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Determination of Nonlinear Stability for Low Order Resonances by a Geometric Criterion

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Abstract—We consider the problem of stability of equilibrium points in Hamiltonian systems of two degrees of freedom under low order resonances. For resonances of order bigger than two there are several results giving stability conditions, in particular one based on the geometry of the phase flow and a set of invariants. In this paper we show that this geometric criterion is still valid for low order resonances, that is, resonances of order two and resonances of order one. This approach provides necessary stability conditions for both the semisimple and non-semisimple cases, with an appropriate choice of invariants.

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1. INTRODUCTION

The problem of determining the stability of the equilibria of dynamical systems is crucial, since the flow evolution strongly depends on it. We could say that this is the first task one have to carry out when studying a system. The question is trivial for one degree of freedom Hamiltonian systems but it turns to be intricate for more degrees of freedom. Much is known for periodic Hamiltonian systems with one degree of freedom and two degrees of freedom autonomous Hamiltonian systems. In fact, there is a strong link between these two cases, as the stability conditions match in both of them. In this work, we will focus on autonomous two degrees of freedom Hamiltonian systems and on necessary conditions of stability for low order resonances, from a geometric point of view.

Without loss of generality we will assume that the origin is an isolated equilibrium of the system. Moreover, we will assume that the Hamiltonian function $\mathcal{H}$ is analytic in a small neighborhood of the origin, so it can be expanded in the form

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3 + \cdots,$$

where each $\mathcal{H}_k$ is a homogeneous polynomial of degree k in coordinates and momenta.

The first term in the expansion, $\mathcal{H}_2$, gives information about linear stability. Indeed, the eigenvalues of the linear system associated to $\mathcal{H}_2$ completely decide the linear stability properties of the origin. In some cases, these properties can be extended to establish nonlinear stability conditions. In this sense, if the eigenvalues have non zero real part, then the origin is both linear and nonlinear unstable. Unfortunately, there is not a similar result when all the eigenvalues have zero real part. This situation belongs to the critical case in the terminology of Lyapunov, when

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higher order terms in the power expansion of $\mathcal{H}$ are necessary to solve the question of stability in rigorous way.

Being $\pm i\omega_1$ and $\pm i\omega_2$ the eigenvalues of the corresponding linear system several results of stability have been established. If $\mathcal{H}_2$ is a definite quadratic form, a theorem of Dirichlet [6] ensures the stability of the origin for the whole Hamiltonian (see [17]). However, if $\mathcal{H}_2$ is not sign-defined there are several methods based on KAM theory to determine the global stability. Thus, we are left with the case of a not sign definite quadratic form $\mathcal{H}_2$.

The way this situation is treated strongly depends on whether ω_1 and ω_2 satisfy a resonant condition or not. We say that ω_1 and ω_2 satisfy a resonant condition of order s, if there exist m and n, coprime integers, such that

$$m\omega_1 - n\omega_2 = 0$$

and $m + n = s$.

If ω_1 and ω_2 do not satisfy a resonant condition, Arnold's theorem [3] ensures the stability of the origin if certain non-degeneracy condition is fulfilled [3, 14]. However, if ω_1 and ω_2 satisfy a resonant condition of order s a case study must be applied. In this way, Markeev [11] and Alfriend [1, 2] provided appropriate results for resonances of third and four order. Later on, Sokolski gave conditions of stability for first and second order resonances [18, 19]. Nevertheless, although the stability conditions for second order resonance were right, the proof had a flaw and was properly demonstrated by Lerman and Markova [10]. These previous results almost cover the totality of situations one can face in the stability question. Only strong degenerate situations are left to be studied, as it was done by Markeev for a degenerate case in the presence of a fourth order resonance [12, 13].

It seems that resonant and non resonant cases must be studied separately. However, Cabral and Meyer [5] gave a very general result including resonant and non resonant cases, except those considered by Sokolski, corresponding to first and second order resonances. Their approach takes advantage of the following fact: if $\omega_1, \omega_2 \neq 0$ and the corresponding linear system is semisimple, that is, the canonic Jordan matrix is diagonal, the normal form of the quadratic part $\mathcal{H}_2$ can be written in Poincaré coordinates as

$$\mathcal{H}_2 = \omega_1\Psi_1 - \omega_2\Psi_2, \tag{1.1}$$

independently if a resonant condition is satisfied or not. Soon after, it was proven by Elipe and coworkers [8, 9] that this result has a nice geometric counterpart giving rise to a *geometric criterion* of stability a bit more general; it is sufficient to characterize the phase flow of the normalized Hamiltonian system, and roughly speaking, the criterion is based on how two surfaces, related with the normal form, intersect one another.

Nevertheless, if either ω_1 or ω_2 are zero, or the corresponding linear system is not semisimple, that is, in the case of first order resonance and second order resonance in the not semisimple case, the normal form of $\mathcal{H}_2$ is no longer as in Eq. (1.1); the previous results are not of applicability and Sokolski's and Lerman's theorems have to be taken into account. The question is whether these theorems have the same geometric counterpart as that of Cabral and Meyer.

In this paper we will show that if we apply the simple ideas of the geometric criterion we find the same conditions of stability that in Sokolski's theorems. To this end, we will consider the *Birkhoff normal form* [4] up to a certain order, and the corresponding set of invariants associated to the reduction that generates the reduced phase space. Finally, we will study the phase flow on the integral manifold where the origin lies.

2. THE GEOMETRIC CRITERION

To begin with, we recall the geometric criterion of stability for resonant cases. Let us suppose that ω_1 and ω_2 satisfy a resonant condition of order greater or equal than two, and that $\mathcal{H}_2$ can be written as in Eq. (1.1). Then, $\mathcal{H}$ can be brought into its Birkhoff's normal form, where $\mathcal{H}_2$ is a formal integral. By doing so, the normal form is generated by a set of invariant quantities, that commute with the new formal integral. This set is easily obtained by using complex variables (u_1, u_2, v_1, v_2) defined as

$$u_k = \frac{1}{\sqrt{2}}(q_k - ip_k), \quad v_k = \frac{-i}{\sqrt{2}}(q_k + ip_k), \qquad k = 1, 2,$$
$$q_k = \sqrt{2\Psi_k}\sin\phi_k, \qquad p_k = \sqrt{2\Psi_k}\cos\phi_k,$$

where $(\phi_1, \phi_2, \Psi_1, \Psi_2)$ are the action-angle Poincaré variables. Indeed, if $n\omega_1 - m\omega_2 = 0$, they are

$$I_1 = u_1 v_1, \quad I_2 = u_2 v_2, \quad I_3 = u_1^n u_2^m, \quad I_4 = v_1^n v_2^m.$$

A counterpart of real invariants results by means of appropriate linear combinations. In this way, we consider the set M_1, M_2, C and S (see [7, 9] for details) defined as

$$M_1 = \frac{i}{2}(mI_1 + nI_2), \qquad M_2 = \frac{1}{2}(mI_1 - nI_2),$$
$$C = \frac{1}{2}m^{n/2}n^{m/2}(I_4 + i^{n+m}I_3), \quad S = \frac{i}{2}m^{n/2}n^{m/2}(I_4 - i^{n+m}I_3).$$

Now, the normal form up to order N is written as

$$\mathcal{H} = \mathcal{H}_2 + \sum_{j=3}^{N}\mathcal{H}_j,$$

where $\mathcal{H}_2 = 2\omega M_2$ $(m = \omega n)$, and

$$\mathcal{H}_j = \sum_{2(\gamma_1+\gamma_2)+(n+m)(\gamma_3+\gamma_4)=j} a_{\gamma_1\gamma_2\gamma_3\gamma_4} M_1^{\gamma_1} M_2^{\gamma_2} C^{\gamma_3} S^{\gamma_4}, \qquad 3 \leqslant j \leqslant N.$$

The invariants are not independent, but they satisfy the equation

$$C^2 + S^2 = (M_1 + M_2)^n (M_1 - M_2)^m, \tag{2.1}$$

together with the constrain

$$M_1 \geqslant |M_2|. \tag{2.2}$$

Equations (2.1) and Eq. (2.2) define the reduced phase space, which is regarded as a fibered three-dimensional space. Each fibre is a two-dimensional space labeled by M_2. Fixed a value for M_2, Eq. (2.1) is a surface of revolution with a vertex on the point $M_1 = |M_2|$, $C = S = 0$. In Fig. 1 we see different surfaces of revolution for a $1:3$ resonance and for several values of M_2.

Once the reduced phase space is determined, it is possible to know the flow of the normalized system, when it is truncated to a prescribed order. Indeed, fixed a value of M_2, the flow results as the intersection of the normalized Hamiltonian function with the surface defined by Eq. (2.1). Based on this idea, the following stability result can be established (for more details, see [9]).

Theorem 1. *Let us assume that the Hamiltonian is normalized up to a certain order $N \geqslant s$, being $\mathcal{H}_N$ the first term does not vanish for $M_2 = 0$. Let us consider the two surfaces*

$$\mathcal{G}_1 = \{(C, S, M_1) \in \mathbb{R}^3; \quad \mathcal{H}_N(C, S, M_1, 0) = 0\},$$

and

$$\mathcal{G}_2 = \{(C, S, M_1) \in \mathbb{R}^3; \quad C^2 + S^2 = M_1^s\}.$$

If the origin is an isolated point of intersection, then it is stable. In other case, and the two surfaces are not tangent, the origin is unstable.

The proof considers the flow on the fibre where the origin lies. If there is a neighborhood of the origin filled with closed orbits, then the origin is stable. On the contrary if there are asymptotic orbits the origin is unstable. It is clear that these conditions are necessary, but what matters is that they are also sufficient. We remark that the simple ideas underlying the geometric criterion lead straightforwardly to stability and instability conditions for equilibrium positions. These conditions

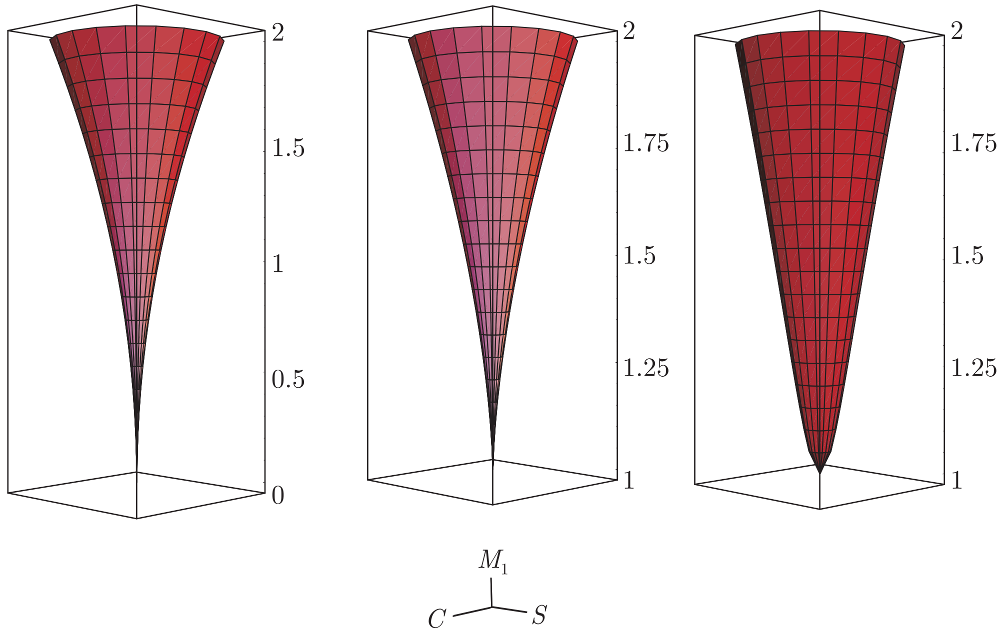

Fig. 1. The reduced phase space in 1:3 resonance for different values of M_2.

are deduced from the flow of a truncated system up to a suitable prescribed order. However, these are necessary conditions and a rigorous proof that they are also sufficient conditions is needed. In this sense, while instability conditions are easily extended to the full system, stability ones need of KAM methods. Moreover, the characterization of the flow in all the fibers of phase space helps to the reconstruction of periodic orbits and invariant tori.

Theorem 1 does not apply for the cases treated by Sokolski and Lerman for first and second order resonances. A natural question is to check if the geometric criterion matches with stability conditions established for these cases. In this way we will show how necessary conditions derived from the geometric criterion are those given in the results of Sokolski and Lerman.

3. LOW ORDER RESONANCES: RESONANCE OF ORDER 2

In this case we focus on the non semisimple case because the semisimple one is solved by the results given in Section 2. To begin with, let us reproduce the Sokolski–Lerman theorem [10, 18] for the non semisimple case.

Theorem 2. *Let us consider a Hamiltonian system under a 1:1 resonance whose normal form up to order 4 is written in terms of the cartesian variables as*

$$\mathcal{H} = \frac{d}{2}(x^2 + y^2) + \omega(xY - yX) + (X^2 + Y^2)[A(X^2 + Y^2) + B(xY - yX) + C(x^2 + y^2)] + \overline{\mathcal{H}},$$

where $d = \pm 1$ *and* $\overline{\mathcal{H}} = O(x, X, y, Y, 6)$*. If* $dA > 0$*, then the origin is stable. If* $dA < 0$*, then the equilibrium is unstable.*

Now, we will show, from a geometric point of view, that necessary stability conditions are precisely those established in theorem 2. To do this, we follow the work of Palacián and Yanguas [15] about the reduction of polynomial planar Hamiltonians with quadratic unperturbed part. Firstly, we introduce the semisimple part of $\mathcal{H}_2$, namely $xY - yX$, as a formal integral. Thus, after a normalization procedure, the Hamiltonian is reduced to another one with only one degree of freedom. The corresponding normal form is generated by four linearly independent invariants I_1, I_2, I_3, I_4, that in terms of Cartesian variables can be written as

$$I_1 = x^2 + y^2, \quad I_2 = X^2 + Y^2, \quad I_3 = xX + yY, \quad I_4 = xY - yX. \tag{3.1}$$

Let us remark that the above invariants are not independent but they verify the relation

$$I_1 I_2 = I_3^2 + I_4^2, \tag{3.2}$$

together with the restriction,

$$I_1, I_2 \geqslant 0. \tag{3.3}$$

Being I_4 a formal integral, Eqs. (3.2) and (3.3) define the reduced phase space as a family of elliptic hyperboloids labeled by I_4 (see Fig. 2). The origin belongs to the elliptic hyperboloid associated to $I_4 = 0$.

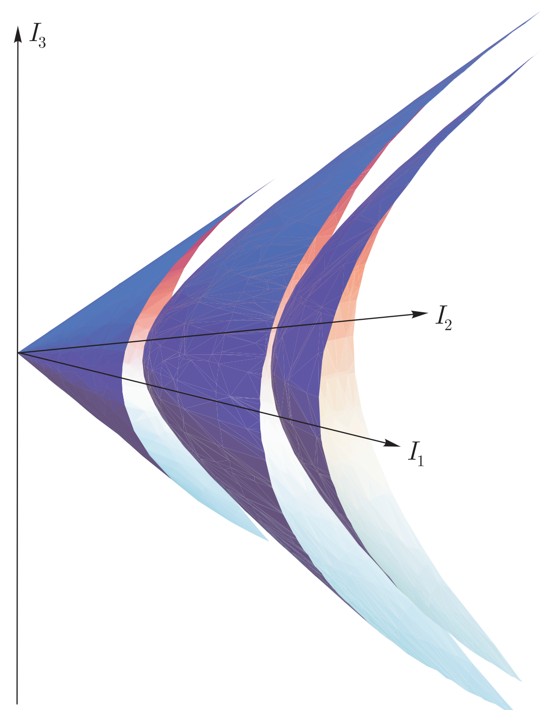

Fig. 2. Three fibres of the reduced phase space $I_1 I_2 = I_3^2 + I_4^2$ for $I_4 = 0$, $I_4 = 0.5$ and $I_4 = 1$.

In terms of the invariants, the Hamiltonian normal form, up to order 4, is written as

$$\mathcal{H} = \frac{d}{2} I_1 + \omega I_4 + A I_2^2 + B I_2 I_4 + C I_1 I_2 + \overline{\mathcal{H}}.$$

To derive a geometric criterion we focus on the phase flow on the manifold where the origin lies, the corresponding to $I_4 = 0$. It is obtained by the intersection of the following two surfaces

$$\mathcal{G}_1 = \{(I_1, I_2, I_3) \in \mathbb{R}^3; \quad \mathcal{H}(I_1, I_2, I_3, 0) = 0\},$$

and

$$\mathcal{G}_2 = \{(I_1, I_2, I_3) \in \mathbb{R}^3; \quad I_1 I_2 = I_3^2, \quad I_1, I_2 \geqslant 0\}.$$

In order to have stability it is necessary the orbits around the origin be closed, which implies the origin is an isolated intersection point of $\mathcal{G}_1$ and $\mathcal{G}_2$.

The intersection of the two surfaces is given by the set of points

$$\mathcal{G}_1 \cap \mathcal{G}_2 = \left\{ (I_1, I_2, I_3) \in \mathbb{R}^3; \quad \frac{d}{2} I_1 + A I_2^2 + C I_1 I_2 = 0, \quad I_1 I_2 = I_3^2, \quad I_1, I_2 \geqslant 0 \right\}.$$

It is clear that a point belonging to $\mathcal{G}_1 \cap \mathcal{G}_2$ must satisfy the second degree equation in I_2,

$$\frac{d}{2} I_1 + A I_2^2 + C I_1 I_2 = 0, \tag{3.4}$$

for $I_1, I_2 \geqslant 0$. As expected, the origin is one of the solutions, but we are interested in checking whether this solution is isolated or not. To solve this question let us consider the discriminant Δ of Eq. (3.4),

$$\Delta = I_1(C^2 I_1 - 2dA).$$

If $A = 0$, the set of points $I_1 = I_3 = 0$ belongs to $\mathcal{G}_1 \cap \mathcal{G}_2$ and the two surfaces are tangent; no information about stability is obtained. If $A \neq 0$, we have intersection points different to the origin if $\Delta \geqslant 0$. Here two cases must be distinguished.

1. On the one hand, if $dA < 0$ it follows that $\Delta \geqslant 0$ for whatever value of $I_1 \geqslant 0$. Consequently, for each value of I_1 we obtain an intersection point and the origin is not isolated. Because the two surfaces $\mathcal{G}_1$ and $\mathcal{G}_2$ are not tangent there are orbits asymptotic to the origin and thus, it is unstable.

2. On the other hand, if $dA > 0$, taking into account that $I_1, I_2 \geqslant 0$, the discriminant Δ is greater or equal than zero when $C \neq 0$ and $I_1 \geqslant 2dA/C^2$ or when $C = 0$ and $I_1 = 0$. Thus, it is possible to find a neighborhood of the origin U such that $U \cap (\mathcal{G}_1 \cap \mathcal{G}_2) = \{(0, 0, 0)\}$ and the origin is an isolated intersection point. In that case, there is a family of closed orbits around the origin and it is stable.

In Fig. 3 we see the intersection of the two surfaces $\mathcal{G}_1$ and $\mathcal{G}_2$ projected onto the plane $I_3 = 0$ in the four possible situations depending on the sign of dA and C. In case the origin is an isolated intersection point, a collection of closed orbits around the origin exists, otherwise there are asymptotic orbits, as it is reflected in Fig. 4.

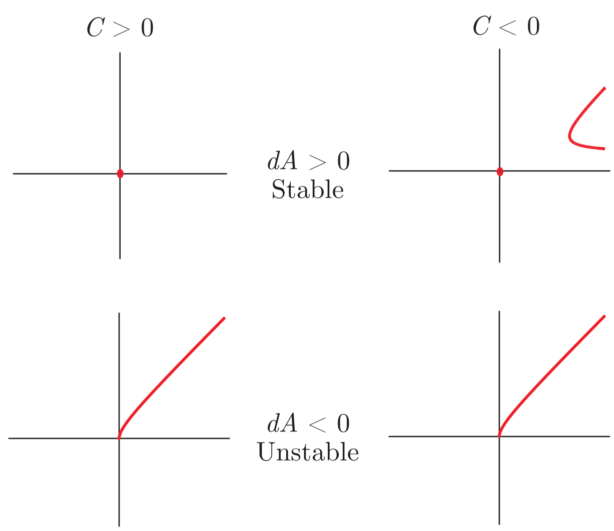

Fig. 3. $\mathcal{G}_1 \cap \mathcal{G}_2$ projected onto the plane $I_3 = 0$.

4. LOW ORDER RESONANCES: RESONANCE OF ORDER 1

For a resonance of order one, two situations must be considered depending if the corresponding linear system is semisimple or not. Both situations were studied by Sokolski [19] providing suitable stability criteria, one for each case. Following a similar reasoning to the one in the previous subsection we will see that necessary conditions for stability and instability can be obtained from a geometric point of view.

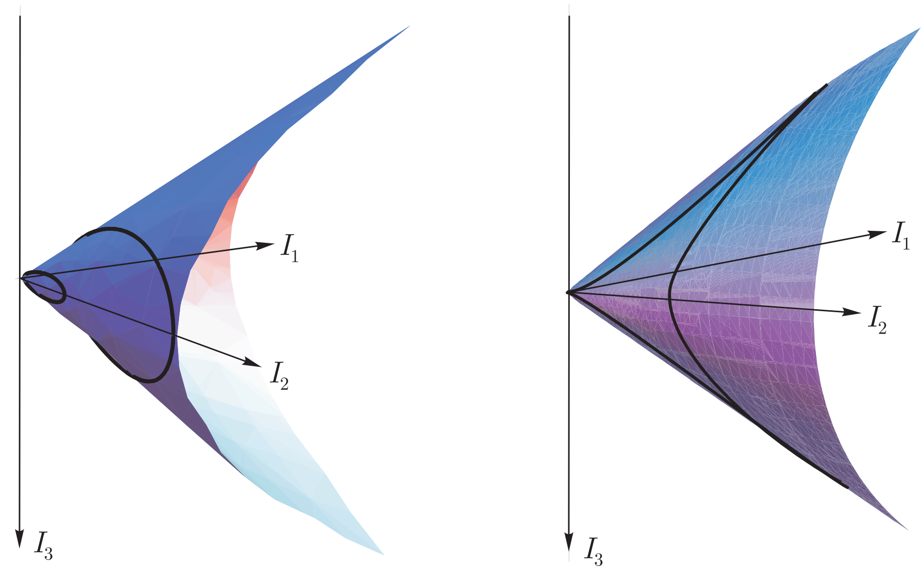

Fig. 4. Orbits around the origin in the case of stability (left) and instability (right).

4.1. Semisimple Case

For the semisimple case Sokolski established the following theorem

Theorem 3. *Let us consider a Hamiltonian system under a 0:1 resonance whose normal form up to order N is written in terms of the cartesian variables as*

$$\mathcal{H}(x,y,X,Y)=\mathcal{H}_2(x,y,X,Y)+\mathcal{H}_3(x,y,X,Y)+\cdots+\mathcal{H}_N(x,y,X,Y)+\overline{\mathcal{H}},$$

where

$$\mathcal{H}_2=\frac{d}{2}\omega_2(y^2+Y^2),\qquad \mathcal{H}_j=\sum_{k=0}^{[j/2]}h^{(k)}_{j-2k}(y^2+Y^2)^k,\quad 3\leqslant j\leqslant N,$$

being $d=\pm1$ and $h^{(k)}_{j-2k}$ a homogeneous polynomial of degree $j-2k$ in x,X and $\overline{\mathcal{H}}=O(x,X,y,Y,N+1)$. If at least one coefficient of the polynomial $h^{(0)}_N$ is nonzero and $h^{(0)}_N$ is a sign–defined function, then the origin is stable. If at least one coefficient of the polynomial $h^{(0)}_N$ is a sign–variable function, then the origin is unstable. In particular, if N is an odd number, then the origin is unstable.

First of all, note that N is not explicitly specified in the theorem, so it is supposed to be the first term in the normal form that is not the null function. Now, we are in conditions to derive the stability conditions of the theorem from the geometric approach. To begin with, we carry out a normalization procedure by reducing the number of degrees of freedom by means of a formal integral. Following [15], we take y^2+Y^2 as the formal integral. Besides, a set of three independent invariants J_1,J_2,J_3 is obtained, that in terms of Cartesian variables can be written as

$$J_1=x,\quad J_2=X,\quad J_3=y^2+Y^2. \tag{4.1}$$

In this case, the quadratic term $\mathcal{H}_2$ becomes

$$\mathcal{H}_2=\frac{d}{2}\omega_2J_3,$$

and it is nothing more than a multiple of the formal integral. Moreover, the reduced phase space is defined by $J_3=c$, with c a constant, and it is regarded to a family of parallel planes, one for each constant value of J_3.

As in the previous section we focus on the phase flow in the plane where the origin lies, that is on the plane $J_3 = 0$. The flow is obtained by the intersection of the two surfaces

$$\mathcal{G}_1 = \{(J_1, J_2, J_3) \in \mathbb{R}^3; \quad \mathcal{H}(J_1, J_2, J_3) = 0\}, \tag{4.2}$$

and

$$\mathcal{G}_2 = \{(J_1, J_2, J_3) \in \mathbb{R}^3; \quad J_3 = 0\}. \tag{4.3}$$

As $J_3 = 0$, it follows that $y = Y = 0$ and therefore, $\mathcal{H}(J_1, J_2, 0) = h_N^{(0)}(J_1, J_2)$. In this way, the intersection of $\mathcal{G}_1$ and $\mathcal{G}_2$ can be described by the set

$$\mathcal{G}_1 \cap \mathcal{G}_2 = \left\{(J_1, J_2, J_3) \in \mathbb{R}^3; \quad h_N^{(0)}(J_1, J_2) = 0, \quad J_3 = 0\right\}.$$

Since $h_N^{(0)}$ is a homogeneous polynomial of degree N in J_1, J_2, it can be written as

$$h_N^{(0)}(J_1, J_2) = a_{N,0} J_1^N + a_{N-1,1} J_1^{N-1} J_2 + \cdots + a_{0,N} J_2^N.$$

It is clear that the origin, $J_1 = J_2 = J_3 = 0$, belongs to $\mathcal{G}_1 \cap \mathcal{G}_2$. Even more, if we fix the value $J_1 = 0$, it must be $J_2 = J_3 = 0$, unless $a_{0,N} = 0$. If $a_{0,N} = 0$ and at least one coefficient in $h_N^{(0)}$ is not zero, $\mathcal{G}_1$ and $\mathcal{G}_2$ intersect transversely along the line $J_1 = J_3 = 0$. Therefore, there is an asymptotic orbit to the origin and it is unstable.

Now, we are interested in intersection points such that $J_1 \neq 0$. In this way, we introduce a new variable z such that $J_2 = zJ_1$ $(z \neq 0)$. Dividing by J_1^N the function $h_N^{(0)}(J_1, J_2)$ we obtain the polynomial

$$p_N(z) = a_{N,0} + a_{N-1,1} z + \cdots + a_{0,N} z^N.$$

We note that if $p_N(z) = 0$ has a real root, z_0, then the straight line defined by $J_3 = 0$, $J_2 = z_0 J_1$, belongs to $\mathcal{G}_1 \cap \mathcal{G}_2$. As a consequence, there are asymptotic lines to the origin and it is unstable. On the contrary, if $p_N(z) = 0$ has no real roots, the origin is the unique intersection point and a family of closed orbits exists around it. Then it is stable.

However, the existence of real roots of $p_N(z) = 0$ depends on how $h_N^{(0)}(J_1, J_2)$ is. In particular, if $h_N^{(0)}(J_1, J_2)$ is sign defined, $p_N(z)$ has no real roots and the origin is stable. On the other hand, if $h_N^{(0)}(J_1, J_2)$ changes the sign, $p_N(z)$ has at least one real root and the origin is unstable. We note that if N is an odd number, the polynomial $p_N(z)$ has at least one real root, and therefore the origin is unstable.

4.2. Non Semisimple Case

For this case the result of Sokolski reads as

Theorem 4. *Let us consider a Hamiltonian system under a 0:1 resonance whose normal form up to order N is written in terms of the cartesian variables as*

$$\mathcal{H}(x, y, X, Y) = \mathcal{H}_2(x, y, X, Y) + \mathcal{H}_3(x, y, X, Y) + \cdots + \mathcal{H}_N(x, y, X, Y) + \overline{\mathcal{H}},$$

where

$$\mathcal{H}_2 = \frac{d_1}{2} x^2 + \frac{d_2}{2} \omega_2 (y^2 + Y^2), \qquad \mathcal{H}_j = \sum_{k=0}^{[j/2]} a_{j-2k,k} X^{j-2k} (y^2 + Y^2)^k, \quad 3 \leqslant j \leqslant N,$$

being $d_1, d_2 = \pm 1$ and $\overline{\mathcal{H}} = O(x, X, y, Y, N + 1)$. If $a_{N,0} \neq 0$, and N is an odd number, then the origin is unstable. If $a_{N,0} \neq 0$, N is an even number and $d_1 a_{N,0} < 0$, then the origin is unstable. If $a_{N,0} \neq 0$, N is an even number and $d_1 a_{N,0} > 0$, then the origin is stable.

Note that, as in theorem 3, it is supposed that $\mathcal{H}_N$ is the first term of the normal form that is not the null function. Under this implicit hypothesis we proceed in the same way as in the previous cases. As it is shown in [15], both the formal integral and the invariants are the same that in the semisimple case, and are given by (4.1).

Now, the quadratic part of the Hamiltonian function $\mathcal{H}_2$ is written as

$$\mathcal{H}_2 = \frac{d_1}{2} J_1^2 + \frac{d_2}{2}\omega_2 J_3,$$

and the reduced phase space is again a collection of parallel planes, $J_3 = c$, with c a constant.

Since the formal integral is J_3 and the origin lies on the plane $J_3 = 0$, we pay attention to the flow on this manifold. Thus, we consider the two surfaces

$$\mathcal{G}_1 = \{(J_1, J_2, J_3) \in \mathbb{R}^3; \quad \mathcal{H}(J_1, J_2, J_3) = 0\}, \tag{4.4}$$

and

$$\mathcal{G}_2 = \{(J_1, J_2, J_3) \in \mathbb{R}^3; \quad J_3 = 0\}. \tag{4.5}$$

To know their intersection, it is worth to note that if $J_3 = 0$, then $y = Y = 0$, and therefore $\mathcal{H}(J_1, J_2, 0) = d_1 J_1^2/2 + a_{N,0} J_2^N$. In this way the intersection is the set of points

$$\mathcal{G}_1 \cap \mathcal{G}_2 = \left\{ (J_1, J_2, J_3) \in \mathbb{R}^3; \quad \frac{d_1}{2} J_1^2 + a_{N,0} J_2^N = 0, \quad J_3 = 0 \right\}.$$

If N is an odd number, then the origin is not an isolated point of the intersection because

$$\mathcal{G}_1 \cap \mathcal{G}_2 = \left\{ (J_1, J_2, J_3) \in \mathbb{R}^3; \quad J_3 = 0, \quad J_2 = \left(\frac{-d_1}{2a_{N,0}} J_1^2 \right)^{1/N} \right\}.$$

Therefore, as the surfaces intersect transversely, the origin is unstable. In Fig. 5 we depict this set of points projected onto the plane $J_3 = 0$ depending on the sign of $d_1 a_{N,0}$.

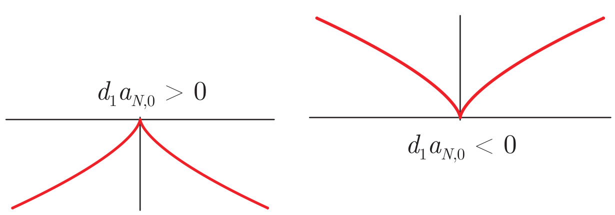

Fig. 5. $\mathcal{G}_1 \cap \mathcal{G}_2$ projected onto the plane $J_3 = 0$ for N an odd number.

If N is an even number, the intersection $\mathcal{G}_1 \cap \mathcal{G}_2$ changes depending on the sign of $d_1 a_{N,0}$. In this way, if $d_1 a_{N,0} < 0$, the intersection is given by

$$\mathcal{G}_1 \cap \mathcal{G}_2 = \left\{ (J_1, J_2, J_3) \in \mathbb{R}^3; \quad J_3 = 0, \quad J_2 = \left(\frac{-d_1}{2a_{N,0}} J_1^2 \right)^{1/N} \right\}.$$

Therefore, the origin is unstable. In Fig. 6 we depict this set of points projected onto the plane $J_3 = 0$.

If $d_1 a_{N,0} > 0$, the origin is an isolated point of intersection and then it is a stable.

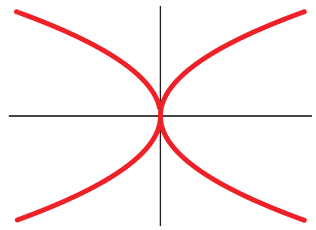

Fig. 6. $\mathcal{G}_1 \cap \mathcal{G}_2$ projected onto the plane $J_3 = 0$ for N an even number.

5. CONCLUSIONS

For a two degrees of freedom Hamiltonian system it was known that stability criteria for resonances of order bigger than two can be obtained from a geometric point of view [9, 16]. In this paper the cases of low order resonances, those of order one and two, have been analyzed from a geometric approach, and it has been shown that the criteria given by Sokolski [18, 19] can be recovered. The idea is based on the structure of the phase flow after a normalization procedure. In this way the normal form of the quadratic part of the Hamiltonian function plays an important role. In fact, this is the reason why the general criterion of Cabral and Meyer [5], and its geometric counterpart [9, 16], is not valid for low order resonances and ad hoc criteria must be given. Nevertheless, the geometric approach is the same does not matter the order of the resonance. In this way it is revealed as a powerful tool for studying stability properties of equilibrium positions.

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