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Resonances and the Stability of Stationary Points Around a Central Body

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Abstract

The question of Lyapunov stability of stationary points around a central body has been studied in the absence of resonances and in the case of resonances of order 3 and 4, by means of the computation of the normal form up to second order. However, some special degenerate cases are not covered, as it happens for resonances of order 5 and 6, when the free parameters of the problem are chosen properly. In this article, we deal with these resonances and apply appropriate results to establish the stability properties of the stationary points.

Introduction

Geostationary orbits are of great interest in astrodynamics, as they are the optimal location for communication satellites and other scientific missions. Indeed, the number of spacecraft in geostationary orbits has rapidly increased since the first launch in 1963 [1].

Assuming the Earth is an oblate rigid body that rotates around the axis of greatest inertia, and considering only the zonal and tesseral harmonics up to second order, there are four equilibrium positions that correspond to geostationary orbits. Linear approximation shows that two of them are stable and the other two unstable [2, 3]. The stable ones belong to the critical case in the terminology of Lyapunov and higher order terms, in a Taylor series approximation around the equilibria, are needed to study their stability properties in the nonlinear sense. For the Earth and Mars, the stable stationary points are also nonlinearly stable [4].

The interest in new missions around other celestial bodies in the solar system could find other interesting stationary orbits around them. Thus, the question of stability of equilibrium solutions around a rotating central body must be solved.

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This was partially performed in [5], where the stability was established, in terms of the harmonic coefficients of the potential function, except for some resonant cases. In this sense, the main goal of this paper is to investigate these resonant cases in order to get a complete chart of stable and unstable solutions.

This problem has much in common with the classical problem of determining the nonlinear stability of libration points, L_4 and L_5, in the restricted three-body problem. In fact, the same techniques [6] can be used, especially in the presence of resonances. However, besides resonances of third and fourth order appearing in the restricted three body problem [7–11], we encounter higher order resonances for stationary solutions around a rotating central body. To deal with these higher order resonances and other degenerate cases, we will use a very general result that gives stability criteria in almost all resonant situations [12–14]. This result takes advantage of geometrical considerations of phase flow after a normalization procedure and, in some sense, constitutes an adaptation of ideas developed for the cases of third and fourth order resonances in the restricted three-body problem.

The article is organized as follows, first, we briefly describe some results on stability, mainly concerning resonances together with the introduction of some sets of canonical variables that we shall use to obtain the normal form and also for the stability criteria. Next, we formulate the problem of a particle around a non-spherical central body, we find the equilibria and their linear stability. Finally, the last section is devoted to the analysis of the stability in the presence of resonances.

On Stability Criteria

Let us consider an autonomous Hamiltonian system of two degrees of freedom, and that the origin is an isolated equilibrium point. It is assumed that the Hamiltonian is an analytic function in the neighborhood of the origin, thus, the Hamiltonian can be expanded in a power series

$$\mathcal{H} = \mathcal{H}_2 + \sum_{n>2} \mathcal{H}_n$$

where each term $\mathcal{H}_j$ is a homogeneous polynomial of degree j.

The linear term $\mathcal{H}_2$ not only provides information about the linear stability, but much more, because if the origin is unstable in the linear sense, and there is at least a characteristic exponent with positive real part, then it is also Lyapunov unstable. On the contrary, if the origin is linearly stable, it may be Lyapunov unstable, because the rest of the terms may contribute to the instability. If the quadratic form given by $\mathcal{H}_2$ is a definite quadratic form, a result of Dirichlet [15] ensures the stability of the origin for the whole Hamiltonian (see [16]). However, if $\mathcal{H}_2$ is not sign-defined there are several methods to determine the global stability. One of the most useful is based on the following result of Arnold [17] that, as it appears in [18], reads:

Theorem 1 (Arnold) *Let us consider a two degree of freedom Hamiltonian system* $\mathcal{H}$ *expressed, in the real action-and-angle canonical coordinates* $(\psi_1, \psi_2, \Psi_1, \Psi_2)$, *as*

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_4 + \cdots + \mathcal{H}_{2N} + \mathcal{H}$$

where:

1. $\mathcal{H}$ is real analytic in a neighborhood of the origin in $\mathbb{R}^4$,

2. $\mathcal{H}_{2k}$, $1 \le k \le N$, *is a homogeneous polynomial of degree k in* Ψ_i, *with real coefficients. In particular,*

$$\mathcal{H}_2 = \omega_1\Psi_1 - \omega_2\Psi_2,\ 0 < \omega_1, 0 < \omega_2$$

$$\mathcal{H}_4 = \frac{1}{2}(A\Psi_1^2 - 2B\Psi_1\Psi_2 + C\Psi_2^2)$$

3. $\bar{\mathcal{H}}$ *has a power expansion in* Ψ_i, *which starts with terms at least of order* $2N + 1$.

Under these assumptions, the origin is stable provided that for some k, $2 \le k \le N$, $\mathcal{H}_2$ *does not divide* $\mathcal{H}_{2k}$ *or likewise, provided that* $D_{2k} = \mathcal{H}_{2k}(\omega_2, \omega_1) \neq 0$.

It is worth noting that there are several implicit assumptions in the above theorem. On the one hand, because $\mathcal{H}_2, \ldots, \mathcal{H}_{2N}$ depends only on the actions Ψ_1 and Ψ_2, $\mathcal{H}$ is in normal form up to degree $2N$. On the other hand, the frequencies ω_1 and ω_2 do not satisfy a resonant condition of order less than or equal to $2N$.

Let us recall that a Hamiltonian $\mathcal{H} = \mathcal{H}_2 + \Sigma_{k\ge3}\,\mathcal{H}_k$, is said to be in normal form [19] up to order N if $\{\mathcal{H}_k; \mathcal{H}_2\} = 0$, with $3 \le k \le N$, where $\{-;-\}$ stands for the Poisson bracket.

Despite the simple definition, it is not an easy task to derive the normal form and usually it is achieved by means of perturbation techniques [20]. Besides, it is necessary to express the quadratic part as in Arnold's Theorem or an equivalent expression. Thus, some canonical transformations are needed. This theorem was successfully applied to determine the stability of Lagrangian point L_4 in the Restricted Three Body Problem (RTBP) by Deprit and Deprit-Bartholomé [6] and, later on, several authors followed their scheme to analyze the stability in other problems.

In case of resonances among the frequencies ω_1 and ω_2, the above theorem cannot be applied, and although there are several results dealing with specific resonances (see e.g., Alfriend [7, 8], Markeev [9–11]), it is not until the work of Cabral and Meyer [21] that the analysis of the resonant cases is not treated one by one for a specific given resonance. Later on, the authors [13] obtained a different method, based on geometric aspects, which generalizes the work of Cabral and Meyer. Because we will use the geometric method, let us briefly describe it. For more details, the reader is referred to the original work [13].

Let us assume that $\mathcal{H}_2$ is expressed as in Hypothesis 2 of Arnold's Theorem, that is to say, $\mathcal{H}_2 = \omega_1\Psi_1 - \omega_2\Psi_2$ and in addition ω_1 and ω_2 satisfy the resonant condition $n\omega_1 = m\omega_2$, where $n + m$ is said to be *the order of the resonance.* Let us define a symplectic transformation

$$\begin{array}{llll} u_1 = \sqrt{\Psi_1}e^{i\psi_1}, & u_2 = \sqrt{\Psi_2}e^{-i\psi_2} \\ U_1 = -i\sqrt{\Psi_1}e^{-i\psi_1}, & U_2 = i\sqrt{\Psi_2}e^{i\psi_2} \end{array} \tag{1}$$

With this, the quadratic part takes the form

$$\mathcal{H}_2 = i\omega_1 u_1 U_1 + i\omega_2 u_2 U_2 \tag{2}$$

whereas each term $\mathcal{H}_k$ is a homogeneous polynomial of degree k in the complex variables u_1, u_2, U_1 and U_2.

Proposition 1 *The following four complex monomials* $I_1 = u_1U_1$, $I_2 = u_2U_2$, $I_3 = u_1^nU_2^m$ *and* $I_4 = U_1^nu_2^m$ *are invariant with respect to the Hamiltonian flow determined by* $\mathcal{H}_2$.

Proof.-The Lie derivative operator of Hamiltonian equation (2) is

$$\mathcal{L}_2 = i\omega_1\left(u_1\frac{\partial}{\partial u_1} - U_1\frac{\partial}{\partial U_1}\right) + i\omega_2\left(u_2\frac{\partial}{\partial u_2} - U_2\frac{\partial}{\partial U_2}\right)$$

thus, the monomial $u_1^{m_1}U_1^{n_1}u_2^{m_2}U_2^{n_2}$ is an eigenvector of $\mathcal{L}_2$, because $\mathcal{L}_2(u_1^{m_1}U_1^{n_1}u_2^{m2}U_2^{n_2}) = i[\omega_1(m_1 - n_1) + \omega_2(m_2 - n_2)]u_1^{m_1}U_1^{n_1}u_2^{m_2}U_2^{n_2}$. Hence the above monomial belongs to the normal form (equivalently, belongs to the kernel of $\mathcal{L}_2$) if the following equation holds

$$\omega_1(m_1 - n_1) + \omega_2(m_2 - n_2) = 0$$

Thus, it is straightforward to check that monomials I_1, I_2, I_3, and I_4 are invariant.

It is worth noting that the invariants I_1, I_2, I_3, and I_4 are not independent, but they are linked through the relation

$$I_1^nI_2^m = I_3I_4 \tag{3}$$

Moreover, the normal form contains monomials, which are products of powers of the four basic monomials. In this way, I_1, I_2, I_3, and I_4 generate the normal form that, up to order *N,* can be expressed as

$$\mathcal{H} = i\omega_1I_1 + i\omega_2I_2 + \sum_{\substack{2(i+j)+(m+n)k=l\\ 3\le l\le N}}(a_{ijk}I_3^k + b_{ijk}I_4^k)I_1^iI_2^j \tag{4}$$

where we have taken into account the relation equation (3).

Note that if no resonant condition is satisfied, the normal form is generated only by I_1 and I_2 and, consequently, the running index l in equation (4) reduces to even integers.

The use of these complex invariants results in an easier normalization, because we only deal with polynomials, and the algebra involving them is simpler and faster by computer than the algebra of trigonometric functions that appears with Poincaré variables (ψ_1, ψ_2, Ψ_1, Ψ_2). By means of the symplectic transformation equation (1), we can express the normalized Hamiltonian in Poincaré variables as

$$\mathcal{H} = \omega_1\Psi_1 - \omega_2\Psi_2 + \sum_{\substack{2(i+j)+(m+n)k=l\\ 3\le l\le N}}(a_{ijk}\cos k\psi + b_{ijk}\sin k\psi)\Psi_1^{i+nk/2}\Psi_2^{j+mk/2} \tag{5}$$

where $\psi = n\psi_1 + m\psi_2$.

As the only angular variable in the normal form in equation (5) is ψ, it is convenient to introduce a new set of action-angle variables that incorporates the angle ψ as one of the coordinates.

To this end, we use the Lissajous variables [22, 23] specially designed to handle resonances. Lissajous variables are related with Poincaré variables through the following equations, which define a symplectic transformation

$$\begin{aligned} \psi_1 &= m(\phi_1 + \phi_2), \qquad & \Psi_1 &= (\Phi_1 + \Phi_2)/(2m) \\ \psi_2 &= n(\phi_1 - \phi_2), \qquad & \Psi_2 &= (\Phi_1 - \Phi_2)/(2n) \end{aligned}$$

In Lissajous variables, the normal form reads as

$$\mathcal{H} = \nu\Phi_2 + \sum_{\substack{2(i+j)+(m+n)k=l \\ 3\le l\le N}} (a_{ijk}\cos 2nmk\phi_1 + b_{ijk}\sin 2nmk\phi_1)(\Phi_1+\Phi_2)^{i+nk/2}(\Phi_1+\Phi_2)^{j+mk/2} \tag{6}$$

where $\nu = \omega_1/m = \omega_2/n$. Note that Φ_2 is a first integral of the system, because ϕ_2 is cyclic in the Hamiltonian function (6).

Finally, we can recover a similar expression to equation (4) by introducing a new set of generators (but now real functions) of the normal form given by

$$M_1 = \frac{1}{2}\Phi_1$$

$$M_2 = \frac{1}{2}\Phi_2$$

$$C = 2^{-(m+n)/2}(\Phi_1-\Phi_2)^{m/2}(\Phi_1+\Phi_2)^{n/2}\cos 2nm\phi_1 \tag{7}$$

$$S = 2^{-(m+n)/2}(\Phi_1-\Phi_2)^{m/2}(\Phi_1+\Phi_2)^{n/2}\sin 2nm\phi_1$$

In terms of these generators, the normal form, up to order N, reads

$$\mathcal{H} = 2\nu M_2 + \sum_{\substack{2(i+j)(n+m)(k+r)=\ell \\ 3\le \ell\le N}} a_{ijkr}M_1^iM_2^jC^kS^r \tag{8}$$

Moreover, the relation in equation (3) becomes

$$C^2 + S^2 = (M_1 + M_2)^n(M_1 - M_2)^m \tag{9}$$

where $M_1 \geq |M_2|$, and equation (9) defines a surface of revolution at each manifold $M_2 = \text{constant}$.

With this, the authors [13] proved the following result, based on geometric properties, and valid when the order of the resonance is greater than two

Theorem 2 *Let us assume that ω_1 and ω_2 satisfy a resonant condition of order $s > 2$. That is, there are two integers n and m such that*

$$n\omega_1 = m\omega_2, \qquad n+m = s$$

Let us also suppose that the Hamiltonian function is normalized up to a certain order $N \geq s$ and expressed in terms of real generators as in equation (8), being $\mathcal{H}_N$ the first term does not vanish for $M_2 = 0$.

Let us consider the two surfaces

$$\mathcal{G}_1 = \{(C,S,M_1) \in \mathbb{R}^3; \qquad \mathcal{H}_N(C,S,M_1,M_2=0)=0\}$$

and

$$\mathcal{G}_2 = \{(C,S,M_1) \in \mathbb{R}^3; \qquad C^2+S^2 = M_1^s\}$$

If the origin is an isolated point of intersection of both surfaces, then, the origin is stable. In the other case, provided that the two surfaces are not tangent, the origin is unstable.

With this theorem, we need to carry out the normalization for the different resonances, and check how the two surfaces cut each other to determine the stability of the equilibrium point.

Hamiltonian and Equilibrium Positions

To describe the motion of a satellite around a planet, we consider a synodic reference frame that rotates with the planet around its greatest axis of inertia with constant velocity ω. We suppose that the origin of the frame is the center of mass of the planet and the axes are the principal axes of inertia. With these assumptions, the problem can be modeled by the Hamiltonian

$$\mathcal{H} = \frac{1}{2}(X^2 + Y^2 + Z^2) - \omega(xY - yX) + \mathcal{U}(x, y, z) \tag{10}$$

where $\mathcal{U}(x,y,z)$ is the potential function

$$\mathcal{U} = \frac{\mu}{r}\left[1 + \left(\frac{R}{r}\right)^2\left\{3G_{2,2}\frac{x^2 - y^2}{r^2} - \frac{1}{2}G_{2,0}\left(1 - 3\frac{z^2}{r^2}\right)\right\}\right] \tag{11}$$

expanded up to second order. Here, μ stands for the Gaussian constant, $r = \sqrt{x^2 + y^2 + z^2}$ is the radial distance to the satellite, R is the planet's equatorial radius and $G_{2,0}$, $G_{2,2}$ are the harmonic coefficients. They satisfy the condition $G_{2,0} < 0 < G_{2,2}$. Due to the choice of the reference frame, they do not coincide with the classical spherical harmonics, although they are related with them by $G_{2,0} = C_{2,0}$ and $G_{2,2} = \sqrt{C_{2,2}^2 + S_{2,2}^2}$.

The stationary points of the dynamical system defined by the Hamiltonian function (10) are obtained by solving the nonlinear system that results from zeroing the equations of motion

$$\begin{aligned}
\dot{x} &= \frac{\partial \mathcal{H}}{\partial X} = X + \omega y, & \dot{X} &= -\frac{\partial \mathcal{H}}{\partial x} = \omega Y - \frac{\partial \mathcal{U}}{\partial x} \\
\dot{y} &= \frac{\partial \mathcal{H}}{\partial Y} = Y - \omega x, & \dot{Y} &= -\frac{\partial \mathcal{H}}{\partial y} = -\omega X - \frac{\partial \mathcal{U}}{\partial y} \\
\dot{z} &= \frac{\partial \mathcal{H}}{\partial Z} = Z, & \dot{Z} &= -\frac{\partial \mathcal{U}}{\partial z}
\end{aligned} \tag{12}$$

Thus, for a stationary solution it must be $X = -\omega y$, $Y = \omega x$, $Z = 0$, and the coordinates (x, y, z) must satisfy

$$\omega^2 x - \frac{\partial \mathcal{U}}{\partial x} = 0, \quad \omega^2 y - \frac{\partial \mathcal{U}}{\partial y} = 0, \quad \frac{\partial \mathcal{U}}{\partial z} = 0$$

A detailed discussion of equilibrium solutions can be found in the work of Howard [24]. Here, as it is made in [4, 5], we will restrict ourselves to the invariant manifold $z = Z = 0$, and we are left with the planar equatorial solutions for a two degree of freedom system. The equilibria located in the equatorial plane satisfy $x = 0$ or $y = 0$. In the first case, from equations (11) and (12), the y coordinate is a root of the polynomial equation

$$\omega^2|y|^5 - \mu|y|^2 + 3R^2\mu\left(\frac{1}{2}G_{2,0} + 3G_{2,2}\right) = 0 \tag{13}$$

Similarly, in the second case, when $y = z = 0$, x is a root of the equation

$$\omega^2|x|^5 - \mu|x|^2 + 3R^2\mu\left(\frac{1}{2}G_{2,0} - 3G_{2,2}\right) = 0 \tag{14}$$

From equations (13) and (14), it follows that equilibria appear by pairs, symmetrically located with respect to the origin along the x- and y-axes. Taking this into account, and restricting ourselves to the equatorial plane (that is $z = Z = 0$), we have the following results:

Proposition 2 There are two equilibrium points along the x-axis

Proof. -It is enough to demonstrate that equation (14) has a unique positive real root. This is a consequence of Descartes' rule of signs and the fact that $1/2G_{2,0} - 3G_{2,2}$ is negative. If r_1 is the positive real root, then there are two equilibrium points on the x-axis with coordinates $(\pm r_1, 0)$, which will be denoted by E_1.

Proposition 3 The number of equilibrium points along the y-axis is

– *two*, *if* $\frac{1}{2}G_{2,0} + 3G_{2,2} < 0.$

– *four*, *if* $0 < \left(\frac{1}{2}G_{2,0} + 3G_{2,2}\right)^3 < \frac{4\mu^2}{3125R^6\omega^4}.$

– *zero*, *if* $\left(\frac{1}{2}G_{2,0} + 3G_{2,2}\right)^3 > \frac{4\mu^2}{3125R^6\omega^4}.$

Proof. -It follows from the discriminant of the polynomial in equation (13), namely

$$81\mu^4R^2\omega^4\left(\frac{1}{2}G_{2,0} + 3G_{2,2}\right)\left(-4\mu^2 + 3125R^6\omega^4\left(\frac{1}{2}G_{2,0} + 3G_{2,2}\right)^3\right)$$

In this case, if r_2 is the greatest real root, and r'_2 (when it exists) the other possible real root, satisfying $0 < r'_2 < r_2$, we can have at most four equilibrium points on the y-axis with coordinates $(0, \pm r_2)$ and $(0, \pm r'_2)$. We denote these points by E_2 and E'_2 respectively.

Linear Stability

On obtaining the equilibrium points, we are interested in their stability properties with respect to perturbations in the equatorial plane. As said before, the linear stability analysis of the equilibria gives a lot of information, because on the one hand, if there are characteristic exponents with positive real part, the point is not only linearly unstable, but also is unstable in the nonlinear sense. On the other hand, inasmuch as the eigenvalues of the linearized Hamiltonian appear in pairs $\pm\lambda$, to have linear stability it is necessary that all the eigenvalues of the linearized system have zero real part. This is also a necessary condition for nonlinear stability.

Let us consider an equilibrium point with coordinates (x_0, y_0, X_0, Y_0). After a symplectic canonical change of variables given by

$$\xi = x - x_0, \quad \eta = y - y_0, \quad \Xi = X - X_0, \quad H = Y - Y_0$$

to shift the origin to the critical point, the linearized system is derived from the second term of the Taylor series expansion of the Hamiltonian function around the origin. This term can be written in the new variational variables (ξ, η, Ξ, H) as

$$\mathcal{H}_2 = \frac{1}{2}(\Xi^2 + H^2) - \omega(\xi H - \eta\Xi) + \frac{1}{2}\omega^2(\alpha\xi^2 + \beta\eta^2) \tag{15}$$

where α and β depend on the parameters of the problem and are different at each critical point. In our case we have

$$\alpha_j = 1 - 12(-1)^j G_{2,2}\frac{\mu R^2}{\omega^2 r_j^5}, \quad \beta_j = 2\left(\frac{\mu}{\omega^2 r_j^3} - 2\right)$$

where j corresponds to the subscript of the critical points, E_1, E_2 and E_2'.

The characteristic exponents of the linear system defined by equation (15) are the eigenvalues of the matrix

$$B = \begin{pmatrix} 0 & \omega & 1 & 0 \\ -\omega & 0 & 0 & 1 \\ -\omega^2\alpha & 0 & 0 & \omega \\ 0 & -\omega^2\beta & -\omega & 0 \end{pmatrix}$$

Thus, they are the solutions of the characteristic equation

$$\lambda^4 + \omega^2(\alpha + \beta + 2)\lambda^2 + \omega^4(1 - \alpha)(1 - \beta) = 0$$

and they are given by

$$\lambda^2 = \frac{\omega^2}{2}[-(\alpha + \beta + 2) \pm \sqrt{(\alpha - \beta)^2 + 8(\alpha + \beta)}]$$

Therefore, a necessary condition to have linear stability is $\lambda^2 \leq 0$. Let $\pm i\omega_1$, $\pm i\omega_2$, with ω_1, $\omega_2 \geq 0$, be the characteristic exponents in the case of linear stability, with

$$\omega_1^2 = \frac{\omega^2}{2}[(\alpha + \beta + 2) + \sqrt{(\alpha - \beta)^2 + 8(\alpha + \beta)}]$$

$$\omega_2^2 = \frac{\omega^2}{2}[(\alpha + \beta + 2) - \sqrt{(\alpha - \beta)^2 + 8(\alpha + \beta)}] \tag{16}$$

These expressions define, in the plane (α, β), the following two regions

$$R_I = \{(\alpha, \beta) \in \mathbb{R}^2 | \alpha, \beta > 1\}$$

$$R_{II} = \{(\alpha, \beta) \in \mathbb{R}^2 | -3 < \alpha, \beta < 1, (\alpha - \beta)^2 + 8(\alpha + \beta) > 0\}$$

in such a way that if $(\alpha, \beta) \in R_I \cup R_{II}$ the corresponding equilibrium point is linearly stable. In our case we have (see [4] for details)

Proposition 4 *For the points E_1 and E_2', parameters $(\alpha, \beta) \notin \bar{R}_I \cup \bar{R}_{II}$. Then, these equilibria are always unstable.*

Proposition 5 *The points E_2, when they exist, are linearly stable if $G_{2,0}$ and $G_{2,2}$ belong to the bounded region defined by:*

$$\begin{aligned} &G_{2,2} > 0, \ G_{2,0} + 6G_{2,2} > 0 \\ &32\mu^2 - 3125(G_{2,0} + 6G_{2,2})^3 R^6 \omega^4 > 0 \\ &\Delta(p_1, p_2) > 0 \end{aligned}$$

where $\Delta(p_1, p_2)$ *is the discriminant of the polynomials*

$$p_1(r_2) = \omega^4 r_2^{10} - 4\mu r_2^5 \omega^2 (54 G_{2,2} R^2 + r_2^2) + 4\mu^2 (6 G_{2,2} R^2 + r_2^2)^2$$

and

$$p_2(r_2) = 2\omega^2 r_2^5 - 2\mu r_2^2 + \mu R^2 (3 G_{2,0} + 18 G_{2,2})$$

Moreover, the above conditions imply $(\alpha, \beta) \in R_{II}$.

From the above propositions it follows that only E_2 can be linearly stable, as it happens in the cases of the Earth and Mars, for example. However, we need to solve whether these points are stable in the Lyapunov sense. The solution would be positive if parameters α and β lie in the interior of region R_I. In this situation, the quadratic part of the Hamiltonian function $\mathcal{H}_2$ is sign defined in a neighborhood of the origin and, from Dirichlet's theorem [15], the origin is stable. Nevertheless, α and β belong to region R_{II} and nonlinear stability properties need specialized theorems.

Normal Form and Nonlinear Stability

As it was described in a preceding section, results about non-linear stability require the Hamiltonian function to be in normal form in a neighborhood of the equilibrium position.

To begin with, let us suppose that the Hamiltonian function has been developed in Taylor series around the equilibrium position where $\mathcal{H}_2$ is given by equation (15) and $\mathcal{H}_k$ are homogeneous polynomials of degree k in the variables ξ and η. Now, a linear symplectic change of variables is introduced in order to transform the quadratic part of the Hamiltonian as in (2), to perform the normalization in a fast way. To this end, we introduce the complex variables $\mathbf{w} = (u_1, u_2, U_1, U_2)$ related to the old ones $\zeta = (\xi, \eta, \Xi, H)$ by the symplectic transformation

$$\zeta = \mathcal{B}\mathbf{w}$$

where $\mathcal{B}$ is the matrix

$$\mathcal{B} = \begin{pmatrix} ia_1 & -ia_2 & a_1 & a_2 \\ -b_1 & b_2 & -ib_1 & -ib_2 \\ b_1\omega - a_1\omega_1 & a_2\omega_2 - b_2\omega & -i(a_1\omega_1 - b_1\omega) & -i(a_2\omega_2 - b_2\omega) \\ i(a_1\omega - b_1\omega_1) & -i(a_2\omega - b_2\omega_2) & a_1\omega - b_1\omega_1 & a_2\omega - b_2\omega_2 \end{pmatrix}$$

and coefficients a_1, a_2, b_1, b_2 given by

$$a_1^2 = \frac{\omega_1^2 + \omega^2(1-\beta)}{2\omega_1(\omega_1^2 - \omega_2^2)}, \quad b_1^2 = \frac{\omega_1^2 + \omega^2(1-\alpha)}{2\omega_1(\omega_1^2 - \omega_2^2)} \tag{17}$$

$$a_2^2 = \frac{\omega_2^2 + \omega^2(1-\beta)}{2\omega_2(\omega_1^2 - \omega_2^2)}, \quad b_2^2 = \frac{\omega_2^2 + \omega^2(1-\alpha)}{2\omega^2(\omega_1^2 - \omega_2^2)}$$

After that, the Hamiltonian function becomes

$$\mathcal{H} = i\omega_1 u_1 U_1 + i\omega_2 u_2 U_2 + \sum_{k \geq 3} \mathcal{H}_k$$

whereas each term $\mathcal{H}_k$ is a homogeneous polynomial of degree k in the complex variables u_1, u_2, U_1 and U_2.

To arrive at the desired normal form, up to a suitable order N, a sequence of near identity canonical changes of variables is performed. This is done step by step by means of Lie transforms [20] deriving a new Hamiltonian

$$\mathcal{K} = \mathcal{H}_2 + \sum_{j=3}^{N} \mathcal{K}_j$$

where each $\mathcal{K}_j$ is a homogeneous polynomial of j degree in the complex variables satisfying

$$\{\mathcal{H}_2, \mathcal{K}_j\} = 0 \tag{18}$$

We do not enter here into details on how the Lie transforms are implemented in an algebraic manipulator in order to satisfy equation (18). For details, see the work of Deprit and López-Moratalla [4].

By applying Theorem 1, it was proved [4, 5] that points E_2 are Lyapunov stable almost everywhere in the interior of region R_{II}. The result follows from the computation of the discriminant D_4 appearing in Theorem 1, which in terms of α and β reads

$$D_4 = \frac{N(\alpha, \beta)}{24(1-\beta)^2 D(\alpha, \beta)} \frac{\omega^2}{r_2^2} \tag{19}$$

where

$$\begin{aligned} N(\alpha, \beta) = {} & -48\beta^7 + 100\alpha\beta^6 - 532\beta^6 + 602\alpha^2\beta^5 - 1956\alpha\beta^5 \\ & + 346\beta^5 - 1476\alpha^3\beta^4 + 5608\alpha^2\beta^4 - 18724\alpha\beta^4 \\ & + 15504\beta^4 + 986\alpha^4\beta^3 + 1404\alpha^3\beta^3 + 7153\alpha^2\beta^3 \\ & - 93886\alpha\beta^3 + 88807\beta^3 - 164\alpha^5\beta^2 - 11452\alpha^4\beta^2 \\ & + 77946\alpha^3\beta^2 - 195053\alpha^2\beta^2 - 60780\alpha\beta^2 + 188207\beta^2 \\ & + 1168\alpha^5\beta + 35711\alpha^4\beta - 186420\alpha^3\beta - 92485\alpha^2\beta \\ & + 105514\alpha\beta + 130032\beta - 5504\alpha^5 - 74745\alpha^4 \\ & - 80454\alpha^3 + 31175\alpha^2 + 191232\alpha - 57816 \end{aligned}$$

and

$$D(\alpha, \beta) = [(\alpha-\beta)^2 + 8(\alpha+\beta)][4(\alpha-\beta)^2 - 9(1+\alpha\beta) + 41(\alpha+\beta)]$$

In fact, if there is not a resonance condition of order less than five, E_2 is stable provided $D_4 \neq 0$. However, if $D_4 = 0$ or if the frequencies satisfy a third or fourth order resonance, further analysis is needed. Moreover, by going to higher order in the normal form, the computation of D_6 reveals the stability of E_2 if no resonances of order less than seven are satisfied [5]. The only cases left to study, and may be unstable cases, are precisely the resonant ones. Although the 1:2 and 1:3 resonances were also treated in [5], for the sake of completeness we will perform a case study of all resonances up to sixth order.

First of all, note that the normal form depends on two free parameters, α and β, or equivalently, ω_1, ω_2 or $G_{2,0}$, $G_{2,2}$, because appropriate units of time, mass and length can be chosen in order to set ω, μ and R equal to one. In this way, several situations must be taken into account depending on the order of the resonance.

Third Order Resonance

For third and fourth order resonances the normal form is no longer as it was computed in the general case from which D_4 was obtained. Now, with resonances, the normal form is as given either in equation (4) or equation (8). Specifically, for the case $\omega_1 = 2\omega_2$, the third order term in the normal form is not zero and it is given by the following expression

$$\mathcal{H}_3 = a_{1002}u_1U_2^2 + a_{0120}U_1u_2^2$$

with $a_{1002} = i\, a_{0120}$. In terms of generators M_1, M_2, S, and C given in equation (7), it is expressed as

$$\mathcal{H}_3 = \frac{\sqrt{2}}{2r_2}\kappa C$$

with

$$\kappa = a_2^2b_1(4 - 5\alpha + \beta) + 2a_1a_2b_2(5\alpha - \beta - 4) + b_1b_2^2(8 + 7\beta)$$

and a_1, a_2, b_1 and b_2 given by equation (17).

From Theorem 2, we only need to see how the two surfaces $\mathcal{G}_1$ and $\mathcal{G}_2$ cut each other. In this case, if $\kappa \neq 0$, the intersection of both surfaces is the planar curve $\{M_1 = S^{2/3}, C = 0\}$. Then, E_2 is unstable.

When $\kappa = 0$, i.e., for

$$\alpha = 0.9823588648575125, \quad \beta = -2.3725645031237670 \tag{20}$$

$\mathcal{H}_3(M_2 = 0) \equiv 0$, then, we need to push forward the normalization up to the fourth order to conclude any result about the stability. With order four, we have

$$\mathcal{H}_4(M_2 = 0) = \frac{5.69311}{r_2^2}M_1^2$$

and, the only common point of $\mathcal{G}_1$ and $\mathcal{G}_2$ is the origin. Hence, from Theorem 2, the point E_2 is stable when α and β are of those values in equation (20).

Fourth Order Resonance

If $\omega_1 = 3\omega_2$, the fourth order term in the normal form can be written as

$$\mathcal{H}_4 = a_{2200}u_1^2U_1^2 + a_{1111}u_1U_1u_2U_2 + a_{0022}u_2^2U_2^2 + a_{1003}u_1U_2^3 + a_{0130}U_1u_2^3$$

where the coefficients are complex numbers. Introducing the generators M_1, M_2, S and C, and setting $M_2 = 0$, there results

$$\mathcal{H}_4(M_2 = 0) = \alpha_4M_1^2 + \gamma C + \eta S$$

where, α_4, γ, η are now real coefficients. From Theorem 2 it is deduced that E_2 is stable if $\alpha_4^2 > \gamma^2 + \eta^2$. On the contrary, if $\alpha_4^2 < \gamma^2 + \eta^2$, the origin is unstable. The case $\alpha_4^2 = \gamma^2 + \eta^2$ is a degenerate one and higher order terms in the normal form are needed to decide the stability. Indeed, the degenerate situation takes place when the two surfaces $\mathcal{G}_1$ and $\mathcal{G}_2$ are tangent and the influence of higher order terms is decisive in order to establish the final disposition of the two surfaces. If the origin

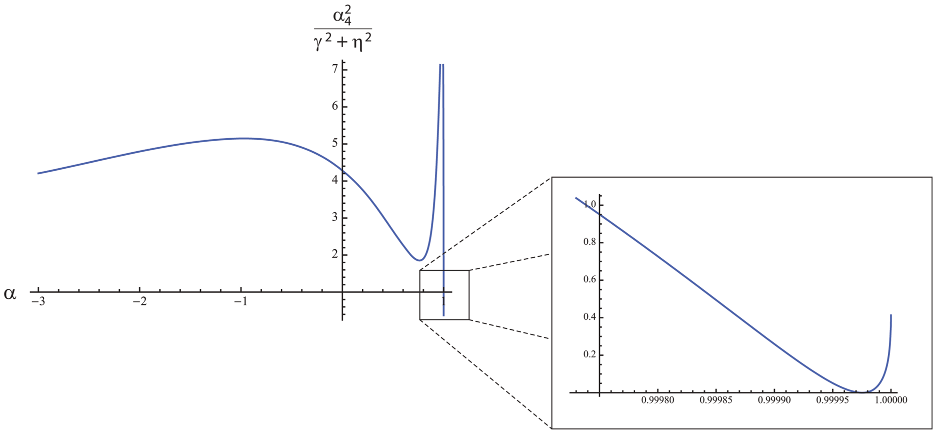

Fig. 1. Evolution of the quotient $\alpha_4^2/(\gamma^2+\eta^2)$ as a function of α. If it is less than one there is instability.

is an isolated intersection point, there will be stability and otherwise the origin will be unstable.

Taking into account the above considerations and equation (16), we proceed to evaluate $\alpha_4^2/(\gamma^2+\eta^2)$ in terms of α, provided the resonance condition ($\omega_1 = 3\omega_2$) is satisfied, that is

$$\beta = \frac{1}{9}(20\sqrt{4\alpha^2 - 17\alpha + 13} + 14\alpha - 68)$$

Depicting $\alpha_4^2/(\gamma^2+\eta^2)$ versus α we obtain the plot in Fig. 1. It can be seen that when α approaches the unit, there is a transition stability-instability as $\alpha_4^2/(\gamma^2+\eta^2)$ takes a value greater or less than one respectively.

The value of α at which $\alpha_4^2 = \gamma^2 + \eta^2$ can be computed by solving the corresponding equation and there results that

$$\alpha = 0.9997387586999709 \tag{21}$$

Thus, the critical point is stable if $-3 < \alpha < 0.9997387586999709$ and unstable if $0.9997387586999709 < \alpha < 1$.

For the critical value $\alpha = 0.9997387586999709$ as given in equation (21), it is necessary to push forward the normalization up to sixth order to determine the stability. However, this case is not covered by Theorem 2 and a specialized result due to Markeev [10, 11] is needed. After applying Markeev's result, it can be concluded that E_2 is unstable for the critical value of α.

Note also that this value of α, so close to unity, was not detected in [5].

Higher Order Resonances

Under the name of higher order resonances, we include those of order greater than four. These resonances are problematic in the case $D_4 = 0$, with D_4 the discriminant given in equation (19). However, from the results in [5], only resonances of order five and six must be taken into account. Thus, the first task is to establish the circumstances of occurrence of a higher order resonance and a zero discriminant. To this end, we pay attention to the curves defined by

$$D(\alpha, \beta) \equiv D_4(\alpha, \beta) = 0 \tag{22}$$

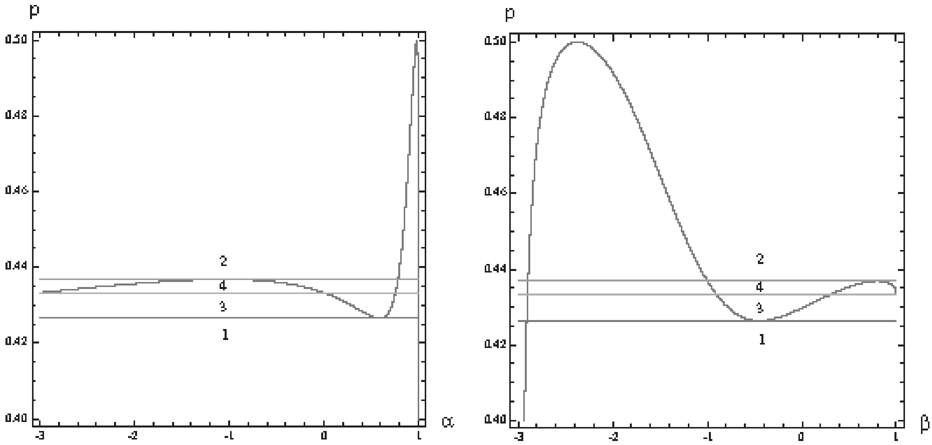

Fig. 2. The Number of Intersection Points Between $D(\alpha, \beta)$ and $R(\alpha, \beta; p)$.

and

$$R(\alpha, \beta; p) \equiv p\omega_1 - \omega_2 = 0 \tag{23}$$

Every intersection point of these two curves, for $p = n/m$ an irreducible fraction and $4 < m + n < 7$, must be considered in detail. It is interesting to see how equation (22) and equation (23) intersect each other for arbitrary values of p with $p \in (0, 1)$; ($p \to 0$ means that the order of the resonance tends to infinity, whereas $p = 1$ means a 1:1 resonance). By computing the resultant of the two polynomials in equation (22) and equation (23), both in α and β, we obtain the following results, which are illustrated in Fig. 2.

- If $p > 1/2$ there is no intersection point between $D(\alpha, \beta)$ and $R(\alpha, \beta; p)$.
- If $p = 1/2$ the two curves are tangent.
- If $0.4367852314804508 < p < 1/2$ the curves intersect twice.
- If $0.4332607423893368 < p < 0.4367852314804508$ there are four intersection points.
- If $0.42642213099660087 < p < 0.4332607423893368$ there are three intersection points.
- If $0 < p < 0.42642213099660087$ the two curves intersect once.

It is worth noting the role played by $p = 1/2$, associated with the 1:2 resonance. It separates the resonances with $p > 1/2$, which are stable, from the rest. Is this a general property of equilibrium positions or it is merely a coincidence for this problem?

Taking into account the above discussion, for the 2:3 resonance (a fifth order resonance), we have stability because $2/3 > 1/2$, and then $D_4 \neq 0$. For the other fifth order resonance, $\omega_1 = 4\omega_2$, we have $1/4 < 0.42642213099660087$ and there is a unique pair (α, β) with vanishing D_4. This pair occurs for the following values of α and β

$$\alpha = 0.9999970158240907, \quad \beta = -2.9853403950624809$$

For these particular values of the parameters we have to perform the normalization up to the fifth order in order to apply Theorem 2. In this case, the surface $\mathcal{G}_1$ is defined by

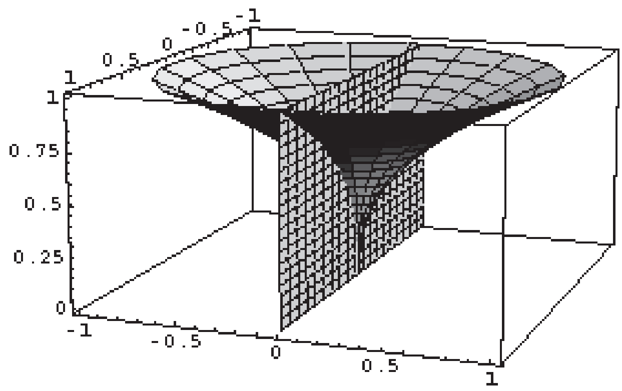

Fig. 3. Intersection Between the Surfaces $\mathcal{G}_1$ and $\mathcal{G}_2$ for the Case of 1:4 Resonance.

$$\mathcal{G}_1 \equiv \mathcal{H}_5(M_2 = 0) = \frac{46814.737737301}{r_2^3} S = 0$$

Consequently, it cuts the surface $\mathcal{G}_2 \equiv C^2 + S^2 = M_1^5$ along the curve (see Fig. 3)

$$S = 0, \quad C^2 = M_1^5$$

that is,

$$\mathcal{G}_1 \cap \mathcal{G}_2 = \{(C, S, M_1) \in \mathbb{R}^3; \ C = M_1^{5/2}, \ S = 0\}$$

hence, the equilibrium is unstable.

The other higher order resonance is a sixth order one, namely the 1:5 resonance ($\omega_1 = 5\omega_2$). As in the above case ($1/5 < 0.42642213099660087$), we find a degenerate case because for

$$\alpha = 0.999999318343538, \quad \beta = -2.9914220312235576$$

the discriminant $D_4 = 0$.

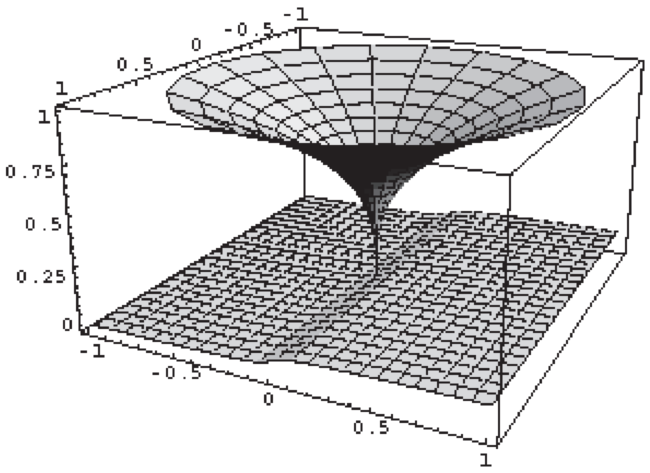

Fig. 4. Intersection between the surfaces $\mathcal{G}_1$ and $\mathcal{G}_2$ for the case of 1:5 resonance. The origin is the only common point.

Now, the normalization process must be carried out up to the sixth order, provided $\mathcal{H}_5 = 0$. In so doing, the two surfaces in Theorem 2 are

$$\mathcal{G}_1 \equiv \frac{2.041547 \times 10^{11}}{r_2^4} M_1^3 + \frac{4.241584 \times 10^7}{r_2^4} C = 0$$

$$\mathcal{G}_2 \equiv C^2 + S^2 = M_1^6$$

It is not difficult to see that they have the origin as the unique common point. In fact, surface $\mathcal{G}_1$ can be roughly approximated by $M_1 = 0$, in view of the size of the coefficients of M_1^3 and C. Consequently, (0, 0, 0) is the unique intersection point (see Fig. 4) and, according to Theorem 2, the equilibrium point is stable.

Conclusions

By means of a theorem based on geometrical aspects of the normalized Hamiltonian and valid for resonances, we characterize the stability of the stationary points of a non-spherical central body, in particular for high order resonances which remained unanalyzed. The algebra has been constructed symbolically and can be applied to bodies with shape different from the Earth.

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