2} & \frac{c_2}{I_3} & 0 & \omega c_1 s_2 & -\omega c_2 \\ \frac{c_1 s_2}{I_1} & 0 & -\frac{s_1 s_2}{I_3} & -\omega c_1 s_2 & 0 & \omega s_1 s_2 \\ -\frac{c_2}{I_1} & \frac{s_1 s_2}{I_2} & 0 & \omega c_2 & -\omega s_1 s_2 & 0 \end{bmatrix},$$

where c_1, s_1, c_2, s_2 stand for $\cos\varphi$, $\sin\varphi$, $\cos\theta$ and $\sin\theta$, respectively. The characteristic polynomial can be written as

$$p(\lambda) = \frac{\lambda^2}{I_1 I_2 I_3}(a\lambda^4 + b\lambda^2 + c),$$

where

$$\begin{aligned} a &= I_1 I_2 I_3, \\ b &= \Big(I_2(I_2^2 - I_1 I_2 + 2I_1 I_3 - I_2 I_3)\cos^2\theta \\ &\qquad + I_3\Big[I_1(3I_2 - 2I_3)\cos^2\varphi + (I_2 - I_3)^2\cos^2\varphi + I_1 I_2 \sin^2\varphi\Big]\sin^2\theta\Big)\omega^2, \\ c &= (I_2 - I_1)(I_2 - I_3)\big(I_2 + 3(I_2 - I_3)\cos^2\varphi\sin^2\theta\big)\omega^4\cos^2\theta. \end{aligned}$$

All the eigenvalues have zero real part if a, b and c are positive and, at the same time, $b^2 - 4ac > 0$. Thus, a necessary condition is $c > 0$ and the statement of the theorem follows. □

Remark 4. Taking into account (3.4), the sufficient and necessary condition for stability, when $I_2 > I_1$, can be written as

$$(I_2 - I_1)(I_2 - I_3)[I_2 + 3(I_2 - I_3)k_3^2] > 0.$$

From this remark, we can establish positions suitable for stable permanent rotations. Indeed, if I_2 is the largest moment of inertia, it is possible to have stable permanent rotations, whatever the orientation of the gyrostat. On the contrary, if $I_2 < I_3 < 4/3 I_2$, there are no stable permanent rotations, due to the fact that $D_4 < 0$. However, as soon as $I_3 > 4/3 I_2$, we find stable permanent rotations in two symmetric spherical caps, along the z axis, whose size increases with I_3, in such a way that, as $I_3 \to \infty$, all the orientations are possible (see Fig. 2). The meridian $y = 0$ must be excluded from our analysis as the points on this curve are those of the family E_0 to be analyzed further.

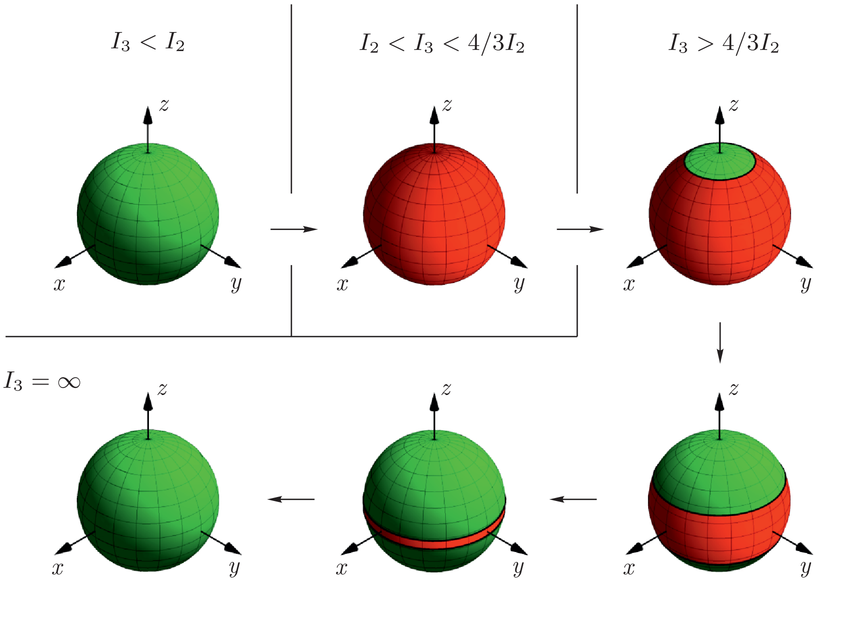

Fig. 2. The orientation of stable permanent rotations as a function of I_3 when $I_2 > I_1$. Green areas stand for stable rotations, while red areas stand for unstable ones.

4.2. Stability of Family E_0

Let us now analyze the E_0 family of equilibrium points. For this family we have the following result:

Theorem 5. *The equilibrium E_0 is stable if the following conditions are satisfied:*

$$-B_1 \cos\varphi > 0, \qquad B_0 B_1 > 0,$$

where

$$\begin{aligned} B_0 &= gmz_0 + (I_2 - I_3)\omega^2 \cos\varphi, \\ B_1 &= 4(I_1 - I_3)I_3\omega^4 \cos^3\varphi + \left[3I_1 + I_3 - (I_1 - I_3)(2\cos 2\varphi + \cos 4\varphi)\right] gmz_0\omega^2 \\ &\quad - 4g^2m^2z_0^2 \cos\varphi \sin^2\varphi. \end{aligned} \tag{4.6}$$

Proof. The proof follows directly from the application of Theorem 2. Indeed, E_0 is a critical point of H_A if

$$\lambda = -\omega, \qquad \mu = \frac{1}{2}(I_3\omega^2 - gmz_0 \sec\varphi).$$

Taking into account that $\theta = \pi/2$ for the family E_0 and

$$l_1 = \frac{(I_3 - I_1)\omega^2 \sin\varphi\cos\varphi - gmz_0 \sin\varphi}{\omega\cos\varphi},$$

we apply the Sylvester criterion to the reduced Hessian matrix $\mathrm{Hess}(H_A)\Big|_{W\times W}$, obtaining for the leading principal minors

$$D_1 = \frac{\cos^2\varphi}{I_1} + \frac{\sin^2\varphi}{I_3}, \quad D_2 = \frac{I_1\cos^2\varphi\sin^2\varphi + I_3\cos^4\varphi}{I_1I_2I_3},$$
$$D_3 = -\frac{\cos\varphi}{4I_1I_2I_3\omega^2}B_1, \quad D_4 = \frac{\cos^2\varphi}{4I_1I_2I_3\omega^2}B_0B_1,$$

where B_0 and B_1 are the expressions given in the statement of the theorem, and the conclusion follows immediately as D_1 and D_2 are positive. $\square$

It is worth noting that the sufficient conditions given by Theorem 5 are, in some cases, necessary conditions too, as it can be deduced from a linear stability analysis. In fact, we have this result:

Theorem 6. *If the equilibrium E_0 is stable, then $B_0 B_1 > 0$, where B_0 and B_1 are given by* (4.6).

Proof. The Jacobian matrix at E_0 is given by

$$\begin{bmatrix} 0 & \frac{(I_2-I_3)}{I_2}\omega\cos\varphi & 0 & 0 & gmz_0 & 0 \\ \frac{(I_3-I_1)}{I_1}\omega\cos\varphi & 0 & \frac{gmz_0}{I_3\omega}\tan\varphi & -gmz_0 & 0 & 0 \\ 0 & \frac{(I_3-I_2)\omega^2\sin^2\varphi-gmz_0\tan\varphi}{I_2\omega} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{I_2}\cos\varphi & 0 & 0 & \omega\cos\varphi & 0 \\ \frac{1}{I_1}\cos\varphi & 0 & -\frac{1}{I_3}\sin\varphi & -\omega\cos\varphi & 0 & \omega\sin\varphi \\ 0 & \frac{1}{I_2}\sin\varphi & 0 & 0 & -\omega\sin\varphi & 0 \end{bmatrix}.$$

The spectral stability is a necessary condition for the Lyapunov stability. However, the characteristic polynomial has the form

$$p(\lambda) = \frac{\lambda^2}{I_1 I_2 I_3 \omega^2}(a\lambda^4 + b\lambda^2 + c),$$

where

$$\begin{aligned} a &= I_1 I_2 I_3 \omega^2, \\ b &= \big(I_1 I_2 + (I_1 - I_2)(I_2 - I_3)\cos^2\varphi\big) I_3 \omega^4 + \\ &\quad + \big(I_1(I_2 - I_3) - I_2(I_1 + I_3)\cos\varphi\big) gmz_0\omega^2 \sec\varphi + g^2 m^2 z_0^2 \tan^2\varphi, \\ c &= \frac{B_0 B_1}{4\cos^2\varphi}. \end{aligned}$$

Spectral stability takes place if a, b and c all have the same sign and, at the same time, $b^2 - 4ac > 0$. Thus, provided $a > 0$, a necessary condition is $c > 0$ and the statement of the theorem follows. $\square$

Theorems 5 and 6 establish sufficient and necessary conditions of stability for E_0. Nevertheless, these conditions need to be analyzed in practice. That is to say, we will try to see if, with fixed φ and mass distribution of the gyrostat, there exist stable permanent rotations. We will focus on the sufficient stability conditions given in Theorem 5, and in doing so, we state the following result.

Theorem 7. *There are stable permanent rotations of the family of equilibrium points E_0 if*

1. $z_0 \cos\varphi < 0$ *or*
2. $z_0 \cos\varphi > 0$ *and, at the same time, I_3 is the largest moment of inertia.*

Otherwise, the sufficient conditions of stability are not satisfied.

Proof. We recall that, from Theorem 5, the sufficient stability conditions are

$$-\cos\varphi B_1 > 0, \qquad B_0 B_1 > 0,$$

where B_0 and B_1 are given by (4.6). First of all, we note that these expressions are invariant under the transformations

$$\begin{array}{ccc} \varphi & \longrightarrow & -\varphi, \\ (\varphi, z_0) & \longrightarrow & (\pi+\varphi, -z_0), \\ (\varphi, z_0) & \longrightarrow & (\pi-\varphi, -z_0). \end{array}$$

From the last two transformations we deduce that all results obtained for $\cos\varphi < 0$ can be extended to the case $\cos\varphi > 0$ by changing z_0 by $-z_0$. Thus, taking this into account we consider the case $\cos\varphi < 0$. Then the two stability conditions are satisfied if both B_0 and B_1 are greater than zero. Let us consider B_0, given by

$$B_0 = gmz_0 + (I_2 - I_3)\omega^2 \cos\varphi.$$

It is made by the sum of two terms, and the sign of B_0 depends on the signs of these two terms. We divide our analysis into four cases, depending on the signs of $(I_2 - I_3)$ and z_0.

Case 1. $(I_2 - I_3) > 0$ and $z_0 < 0$.

In this case the two terms in B_0 are negative (recall $\cos\varphi < 0$) and then $B_0 < 0$ for all ω. Then the sufficient stability conditions cannot be satisfied.

Case 2. $(I_2 - I_3) > 0$ and $z_0 > 0$.

We note that, in this case, $B_0 > 0$ if ω is small enough provided that

$$B_0(\omega = 0) = gmz_0 > 0.$$

On the other hand,

$$B_1(\omega = 0) = -4g^2m^2z_0^2 \sin^2\varphi \cos\varphi > 0, \tag{4.7}$$

because $\cos\varphi < 0$. Thus, for ω small enough, both B_0 and B_1 are positive and the sufficient stability conditions are satisfied. Once ω is fixed to satisfy the two stability conditions, l_1 is determined from the equilibrium relation (3.2) to complete the parameters for the stable permanent rotation.

Case 3. $(I_2 - I_3) < 0$ and $z_0 > 0$.

In this case it follows immediately that $B_0 > 0$ for all ω. Thus, taking into account (4.7), it follows that for ω small enough there exist stable permanent rotations.

Case 4. $(I_2 - I_3) < 0$ and $z_0 < 0$.

In this case we focus on the behavior of B_0 and B_1 at infinity, as $B_0(\omega = 0) < 0$ but $\lim_{\omega\to\infty} B_0 = +\infty$. However, the leading coefficient of B_1 as a polynomial in ω is given by

$$-4(I_3 - I_1)I_3 \cos^3\varphi > 0$$

if $I_3 > I_1$. Then, for ω greater enough, both B_0 and B_1 are positive and there exist stable permanent rotations. Note that, in this case, I_3 is the largest moment of inertia.

If, on the contrary, $I_3 - I_1 < 0$, the leading coefficient of B_1 is negative. To analyze this case, we introduce the new variable $z = \omega^2$, as both B_0 and B_1 are even functions of ω. After this change of variable, B_0 becomes linear in z and B_1 quadratic. B_0 is negative for $z = 0$, but B_1 is positive. However, for z going to infinity, B_0 is positive, while B_1 is negative. Both functions have a unique real root in between zero and infinity. Let z_0 and z_1 be the roots of B_0 and B_1, respectively. If we prove that $z_1 < z_0$, then B_0 and B_1 cannot be positive at the same time. Let us consider

$$\bar{z} = \frac{gmz_0}{I_2 - I_3}, \qquad \bar{z} \in (0, \infty).$$

On the one hand, we have

$$B_0(\bar{z}) = (1 + \cos\varphi) gmz_0 \leqslant 0,$$

provided $z_0 < 0$. This means that $\bar{z} \leqslant z_0$. On the other hand,

$$B_1(\bar{z}) = a + (I_1 - I_3)m,$$

where

$$a = B_1(\bar{z})_{I_1=I_3} = 4g^2m^2z_0^2 \frac{I_3 + (I_3 - I_2)\sin^2\varphi\cos\varphi}{I_2 - I_3} < 0,$$

and

$$m = 4g^2m^2z_0^2 \frac{I_3\cos^3\varphi + (I_2 - I_3)(2 + \cos 2\varphi)\sin^2\varphi}{(I_2 - I_3)^2} < 0.$$

Taking into account that $I_1 > I_3$, and the above facts, it follows that $B_1(\bar{z}) < 0$ for all $I_1 > I_3$, and then $z_1 < \bar{z}$. In summary, we have $z_1 < \bar{z} \leqslant z_0$, and then B_0 and B_1 cannot be positive at the same time. This completes the proof. □

We remark that Theorems 5 to 7 are valid if $\cos\varphi \neq 0$. The case $\cos\varphi = 0$ deserves a special treatment, namely, computing the reduced Hessian matrix from the very beginning. However, for the sake of conciseness, this time we omit the calculations and present the final results.

Theorem 8. *Let us assume that, for the equilibrium E_0, $\cos\varphi = 0$, then it must be $z_0 = 0$ and the sufficient stability conditions are given by*

$$\varphi = \pi/2, \quad (l_1 + (I_1 - I_2)\omega)\omega > 0, \quad (l_1 + (I_1 - I_2)\omega)(l_1 + (I_1 - I_3)\omega) > 0,$$

$$\varphi = -\pi/2, \quad -(l_1 - (I_1 - I_2)\omega)\omega > 0, \quad (l_1 - (I_1 - I_2)\omega)(l_1 - (I_1 - I_3)\omega) > 0.$$

On the other hand, a necessary stability condition is given by

$$\varphi = \pi/2, \quad \big(l_1 + (I_1 - I_2)\omega\big)\big(l_1 + (I_1 - I_3)\omega\big) > 0,$$

$$\varphi = -\pi/2, \quad \big(l_1 - (I_1 - I_2)\omega\big)\big(l_1 - (I_1 - I_3)\omega\big) > 0.$$

Remark 5. Note that, if $\cos\varphi = 0$, it is always possible to have stable permanent rotations if the absolute value of gyrostatic momentum l_1 is larger enough.

Remark 6. It is worth noting that, in this case, as $z_0 = 0$, the center of mass G of the gyrostat coincides with the fixed point O. Therefore, although the gyrostat is under the action of a uniform gravitational field, the weight of the system is applied in the fixed point O. Thus, the gravitational torque $\boldsymbol{N}$ about O is zero, and then we have the case of an asymmetric gyrostat in free rotational motion about O [16, 24, 29]. Moreover, for $l_1 = 0$ we have the well-known problem of the free triaxial rigid body. In this case, the sufficient and necessary stability conditions of Theorem 8 coincide with the classical results on the rotation stability of a free asymmetric rigid body [19].

5. CONCLUSIONS

In a previous work [25], we studied, by using the Energy–Casimir method, the stability of permanent rotations of a gyrostat with a fixed point in such a way that the center of mass is placed on one principal axis of inertia and, moreover, the gyrostatic momentum spins about the same axis. However, the problem becomes rather more involved when the rotor is spinning about one of the other two principal axes, which is the case considered here. Indeed, the number of permanent rotations is different; there are only two families. One of them, on one of the coordinate planes, is similar to cases analyzed in [25], whereas the other family lies on the space, which makes the analysis more complex. For this family we have proved the existence of stable permanent rotations when the gyrostat is oriented in any direction of space, for some disposition of the moments of inertia. This is the case if I_2 is the largest moment of inertia. For the other family, we also have established sufficient and necessary stability conditions and we have proved that in the limit situation, when the fixed point is the center of mass of the gyrostat, we recover the classical stability conditions.

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