



CHAOTIC PITCH MOTION OF A MAGNETIC SPACECRAFT WITH VISCOUS DRAG IN AN ELLIPTICAL POLAR ORBIT

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We study the pitch attitude dynamics of an asymmetric magnetic spacecraft in an almost circular orbit under the influence of a gravity gradient torque. The spacecraft is also subject to the influence of three perturbations: the small eccentricity of the elliptical orbit, a small magnetic torque due to the interaction with the Earth's magnetic field, and a small aerodynamic viscous drag generated by the action of the Earth's atmosphere. Under these perturbations, we show that the pitch motion exhibits heteroclinic chaotic behavior by means of the Melnikov method. This method gives us an analytical criterion for the existence of heteroclinic chaos in terms of the system parameters. This analytical criterion is confirmed numerically with good agreement. In spite of the chaos generated by the perturbations, we also find, by means of Poincaré surfaces of section that some periodic pitch motions persist in the perturbed system with the same period as the orbital motion of the spacecraft. Finally, we carry out a bifurcation analysis of these periodic motions by numerical continuation of them in terms of the perturbation parameters.

Keywords: Spacecraft; attitude dynamics; chaos; Melnikov method.

1. Introduction

The dynamics of rotating bodies is a very interesting topic in astrodynamics and space engineering because it is a useful model for studying, as a first approximation, the attitude dynamics of spacecraft [Hughes, 1986; Sidi, 1997]. Any spacecraft in orbit is under the influence of the action of several kinds of external disturbance torques, such as solar radiation pressure, gravity gradient torque, magnetic torque caused by the Earth's magnetic field, or aerodynamic drag torque [Beletskii, 1966]. Although all these external disturbances are not large compared to the weight of the vehicle, their influence on the real orientational motion of the vehicle may be significant. Gravity gradient torque is related to an interesting aspect of the attitude dynamics of a spacecraft: the so-called pitch motion [Hughes,

1986]. In this paper, we study the pitch attitude dynamics of an asymmetric magnetic spacecraft in a polar, almost circular orbit under the influence of gravity gradient torque. The spacecraft studied in this work is also subject to the influence of three different perturbations: (i) the small eccentricity of the elliptical orbit, (ii) a small magnetic torque due to the interaction between the Earth's magnetic field and the magnetic moment of the spacecraft, and (iii) a small aerodynamic viscous drag generated by the action of the Earth's atmosphere. In this study, we ignore the coupling between the orbital and the attitude motion of the spacecraft. Thus, we assume that the orbit of the spacecraft is not affected by the orientational motion.

Data obtained from the flight experiences of different satellites throughout aerospace history show

that unexpected behaviors have arisen in the attitude motion of several spacecrafts. These undesirable or chaotic orientational motions were frequently due to the action of external torques that had not been taken into account in the spacecraft design [NASA, 1969a, 1969b, 1971]. Chaotic attitude motions have also been observed and studied for natural satellites as Hyperion [Wisdom *et al.*, 1984]. These unexpected behaviors give cause for study and theoretical understanding of the attitude motion of spacecraft in different conditions, in order to detect and prevent undesirable orientational motions in advance.

During the second part of the last century, several authors studied the effects of gravity gradient torque on the attitude motion of spacecrafts. Klemperer and Baker [1956], Schindler [1959] and Klemperer [1960], studied the librations of dumbbell and ellipsoid of revolution satellites in circular orbit. Moran [1961] analyzed the effects of the planar librations on the orbital motion of an asymmetric spacecraft. Modi and Brereton [1969a, 1969b] investigated the libration periodic solutions of a gravity-gradient oriented satellite in circular and elliptic orbits. The magnetic torque is generated by the interaction between the magnetic features of the spacecraft and the magnetic field of the Earth. The cases of Vanguard I and Tiros I satellites can be cited as examples of the effects of magnetic torques on the attitude motion of spacecrafts [Hughes, 1986; NASA, 1969a]. The strength of the magnetic torque depends on the intrinsic magnetic moment of the spacecraft, but it is usually smaller than one tenth of the gravity gradient torque [Hughes, 1986; Bryson, 1994]. In relation to the aerodynamic drag torque, it is worth noting that there is a range of altitudes with operative satellites at which aerodynamic drag is not only not negligible but it may also be dominant [Hughes, 1986]. Several authors, such as Wainwright [1927], Deimel [1952], Gray [1959], and others cited by Leimanis [1965] have studied the dynamics of a revolving symmetric body under the influence of an aerodynamic drag.

During recent decades, numerous theoretical studies have indicated the existence of chaotic attitude behaviors in several kinds of satellites under the action of different perturbations. In this way, Tong and Rimrott [1991] numerically investigated the planar libration of an asymmetric satellite in elliptic orbit under gravity gradient torque. Teofilatto and Graziani [1996] have studied the same system but considering the three-dimensional libration

motion of the spacecraft. Holmes and Marsden [1983], Koiller [1984], and Peng and Liu [2000] have analyzed free gyrostats with a slightly asymmetric rotor. Karasopoulos and Richardson [1992, 1993] have analytically and numerically studied the attitude dynamics of a satellite under gravity gradient torque. Nixon and Misra [1993], Fujii and Ichiki [1997] have numerically researched the orientational motion of tethers. Tong *et al.* [1995] have also discussed the case of an asymmetric gyrostat under a uniform gravitational field. Maciejewski [1995] has analytically and numerically studied the planar oscillations of a rigid satellite under nongravitational perturbations. Meehan and Asokanthan [1997] and Gray *et al.* [1999] have analyzed the attitude motion of satellites with internal dissipation of energy. Beletsky *et al.* [1999] have numerically treated the case of a magnetic spacecraft in circular polar orbit subject only to geomagnetic torque. Lanchares *et al.* [1998], Iñarrea and Lanchares [2000], Iñarrea *et al.* [2003], Iñarrea and Lanchares [2006] have analytically and numerically investigated the chaotic orientational motions of several kinds of asymmetric spacecraft with time-dependent moments of inertia in different external conditions.

This study is a continuation of the author's previous works. We have considered that the spacecraft is affected by several kinds of perturbations, such as magnetic torque and the eccentricity of the elliptic orbit. We have studied the effects of the perturbations on the attitude dynamics of the vehicle. To this end, we have made use of analytical and numerical tools already applied in the author's former papers, such as the Melnikov method and Poincaré surfaces of section. Moreover, we have also performed a bifurcation analysis by numerical continuation of periodic attitude motions of the spacecraft. It is worth noting that this technique had not been used by the author in his previous works [Iñarrea & Lanchares, 2000; Iñarrea *et al.*, 2003; Iñarrea & Lanchares, 2006]. In this way, we have found that, although the perturbations generate chaotic behaviors in the orientational dynamics, regular periodic motions persist with the same period of the orbital motion of the spacecraft.

The present paper is organized in the following way. In Sec. 2, we describe the perturbed system and express the equation of the spacecraft pitch motion. In Sec. 3, we calculate the Melnikov function of the perturbed spacecraft obtaining an analytical criterion for the existence of chaos.

The validity of this criterion is numerically confirmed. In Sec. 4, we study the persistency of periodic pitch motions with the same period of the orbital motion. Finally, in Sec. 5, we perform a bifurcation analysis of these periodic motions by a numerical continuation of them.

2. Description of the System and Equations of Motion

We consider an asymmetric magnetic spacecraft in an almost circular polar orbit in the Earth's gravitational and magnetic fields subject to the viscous aerodynamic drag due the Earth's atmosphere. The spacecraft has its own magnetic moment generated by permanent magnets or electric current loops. We assume that the aerodynamic drag only affects the attitude motion of the spacecraft. Therefore, we neglect any decay or rise in the orbit followed by the spacecraft and we focus the analysis on the system attitude dynamics.

In this study, three different right oriented orthonormal reference frames are used:

- The inertial geocentric frame $\mathcal{E}\{O_E, X_E, Y_E, Z_E\}$ with the origin O_E at the center of mass of the Earth, the $X_E Y_E$ plane coincident with the equatorial plane, the X_E axis passing through the ascending node N , and the Z_E axis aligned with the Earth's rotation axis.
- The orbital frame $\mathcal{R}\{O, X, Y, Z\}$ with origin O at the mass center of the spacecraft, the Z axis along the local vertical pointing to the mass center of the Earth O_E , the Y axis is normal to the orbital plane and the X axis is in the orbital plane but it does not coincide exactly with the velocity vector of the spacecraft due to the eccentricity of the orbit. See Fig. 1. In the usual aircraft and spacecraft terminology, the X, Y, Z axes are called respectively *roll*, *pitch* and *yaw* axes [Hughes, 1986; Sidi, 1997].
- The body frame $\mathcal{B}\{O, x, y, z\}$, is established with the directions of the axes coincident with the principal axes of the spacecraft.

As it is well known, the relative orientation between the last two reference frames results by means of three consecutive rotations involving the Euler angles (ψ, θ, ϕ) . To move from the orbital axes $\{X, Y, Z\}$ to the body axes $\{x, y, z\}$, the first rotation is about the Z axis through an angle ψ (yaw). The second rotation is about

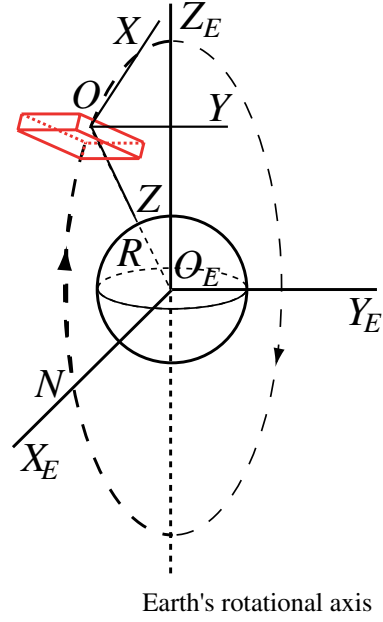


Fig. 1. The inertial geocentric frame $\mathcal{E}$ and the orbital reference frame $\mathcal{R}$.

the new axis Y' by an angle θ (pitch). Finally, the third rotation is about the new axis x through an angle ϕ (roll), reaching the body axes $\{x, y, z\}$. This particular set of Euler angles is commonly used in aircraft and spacecraft attitude and they are also known as Tait–Bryan or Cardan angles [Hughes, 1986; Hale, 1994; Wiesel, 1997]. We do not use the classical Euler angles [Goldstein *et al.*, 2002] because they have a singularity in the particular orientation that is studied in this paper.

The attitude dynamics of the spacecraft is governed by three different torques: (i) the one provided by the gravity gradient, (ii) the magnetic torque generated by the interaction between the spacecraft and the Earth's magnetic field, and (iii) the aerodynamic drag caused by the Earth's atmosphere. Taking these torques into account, the classical theorem of angular momentum about the mass center O of the spacecraft, expressed in the noninertial body frame $\mathcal{B}$, is

$$\mathbb{I} \frac{d\boldsymbol{\omega}_T}{dt} + \boldsymbol{\omega}_T \times \mathbb{I} \boldsymbol{\omega}_T = \mathbf{N}_g + \mathbf{N}_m + \mathbf{N}_d, \quad (1)$$

where $\boldsymbol{\omega}_T$ is the total angular velocity of the spacecraft about its mass center, $\mathbf{N}_g$ is the gravitational torque, $\mathbf{N}_m$ is the magnetic torque, $\mathbf{N}_d$ is the drag torque, and $\mathbb{I}$ is the tensor of inertia of the spacecraft. As it is expressed in the body frame $\mathcal{B}$ of

the principal axes of the spacecraft, this tensor is a diagonal one, that is, $\mathbb{I} = \text{diag}(I_x, I_y, I_z)$, where I_x, I_y and I_z are the moments of inertia of the spacecraft. We assume an asymmetric spacecraft with this specific relation $I_x > I_y > I_z$.

Taking into account that in the total angular velocity $\boldsymbol{\omega}_T$ of the spacecraft there are two contributions: one from the orbital motion and other from the attitude motion, thus this total angular velocity $\boldsymbol{\omega}_T$ can be written in the body frame $\mathcal{B}$ as

$$\boldsymbol{\omega}_T = \boldsymbol{\omega} + C_{\mathcal{R}\mathcal{B}} \boldsymbol{\omega}_o. \quad (2)$$

Here $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$ is the attitude angular velocity of the body about its mass center O in the body frame $\mathcal{B}$. Besides, $\boldsymbol{\omega}_o = (0, -\dot{\nu}, 0)$ is the orbital angular velocity of the spacecraft expressed in the orbital frame $\mathcal{R}$, where ν is the true anomaly that gives us the angular position of the spacecraft in its orbit. Finally, $C_{\mathcal{R}\mathcal{B}}$ is the transformation matrix from the orbital frame $\mathcal{R}$ to the body frame $\mathcal{B}$, that is, the matrix of the three consecutive rotations involving the Euler angles (ψ, θ, ϕ) .

Due to the gravity gradient and the finite dimension of the spacecraft, it is under the action of a gravitational torque $\mathbf{N}_g$ about the body mass center O . The components of this torque $\mathbf{N}_g$ in the body frame $\mathcal{B}$ are given in [Hughes, 1986; Sidi, 1997; Wiesel, 1997]:

$$\begin{cases} N_{gx} = \frac{3\mu_g}{R^3} (I_z - I_y) \sin \phi \cos \phi \cos^2 \theta, \\ N_{gy} = \frac{3\mu_g}{R^3} (I_z - I_x) \cos \phi \sin \theta \cos \theta, \\ N_{gz} = \frac{3\mu_g}{R^3} (I_x - I_y) \sin \phi \sin \theta \cos \theta, \end{cases} \quad (3)$$

where $\mu_g = Gm_e = 3.986 \cdot 10^{14} \text{ Nm}^2/\text{kg}$ is the mass parameter of the Earth, and R is the distance between the mass centers of the spacecraft and Earth.

As we consider that the spacecraft has its own magnetic moment, it is also under the action of another torque generated by the interaction with the Earth's magnetic field. We suppose that the terrestrial magnetic field $\mathbf{B}$ is generated by a perfect dipole located at the mass center of the Earth and aligned with its rotation axis [Hughes, 1986; Sidi, 1997; Wiesel, 1997]. In this way, the components of the magnetic field $\mathbf{B} = (B_x, B_y, B_z)$ are expressed in the orbital frame $\mathcal{R}$ in SI units as

$$\begin{cases} B_x = \frac{\mu_o \mu_m}{4\pi R^3} \sin i \cos(\nu + \Omega), \\ B_y = -\frac{\mu_o \mu_m}{4\pi R^3} \cos i \\ B_z = \frac{\mu_o \mu_m}{4\pi R^3} 2 \sin i \sin(\nu + \Omega), \end{cases} \quad (4)$$

where μ_o is the magnetic permeability of free space, $\mu_m \approx 7.8 \cdot 10^{22} \text{ A} \cdot \text{m}^2$ is the geomagnetic dipole moment [Korte & Constable, 2005], and i and Ω are the inclination and the argument of perigee of the spacecraft orbit respectively.

The magnetic torque $\mathbf{N}_m$ acting over the spacecraft, calculated in the body frame $\mathcal{B}$, is given by the cross product,

$$\mathbf{N}_m = \mathbf{M} \times C_{\mathcal{R}\mathcal{B}} \mathbf{B}, \quad (5)$$

where $\mathbf{M} = (M_x, M_y, M_z)$ is the own magnetic moment of the spacecraft expressed in the body frame $\mathcal{B}$, and the geomagnetic field $\mathbf{B} = (B_x, B_y, B_z)$ is expressed in the orbital frame (4).

On the other hand, we also consider that the spacecraft is in a high enough orbit to consider the Earth's atmosphere as a lightly resisting medium and its action on the rotating body is a small drag torque $\mathbf{N}_d$ opposite to the attitude motion about O . We also assume that the torque is directly proportional to the total angular velocity $\boldsymbol{\omega}_T$ of the spacecraft, that is,

$$\mathbf{N}_d = -\gamma \boldsymbol{\omega}_T = -\gamma (\boldsymbol{\omega} + C_{\mathcal{R}\mathcal{B}} \boldsymbol{\omega}_o), \quad (6)$$

where $\gamma > 0$ is the coefficient of the viscous drag.

As it is well known, the components $(\omega_x, \omega_y, \omega_z)$ of the angular velocity $\boldsymbol{\omega}$ in the body frame $\mathcal{B}$, can be written in terms of the Euler angles (ψ, θ, ϕ) and their velocities $(\dot{\psi}, \dot{\theta}, \dot{\phi})$ [Hughes, 1986; Hale, 1994; Sidi, 1997; Wiesel, 1997]. Making use of those well known relations and applying Eqs. (2)–(6), the equation of motion (1) could be explicitly written in terms of the Euler angles (ψ, θ, ϕ) , their velocities $(\dot{\psi}, \dot{\theta}, \dot{\phi})$ and their accelerations $(\ddot{\psi}, \ddot{\theta}, \ddot{\phi})$, resulting in quite cumbersome expressions.

Nevertheless, in this paper we adopt the following assumptions: (i) the spacecraft is tracing a polar orbit, that is, its inclination is $i = \pi/2$; (ii) the magnetic moment $\mathbf{M}$ of the spacecraft keeps constant and aligned with the principal axis z of the spacecraft, that is, $\mathbf{M} = (0, 0, M)$ in the body frame $\mathcal{B}$; and (iii) the roll and yaw motions are initially quiescent, that is, $\psi(0) = \dot{\psi}(0) = 0$ and $\phi(0) = \dot{\phi}(0) = 0$. In this situation, roll and yaw motions are not

excited by the pitch motion. The direction of the principal axis y of the spacecraft is fixed in space and it is always normal to the orbital plane. The orientation of the spacecraft can be described with only one angle θ , the pitch angle. Thus, there is only one nontrivial equation of motion for the attitude dynamics of the system,

$$\begin{aligned} \frac{d^2\theta}{dt^2} = & \frac{d^2\nu}{dt^2} - \frac{3\mu_g(I_x - I_z)}{I_y R^3} \sin\theta \cos\theta \\ & + \frac{\mu_o M \mu_m}{4\pi I_y R^3} [\cos\theta \cos(\nu + \Omega) \\ & - 2\sin\theta \sin(\nu + \Omega)] \\ & + \frac{\gamma}{I_y} \left(\frac{d\nu}{dt} - \frac{d\theta}{dt} \right). \end{aligned}$$

Now, performing a change of variable where time t is replaced by the true anomaly ν as the independent variable of the problem, and introducing the following new dimensionless parameters

$$K = \frac{3(I_x - I_z)}{I_y}, \quad \beta = \frac{\mu_o M \mu_m}{4\pi \mu_g I_y}, \quad \alpha = \frac{p^{3/2} \gamma}{I_y \mu_g^{1/2}},$$

the equation of the pitch motion becomes

$$\begin{aligned} \ddot{\theta} = & -\frac{K \sin\theta \cos\theta}{1 + e \cos\nu} + \frac{2e \sin\nu}{1 + e \cos\nu} (\dot{\theta} - 1) \\ & + \beta \frac{[\cos\theta \cos(\nu + \Omega) - 2\sin\theta \sin(\nu + \Omega)]}{1 + e \cos\nu} \\ & + \frac{\alpha}{(1 + e \cos\nu)^2} (1 - \dot{\theta}), \end{aligned}$$

where ν is the independent variable, e is the eccentricity and $p = a(1 - e^2)$ is the parameter of the orbit traced by the spacecraft. From this equation and in the rest of the paper, the dot means derivation with respect to the true anomaly ν . In this equation, the first term on the right-hand side comes from the gravity gradient, the second one arises from the inertial Coriolis forces, the third one from the interaction with the Earth's magnetic field, and the last one from the aerodynamic drag.

Therefore, the attitude dynamics of the spacecraft depends on four parameters: K which describes the spacecraft's asymmetry, e the orbit eccentricity, β which describes the strength of the magnetic interaction, and α which describes the strength of the aerodynamic drag.

As we consider that the spacecraft is tracing an almost circular orbit, and we also assume that the magnetic interaction and the aerodynamic drag are much weaker than the gravitational interaction, in this case we can suppose that the parameters e , β and α are small, that is, $e \ll 1$, $\beta \ll 1$ and $\alpha \ll 1$. Hence, making use of the expansion $(1 + e \cos\nu)^{-1} \approx 1 - e \cos\nu$, and omitting terms of second order in the small parameters e , β and α , the equation of the pitch motion results in

$$\begin{aligned} \ddot{\theta} = & -K \sin\theta \cos\theta + K e \cos\nu \sin\theta \cos\theta \\ & + 2e(\dot{\theta} - 1) \sin\nu \\ & + \beta [\cos\theta \cos(\nu + \Omega) - 2\sin\theta \sin(\nu + \Omega)] \\ & + \alpha(1 - \dot{\theta}). \end{aligned} \quad (7)$$

The terms in e , β and α in this equation can be considered as small perturbations. In this way, the unperturbed system ($e = \beta = \alpha = 0$) coincides with an asymmetric spacecraft in circular orbit under only the gravity gradient torque. Thus, the equation of motion of the unperturbed spacecraft may be rewritten in the form of a system of two first order differential equations as,

$$\begin{cases} \dot{\theta} = \omega = f_1, \\ \dot{\omega} = -K \sin\theta \cos\theta = f_2. \end{cases} \quad (8)$$

As it can be seen, the unperturbed spacecraft is a one-degree-of-freedom Hamiltonian system and, therefore, it is integrable. In fact, Eqs. (8) are those corresponding to a nonlinear pendulum taking 2θ as the angular variable. Therefore, it is known that the system has unstable equilibria at $(\theta, \omega) = (\pm(2n + 1)\pi/2, 0)$, and stable equilibria at $(\pm n\pi, 0)$. Figure 2 shows the main features of the phase flow for the unperturbed system (8) for $K = 1$. The two unstable equilibria, denoted by E_1 and E_2 , are connected by four heteroclinic trajectories. These orbits are the separatrices of the phase space, the thick continuous lines in Fig. 2.

The energy of the system corresponding to the unstable equilibria and the separatrices is $\mathcal{E}_{\text{sep}} = K/2$. These separatrices divide the phase space in two different classes of pitch motion. On the one hand, oscillations, the dotted lines inside the separatrices, when the energy of the spacecraft is $\mathcal{E} < \mathcal{E}_{\text{sep}}$. On the other hand, tumbling rotations, the dashed lines outside the separatrices, when the energy of the body is $\mathcal{E} > \mathcal{E}_{\text{sep}}$. Likewise, the solutions corresponding to the four asymptotic heteroclinic

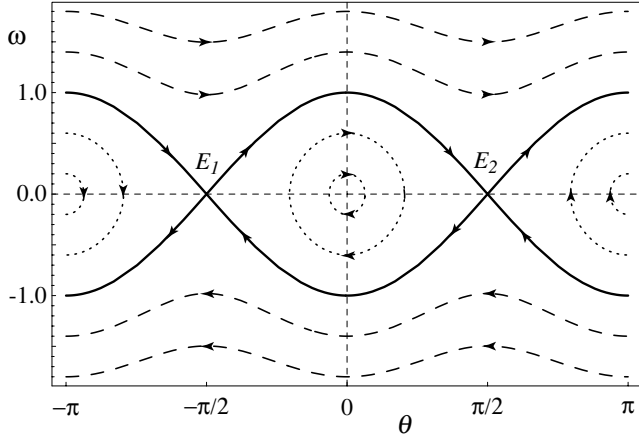


Fig. 2. The phase space of the unperturbed pitch motion of an asymmetric spacecraft in circular orbit under the gravity gradient torque for $K = 1$.

trajectories, are

$$\begin{aligned} [\theta^\pm(\nu), \omega^\pm(\nu)] \\ = \{\pm \arcsin[\tanh(\sqrt{K}\nu)], \pm \sqrt{K} \operatorname{sech}(\sqrt{K}\nu)\}, \end{aligned} \quad (9)$$

subject to the initial conditions $(\theta_o^\pm(0), \omega_o^\pm(0)) = (0, \pm\sqrt{K})$. The four heteroclinic trajectories form the stable $W_s(E_1), W_s(E_2)$ and unstable $W_u(E_1), W_u(E_2)$ manifolds corresponding to the two unstable equilibria, that join smoothly together. So it holds that $W_s(E_1) = W_u(E_2)$ and $W_u(E_1) = W_s(E_2)$. The positive (negative) solutions of (9) form the upper (lower) branches of the invariant manifolds.

3. Chaotic Pitch Motion. The Melnikov Function

Let us consider the perturbed system. Now the stable and unstable manifolds are not forced to coincide and it is possible that they intersect transversally in the corresponding Poincaré surface of section, leading to an infinite number of new heteroclinic points. Then, a heteroclinic tangle is generated. In such a case, because of the perturbations, the pitch motion of the spacecraft near the unperturbed separatrices becomes extremely complicated and chaotic in the sense that the system exhibits Smale's horseshoes and a stochastic layer appears near the unperturbed separatrices. Inside this chaotic layer, small isolated regions of regular motion with periodic orbits can also appear.

The existence of heteroclinic intersections may be proved, at first order, by means of the Melnikov

method [Melnikov, 1963; Guckenheimer & Holmes, 1983]. In order to apply the Melnikov method, Eq. (7) can be expressed as the following system of two differential equations of first order

$$\begin{cases} \dot{\theta} = \omega = f_1 + g_1, \\ \dot{\omega} = -K \sin \theta \cos \theta + Ke \cos \nu \sin \theta \cos \theta \\ \quad + 2e(\omega - 1) \sin \nu \\ \quad + \beta[\cos \theta \cos(\nu + \Omega) - 2 \sin \theta \sin(\nu + \Omega)] \\ \quad + \alpha(1 - \omega) = f_2 + g_2, \end{cases} \quad (10)$$

where $g_1 = 0$ and $g_2 = Ke \cos \nu \sin \theta \cos \theta + 2e \sin \nu(\omega - 1) + \beta[\cos \theta \cos(\nu + \Omega) - 2 \sin \theta \sin(\nu + \Omega)] + \alpha(1 - \omega)$.

The Melnikov function, $M^\pm(\nu_0)$, for the system (10) is given by

$$\begin{aligned} M^\pm(\nu_0) &= \int_{-\infty}^{\infty} \mathbf{f}[\mathbf{z}^\pm(\nu)] \wedge \mathbf{g}[\mathbf{z}^\pm(\nu), \nu + \nu_0] d\nu \\ &= \int_{-\infty}^{\infty} \{f_1[\mathbf{z}^\pm(\nu)] g_2[\mathbf{z}^\pm(\nu), \nu + \nu_0] \\ &\quad - f_2[\mathbf{z}^\pm(\nu)] g_1[\mathbf{z}^\pm(\nu), \nu + \nu_0]\} d\nu \\ &= \int_{-\infty}^{\infty} f_1[\mathbf{z}^\pm(\nu)] g_2[\mathbf{z}^\pm(\nu), \nu + \nu_0] d\nu \\ &= \int_{-\infty}^{\infty} \omega^\pm(\nu) \{Ke \cos(\nu + \nu_0) \\ &\quad \times \sin \theta^\pm(\nu) \cos \theta^\pm(\nu) \\ &\quad + 2e(\omega^\pm(\nu) - 1) \sin(\nu + \nu_0) \\ &\quad + \beta[\cos \theta^\pm(\nu) \cos(\nu + \nu_0 + \Omega) \\ &\quad - 2 \sin \theta^\pm(\nu) \sin(\nu + \nu_0 + \Omega)] \\ &\quad + \alpha(1 - \omega^\pm(\nu))\} d\nu, \end{aligned} \quad (11)$$

where $\mathbf{z}^\pm(\nu) = (\theta^\pm(\nu), \omega^\pm(\nu))$ are the solutions of the unperturbed heteroclinic orbits (9).

The Melnikov function $M^\pm(\nu_0)$ gives us a measure of the distance between the stable and unstable manifolds of the perturbed hyperbolic fixed points. Thus, if $M^\pm(\nu_0)$ has simple zeroes, there are transverse intersections between the stable and unstable manifolds in the corresponding Poincaré surface of section.

Now, by substitution of the positive solutions $(\theta^+(\nu), \omega^+(\nu))$ of Eq. (9) into (11) we obtain the Melnikov function $M^+(\nu_0)$ for the upper

branches of the invariant manifolds of the hyperbolic equilibria

$$\begin{aligned}
 M^+(\nu_0) &= M_1^+ + M_2^+ + M_3^+ + M_4^+ \\
 &= K^{3/2}e \int_{-\infty}^{\infty} \text{sech}^2(\sqrt{K}\nu) \tanh(\sqrt{K}\nu) \cos(\nu + \nu_0) d\nu \\
 &\quad + 2e\sqrt{K} \int_{-\infty}^{\infty} \text{sech}(\sqrt{K}\nu) [\sqrt{K} \text{sech}(\sqrt{K}\nu) - 1] \sin(\nu + \nu_0) d\nu \\
 &\quad + \beta\sqrt{K} \int_{-\infty}^{\infty} \left[\text{sech}^2(\sqrt{K}\nu) \cos(\nu + \nu_0 + \Omega) - 2 \frac{\sinh(\sqrt{K}\nu)}{\cosh^2(\sqrt{K}\nu)} \sin(\nu + \nu_0 + \Omega) \right] d\nu \\
 &\quad + \alpha\sqrt{K} \int_{-\infty}^{\infty} \text{sech}(\sqrt{K}\nu) [1 - \sqrt{K} \text{sech}(\sqrt{K}\nu)] d\nu,
 \end{aligned} \tag{12}$$

M_1^+ and M_2^+ being the Melnikov function corresponding to the perturbation coming from the elliptic orbit, M_3^+ the one arising from the magnetic interaction, and M_4^+ the one coming from the aerodynamic drag.

These four integrals, M_i^+ can be calculated integrating them by parts and arriving at other simpler integrals tabulated by Gradshteyn and Ryzhik [1980]. In this way, we obtain

$$\begin{aligned}
 M_1^+ &= -\frac{\pi e}{2} \text{cosech}\left(\frac{\pi}{2\sqrt{K}}\right) \sin(\nu_0), \\
 M_2^+ &= 2\pi e \left[\text{cosech}\left(\frac{\pi}{2\sqrt{K}}\right) - \text{sech}\left(\frac{\pi}{2\sqrt{K}}\right) \right] \sin(\nu_0), \\
 M_3^+ &= \frac{\pi\beta}{\sqrt{K}} \left[\text{cosech}\left(\frac{\pi}{2\sqrt{K}}\right) - 2\text{sech}\left(\frac{\pi}{2\sqrt{K}}\right) \right] \\
 &\quad \times \cos(\nu_0 + \Omega), \\
 M_4^+ &= \alpha(\pi - 2\sqrt{K}).
 \end{aligned} \tag{13}$$

Thus, the complete Melnikov function $M^+(\nu_0)$ results in

$$\begin{aligned}
 M^+(\nu_0) &= M_A^+(\nu_0) + M_B^+(\nu_0) + M_C^+ \\
 &= C_A^+ \sin(\nu_0) + C_B^+ \cos(\nu_0 + \Omega) \\
 &\quad + \alpha(\pi - 2\sqrt{K}),
 \end{aligned} \tag{14}$$

where the coefficients C_A^+ and C_B^+ , which depends on the system parameters K, e and β , are given by

$$\begin{aligned}
 C_A^+(K, e) &= \pi e \left[\frac{3}{2} \text{cosech}\left(\frac{\pi}{2\sqrt{K}}\right) - 2\text{sech}\left(\frac{\pi}{2\sqrt{K}}\right) \right],
 \end{aligned}$$

$$\begin{aligned}
 C_B^+(K, \beta) &= \frac{\pi\beta}{\sqrt{K}} \left[\text{cosech}\left(\frac{\pi}{2\sqrt{K}}\right) - 2\text{sech}\left(\frac{\pi}{2\sqrt{K}}\right) \right].
 \end{aligned} \tag{15}$$

In the same way, using the negative solutions $(\theta^-(\nu), \omega^-(\nu))$ of Eq. (9) in (11) we obtain the Melnikov function $M^-(\nu_0)$ for the lower branches of the invariant manifolds of the hyperbolic fixed points E_1 and E_2

$$\begin{aligned}
 M^-(\nu_0) &= M_A^-(\nu_0) + M_B^-(\nu_0) + M_C^- \\
 &= C_A^- \sin(\nu_0) + C_B^- \cos(\nu_0 + \Omega) \\
 &\quad - \alpha(\pi + 2\sqrt{K}),
 \end{aligned} \tag{16}$$

where the coefficients C_A^- and C_B^- are given by

$$\begin{aligned}
 C_A^-(K, e) &= \pi e \left[\frac{3}{2} \text{cosech}\left(\frac{\pi}{2\sqrt{K}}\right) + 2\text{sech}\left(\frac{\pi}{2\sqrt{K}}\right) \right], \\
 C_B^-(K, \beta) &= -\frac{\pi\beta}{\sqrt{K}} \left[\text{cosech}\left(\frac{\pi}{2\sqrt{K}}\right) + 2\text{sech}\left(\frac{\pi}{2\sqrt{K}}\right) \right].
 \end{aligned} \tag{17}$$

It is important to note that Eqs. (14) and (16) give us analytical criterions for the existence of heteroclinic chaos in terms of the system parameters. Indeed, the first two terms $M_A^\pm$ and $M_B^\pm$ of the Melnikov functions (14) and (16) form bounded functions $M_{AB}^\pm(\nu_0) = M_A^\pm(\nu_0) + M_B^\pm(\nu_0)$. It is easy to obtain that the extreme values of these functions

$M_{AB}^\pm(\nu_0)$ are given by

$$M_{AB\text{ext}}^\pm = \pm \sqrt{(C_A^\pm)^2 + (C_B^\pm)^2 - 2C_A^\pm C_B^\pm \sin \Omega}.$$

Therefore, as the drag parameter is positive defined ($\alpha > 0$), it is straightforward to conclude that the positive Melnikov function $M^+(\nu_0)$ has

simple zeroes for

$$\begin{aligned} \alpha < \alpha_c^+ &= \frac{M_{AB\text{ext}}^+}{\pi - 2\sqrt{K}} \\ &= \frac{\sqrt{(C_A^+)^2 + (C_B^+)^2 - 2C_A^+ C_B^+ \sin \Omega}}{\pi - 2\sqrt{K}}, \end{aligned} \quad (18)$$

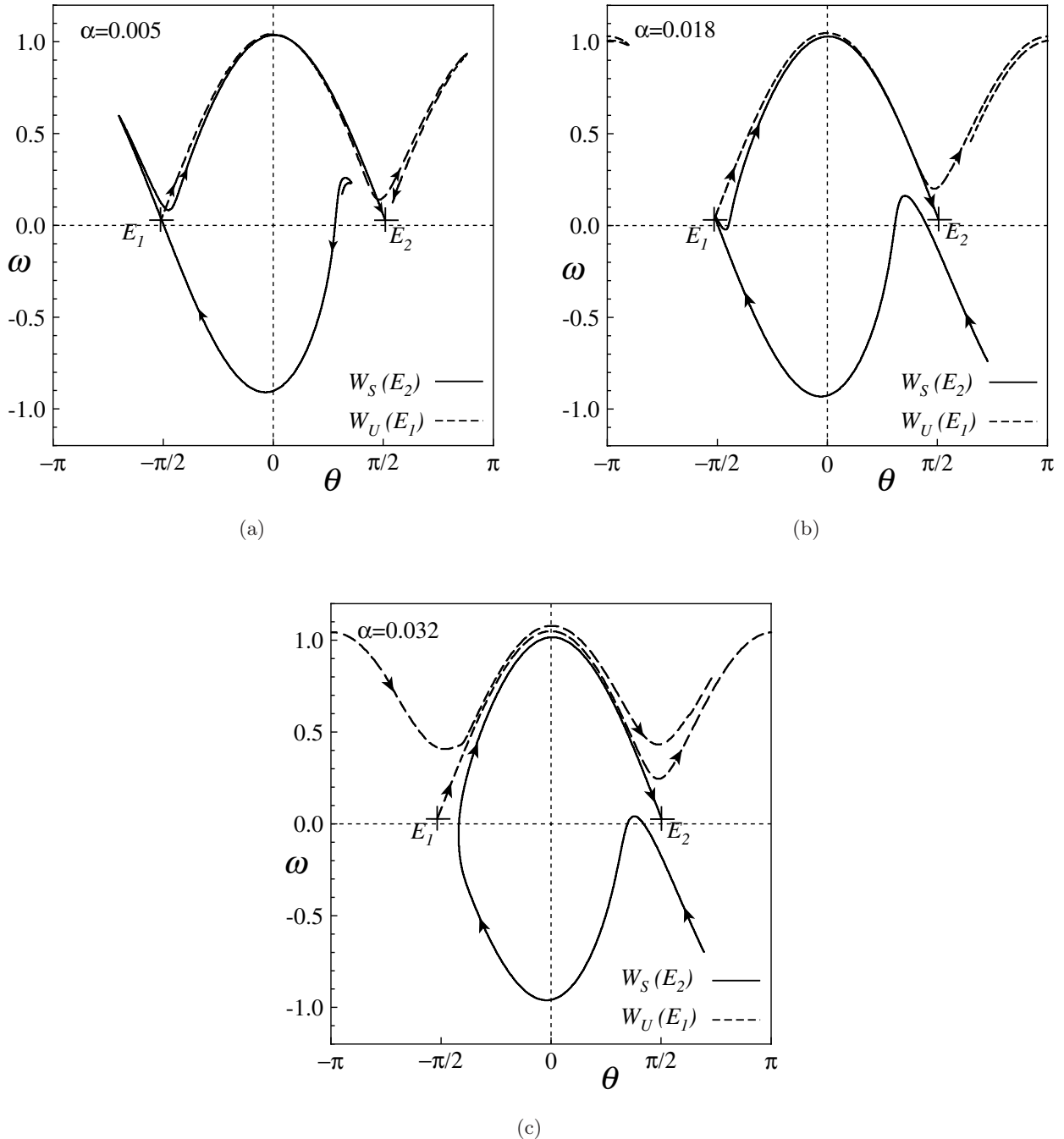


Fig. 3. Evolution of the stable and unstable manifolds $W_s(E_2)$ and $W_u(E_1)$ of the saddle fixed points E_i in the $\nu = 2\pi$ Poincare map as a function of the drag parameter α for $K = 1, \epsilon = \beta = 0.03, \Omega = \pi/2$.

and that the negative Melnikov function $M^-(\nu_0)$ has simple zeroes for

$$\begin{aligned} \alpha < \alpha_c^- &= \frac{M_{AB\text{ext}}^-}{-(\pi + 2\sqrt{K})} \\ &= \frac{\sqrt{(C_A^-)^2 + (C_B^-)^2 - 2C_A^-C_B^- \sin \Omega}}{\pi + 2\sqrt{K}}. \end{aligned} \quad (19)$$

Hence, for $\alpha < \max(\alpha_c^\pm)$ the perturbations produce heteroclinic intersections between the stable and unstable manifolds of the hyperbolic equilibria E_1 and E_2 in the corresponding Poincaré surface of section. Therefore the perturbed spacecraft shows chaotic pitch motions near the unperturbed separatrices. On the other hand, for $\alpha > \max(\alpha_c^\pm)$, the Melnikov functions $M^\pm(\nu_0)$ are bounded away from zero, and thus there are no heteroclinic intersections and no chaos in the pitch motion of the

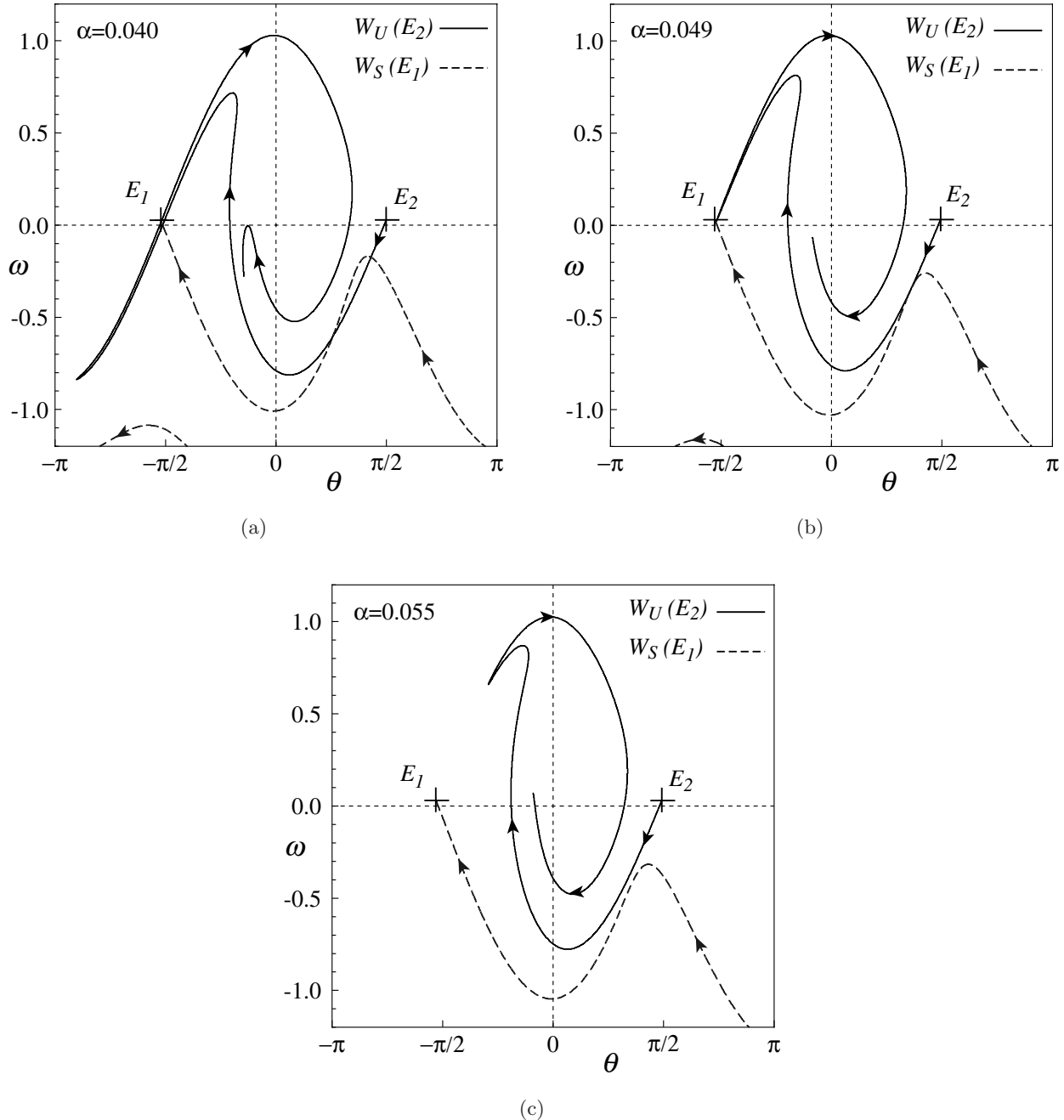


Fig. 4. Evolution of the stable and unstable manifolds $W_s(E_1)$ and $W_u(E_2)$ of the saddle fixed points E_i in the $\nu = 2\pi$ Poincaré map as a function of the drag parameter α for $K = 1, \epsilon = \beta = 0.03, \Omega = \pi/2$.

perturbed spacecraft. From Eqs. (15) and (17)–(19), it is clear that the critical values $\alpha_c^\pm$ of the aerodynamic drag parameter, which state the existence of chaotic behavior, depends on the rest of the system parameters K, e, β and Ω .

In order to check the validity of the analytical criteria (18) and (19) given by the Melnikov method, we have numerically calculated the stable $W_s(E_i)$ and unstable $W_u(E_i)$ manifolds associated to the saddle fixed points E_1, E_2 of the Poincaré map. The Poincaré surface of section consists of sections $\nu = \text{cte.}(\text{mod } 2\pi)$ of the three-dimensional (θ, ω, ν) extended phase space, that is, we have the Poincaré sections in the ν variable with the angular period of the orbital motion of the spacecraft. This computation was carried out by means of the commercial software DYNAMICS [Nusse & Yorke, 1998], integrating numerically the equations of motion (10) by means of a Runge–Kutta algorithm of fifth order with fixed step.

We have focused on the evolution of the invariant manifolds $W_{s/u}(E_i)$ as a function of the drag parameter α , for fixed parameters $K = 1, e = \beta = 0.03$ and $\Omega = \pi/2$. For these parameter values, the analytical criteria (18) and (19) give us the critical values of the drag parameter $\alpha_c^+ \approx 0.01793$ and $\alpha_c^- \approx 0.04913$. We have tuned α from values less than $\alpha_c^\pm$ to greater ones. In Figs. 3(a)–3(c), we show the evolution of the upper branches of the invariant manifolds of the fixed points E_i , that is, $W_u(E_1)$ and $W_s(E_2)$. It can be observed clearly that, for $\alpha < \alpha_c^+$ ($\alpha = 0.005$), the stable and unstable manifolds transversally intersect each other [Fig. 3(a)]. However, when $\alpha > \alpha_c^+$ ($\alpha = 0.032$), the invariant manifolds do not intersect [Fig. 3(c)]. Finally, Fig. 3(b) shows just the situation for the critical value α_c^+ , where the tangency of the stable and unstable manifolds can be seen.

In the same way, Figs. 4(a)–4(c) show the evolution of the lower branches of the invariant manifolds, that is, $W_s(E_1)$ and $W_u(E_2)$. For $\alpha < \alpha_c^-$, [Fig. 4(a), $\alpha = 0.04$] the stable and unstable manifolds suffer heteroclinic intersections with each other, whereas for $\alpha > \alpha_c^-$, [Fig. 4(c), $\alpha = 0.055$] both manifolds do not intersect. In the critical case $\alpha_c^- \approx 0.04913$ [Fig. 4(b)], the manifolds are just tangents of each other.

These evolutions of the invariant manifolds, based on numerical calculations for specific parameter values, confirm with very good agreement the analytical results (18) and (19) obtained with the Melnikov method. Therefore, both numerical and

analytical studies show, in good accordance, that the perturbations of the system generate heteroclinic intersections between the invariant manifolds, and hence, the arise of chaotic behavior in the pitch motion of the spacecraft.

4. Poincaré Surfaces of Sections

In order to globally visualize the effect of the perturbations on the pitch motion dynamics of the spacecraft, we have generated Poincaré surfaces of section of the three-dimensional (θ, ω, ν) extended phase space of the perturbed system. To this end, we have made use of the appropriate algorithm [Weisstein, 2008] implemented with the symbolic manipulator MATHEMATICA [Wolfram, 2003].

As we are interested in the existence of periodic pitch motions with the same period as the orbital motion of the spacecraft, the Poincaré surfaces of sections consist of sections (θ, ω) in the independent angular variable ν , the true anomaly of the orbit, with $\nu = \text{cte.}(\text{mod } 2\pi)$.

Figure 5 shows three surfaces of sections corresponding to a spacecraft with $K = 1, \Omega = \pi/2$, and different values of the perturbation parameters e and β , and in the absence of aerodynamic drag ($\alpha = 0$). Figure 5(a) stands for a nonmagnetic spacecraft ($\beta = 0$) in a slightly elliptic orbit ($e = 0.02$), Fig. 5(b) corresponds to a magnetic spacecraft ($\beta = 0.02$) in circular orbit ($e = 0$), and finally, Fig. 5(b) is the one corresponding to a magnetic spacecraft in elliptic orbit ($e = \beta = 0.02$).

These figures confirm that both perturbations, those coming from the elliptic orbit and the magnetic interaction, separately or together, cause the appearance of chaotic attitude motions. These irregular motions appear as a cloud of disordered points located at a stochastic layer around the unperturbed separatrix, where the transversal heteroclinic intersections between the invariant manifolds $W_u(E_i)$ and $W_s(E_i)$ take place. Out of this stochastic layer, regular pitch motions, such as oscillations and tumbling rotations, persist in spite of the perturbations. These regular motions appear as invariant curves for quasiperiodic motions, or as fixed points (centers or saddles) for periodic motions with the same period 2π as the orbital motion.

Figure 5(a) ($e = 0.02, \beta = 0$), shows the presence of four center points labeled with letters A–D, as well as two saddle points labeled with letters E–F. As it is well known, the center

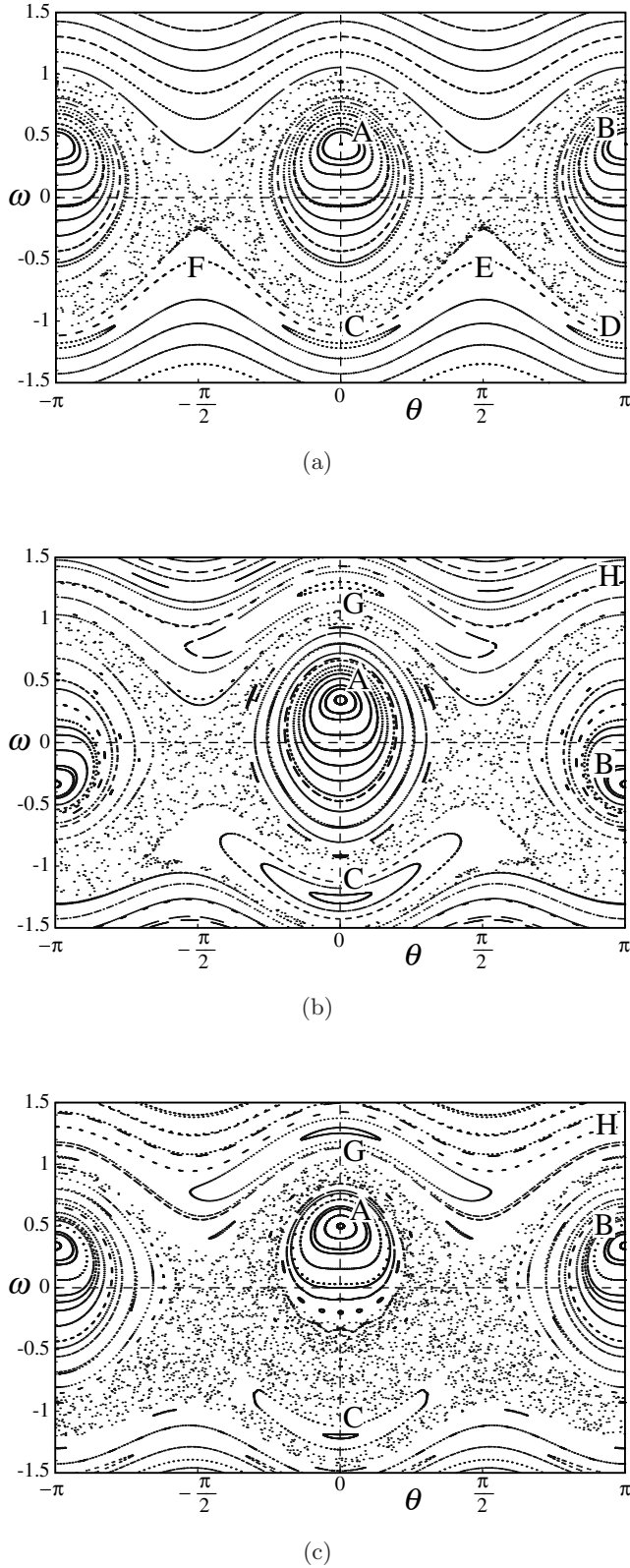


Fig. 5. $\nu = 2\pi$ Poincaré surfaces of section of a spacecraft with $K = 1$ and $\Omega = \pi/2$ in the absence of viscous drag ($\alpha = 0$). (a) Perturbed only by the elliptic orbit ($e = 0.02, \beta = 0$). (b) Perturbed only by the magnetic interaction ($e = 0, \beta = 0.02$). (c) Affected by both perturbations ($e = 0.02, \beta = 0.02$).

points correspond to stable periodic motions, and the saddle points correspond to unstable periodic ones. In Fig. 5(b) ($e = 0, \beta = 0.02$), it seems that the center point D has disappeared, whereas a new center point and a new saddle point have arisen in the surface of section. We have labeled these new fixed points as G and H respectively. The previous saddle points E and F have also disappeared or perhaps, persist inside the stochastic layer of chaotic motions. When both perturbations go into action simultaneously, Fig. 5(c) ($e = \beta = 0.02$) shows that, apart from the stochastic region increase, the surface of section exhibits the same basic features as that corresponding to the magnetic perturbation alone.

To make visible the effect of the viscous drag perturbation on the dynamical attitude behavior of the spacecraft, in Fig. 6 we have plotted three Poincaré surfaces of section of the same spacecraft ($K = 1, \Omega = \pi/2$) for increasing values of the drag parameter α , keeping the other two perturbations constant ($e = \beta = 0.02$). As it is well known, the main contribution of the viscous drag in a dynamical system is the opposing motion. Therefore, in this case it could be expected that, it does not matter what the initial conditions, the oscillations and rotations of the pitch motion would decay, and the final state of the perturbed spacecraft would be a constant pitch angle $\theta = 0$ or $\theta = \pi$. That is to say, the two fixed equilibria located at $(\theta, \omega) = (0, 0)$ or $(\pi, 0)$ in the unperturbed phase space (Fig. 2) would be the only two sinks for the system affected by the aerodynamic drag. In this sense, the action of the drag on the spacecraft destroys the regular structures of the previous surfaces shown in Fig. 5. But nevertheless, Fig. 6 shows us that, under the viscous drag, the system does not exhibit the two aforementioned sinks, but another four sinks situated at different locations. We have labeled these sinks by A, B, C and G, the same letters as the center fixed points located near these sinks in the previous surfaces of section of the spacecraft without drag. In fact, as we will see in the next section, these four sinks correspond to four stable periodic pitch motions with the same period as the orbital motion, $\nu = 2\pi$. That is, the system with aerodynamic drag exhibits four limit cycles.

At first, the existence of these periodic pitch motions under a viscous drag may seem paradoxical, as the drag produces a dissipation of the kinetic energy related to the attitude motion. However, the other two perturbations on the spacecraft, the

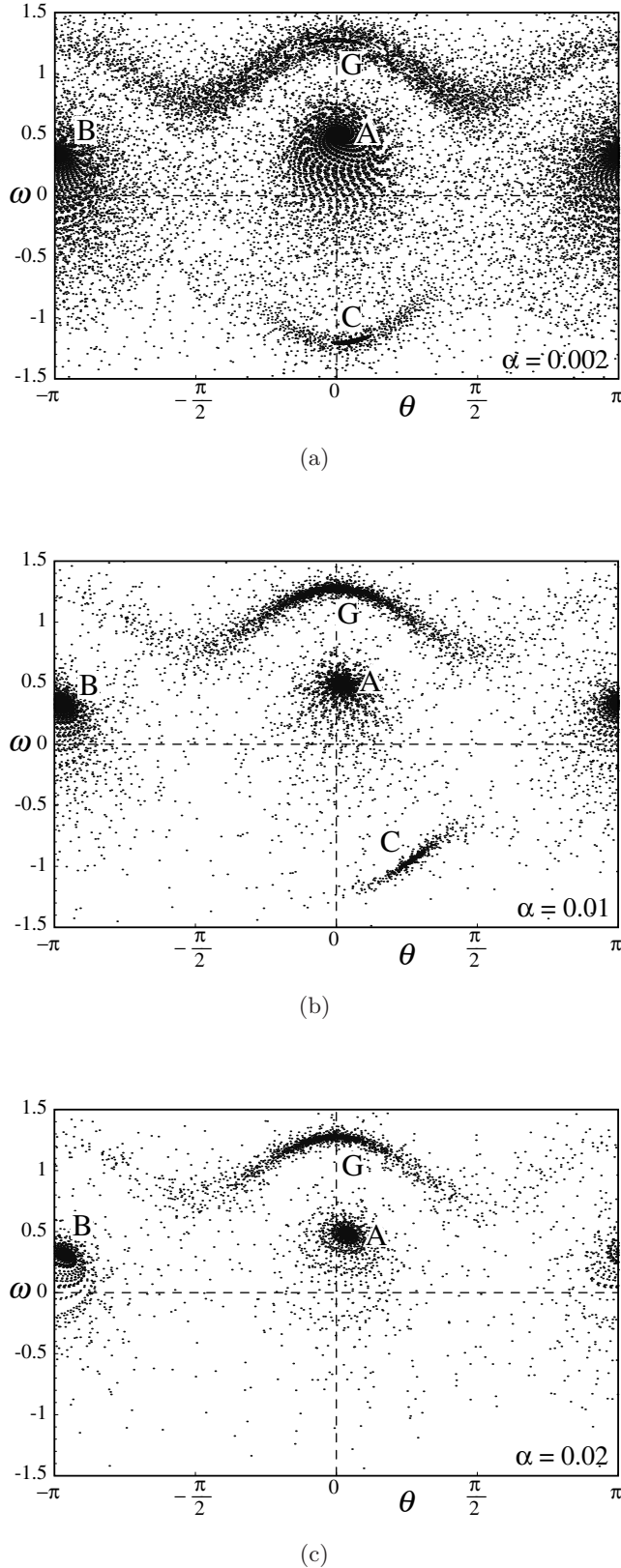


Fig. 6. $\nu = 2\pi$ Poincaré surfaces of section of a spacecraft with $K = 1$ and $\Omega = \pi/2$ for increasing values of the drag parameter, (a) $\alpha = 0.002$, (b) $\alpha = 0.01$ and (c) $\alpha = 0.02$, keeping constant the other two perturbations: the elliptic orbit ($e = 0.02$) and the magnetic interaction ($\beta = 0.02$).

magnetic interaction and the elliptic orbit, may work as sources of energy for the system, depending on the initial conditions, the parameter values and the frequency of the pitch motion. These sources of energy are due, on one hand to the action of the geomagnetic field, and on the other hand to the coupling between the attitude motion of the spacecraft and its orbital motion through the gravity gradient torque. In this way, those periodic limit cycles A–C and G, that persist under the action of the drag, are determined by a balance between the energy added by the elliptic and magnetic perturbations, and the energy dissipated by the drag. This phenomenon of the existence of periodic motions under viscous drag also appears in other systems like the well known problem of the driven damped simple pendulum [Giordano, 1997], or in the pitch motion of a nonrigid spacecraft with drag in circular orbit [Iñarraea & Lanchares, 2006].

The three surfaces of section have been generated with the same initial conditions and with the same integration time of the equations of motion (10). As it is expected, for small values of the drag parameter α , most of the surface of section is covered by a dense cloud of disordered points [Fig. 6(a)], whereas for bigger drag, the points are much more concentrated around the sinks, in such a way the surface of section shows a clearer aspect [Fig. 6(c)]. For small values of α , most of the pitch motions with initial conditions outside or around the unperturbed separatrix exhibit a long transient chaotic regime previous to reaching the corresponding limit cycle. See Fig. 7(a), which shows the time evolution of the pitch angle θ of a slightly dragged attitude motion with $\alpha = 0.003$. On the other hand, the bigger the drag the shorter that transient chaotic regime. In this way, for big drags, the pitch motion becomes a regular one decaying more quickly to one of the limit cycles. See Fig. 7(b), where the time evolution of the pitch angle θ of a motion is plotted with same initial conditions as of Fig. 7(a) for a bigger drag $\alpha = 0.035$.

This feature is not only a consequence of the well known relation between the drag strength and the decay time in a damped system, but also a consequence of the previously studied dependence of the chaotic behavior of this system on the drag parameter α . Indeed, for small drags, $\alpha < \alpha_c^\pm$, when the trajectories of the pitch motion reach the region surrounding the unperturbed separatrix, where the heteroclinic intersections between the invariant manifolds $W_u(E_i)$ and $W_s(E_i)$ take place,

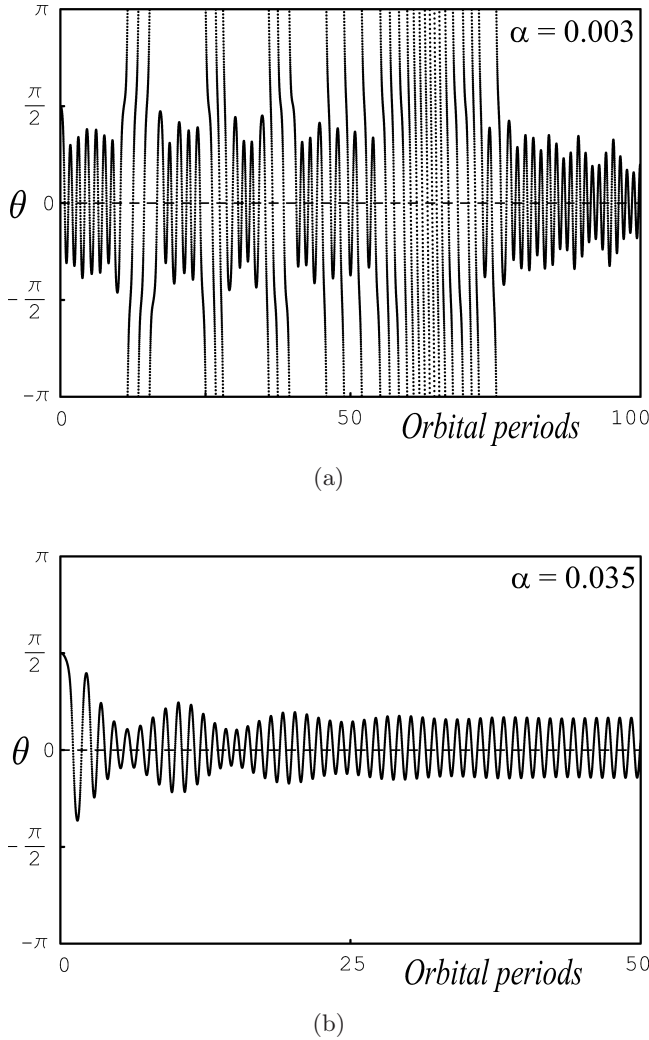


Fig. 7. Time evolution of the pitch angle θ of the spacecraft attitude motion for $K = 1, \Omega = \pi/2, e = \beta = 0.02$ and for the same initial condition $(\theta_o, \omega_o) = (\pi/2, 0)$. (a) Under small drag $\alpha = 0.003$. (b) Under bigger drag $\alpha = 0.035$.

the attitude motions suffer a long transient chaotic regime until they leave that chaotic region and decay to one of the limit cycles [see Fig. 7(a)]. Nevertheless, for bigger drags, $\alpha > \alpha_c^+$, the trajectories of the pitch motions are regular decays to the limit cycles without any transient chaotic phase [see Fig. 7(b)].

Therefore, for fixed parameters K, Ω, e and β , the dynamical behavior of the spacecraft near the unperturbed separatrix suffers a transition from a chaotic regime to a regular one, when the viscous drag parameter α is increased. This transition from chaos to order is in a qualitative good agreement not only with the analytical criterions (18), (19) obtained from the Melnikov method, but also with the numerical study of the heteroclinic intersections between the invariant

manifolds $W_u(E_i)$ and $W_s(E_i)$ explained in a previous section.

It is also important to note that, as Fig. 6(c) points out, for a value of the drag parameter $\alpha = 0.02$, the limit cycle labeled with C has disappeared, and only the other three limit cycles persist. This situation continues even for stronger drags. Thus, there must be a critical value of α for which the C limit cycle is destroyed in some kind of bifurcation.

5. Periodic Pitch Motions and Bifurcations

In the previous section we have seen by the $\nu = 2\pi$ -Poincaré surfaces of section, that despite not only the chaotic behavior generated by the elliptic orbit and magnetic perturbations, but also the kinetic energy dissipation produced by the viscous drag, some periodic pitch motions persist with the same period as the orbital motion, that is, $\nu = 2\pi$.

In the surface of section of Fig. 5(a) ($e = 0.02, \beta = \alpha = 0$), we have detected six different $\nu = 2\pi$ -periodic attitude motions. The two center points labeled with letters A and B correspond to periodic oscillations. In motion A, the positive z axis of the spacecraft is oscillating pointing towards Earth, whereas in motion B, the positive z axis is oscillating pointing away from Earth. The two other center points labeled C and D, as well as the saddle points E and F, correspond to periodic tumbling rotations, all of them with the same direction. As it is well known, as the motions C and D appear as center points, they are stable rotations, whereas motions E and F are unstable rotations, because they appear as saddle points.

On the other hand, in Figs. 5(b) ($\beta = 0.02, e = \alpha = 0$) and 5(c) ($e = \beta = 0.02, \alpha = 0$), two other $\nu = 2\pi$ -periodic pitch motions have been found. The center point labeled G, corresponds to a stable periodic tumbling rotation, while the saddle point labeled H, corresponds to an unstable periodic tumbling rotation. In these two attitude motions the spacecraft rotates with a direction opposite to the aforementioned C–F rotations. However, it seems that in these two surfaces of section, neither the stable periodic rotation (center point) D, nor the unstable periodic rotations (saddle points) E and F appear.

When the aerodynamic drag goes into action ($\alpha \neq 0$), see Fig. 6, the unstable periodic rotation H does not appear, whereas for small drags, the two stable periodic oscillations A and B, and also the

two stable periodic rotations C and G, all continue as limit cycles in the damped system. Nevertheless, as we have already mentioned above, when the drag increases, only the periodic motions A, B and G persist, while the limit cycle C disappears, see Fig. 6(c).

It is worth noting that the persistence of these periodic motions may be useful for practical purposes in the control of spacecraft orientation. As it is well known, many satellites are intended to maintain a particular fixed orientation with respect to Earth. In this sense, we have found periodic oscillations (motions A and B) with small amplitudes around the local vertical. In these periodic motions the spacecraft orientation does not suffer large variations. In such a way, the z axis of the vehicle keeps pointing around the local vertical all over the orbit. Therefore, these oscillations could be considered as target attitude motions for a suitable active control method that could be applied in order to eliminate the initially chaotic motion of the spacecraft. With respect to the periodic tumbling rotations, although the vehicle orientation is not fixed in these attitude motions, at least they are regular ones and thus, it is possible to know the future orientation of the spacecraft along the orbit.

In order to understand and get a more precise view of the evolution of these periodic pitch motions with the values of the perturbation parameters, we have made a numerical bifurcation analysis of these periodic motions by means of the freely distributed software package AUTO2007 [Doedel *et al.*, 2002; Kamthan, 2008]. This software carries out the continuation of solutions of systems of differential equations with respect to the parameters of the problem. We have performed the bifurcation analysis in the interval $[0, 0.04]$ for the perturbation parameters e, β and α , and we have found the following results.

The periodic oscillations A and B, as well as the periodic rotation G, are always stable attitude motions for all values of e, β and α in the interval considered. When the spacecraft is under the action of the drag ($\alpha \neq 0$), all these stable motions transform into limit cycles.

The periodic rotation H is a stable motion as long as the system is only affected by the elliptic orbit perturbation ($e \neq 0, \beta = \alpha = 0$). Nevertheless, the magnetic perturbation ($\beta \neq 0$) convert this rotation into an unstable motion, see Fig. 5(b). The drag perturbation ($\alpha \neq 0$) does not change the unstable nature of this pitch motion.

With regard to the periodic rotations E and F, both always exist as unstable motions in the interval

studied for ($e \neq 0, \beta = \alpha = 0$). However, when the magnetic interaction goes into action, ($\beta \neq 0$), both motions suffer a subcritical symmetry breaking bifurcation [Nayfeh & Balachandran, 1995], when they meet together with the periodic rotation D. In the bifurcation point, the asymmetric periodic motions E and F disappear, and the symmetric and previously stable rotation D persists being transformed into unstable one, which is absorbed by the stochastic layer of chaotic motions, see Fig. 5(b). The diagram of this bifurcation is shown in Fig. 8. As a measure of the periodic motions, we have used the L2-norm of the corresponding orbit in the extended phase space (θ, ω, ν) . In this figure, the L2-norm of the periodic rotations E, F and D is plotted versus the parameter β , for $e = 0.02$ and $\alpha = 0$. Solid and dashed lines denote the branches of stable and unstable periodic pitch motions respectively. The bifurcation point takes place for $\beta \approx 2.59 \cdot 10^{-3}$.

In relation to the periodic rotation C, it is always a stable motion for any value of e and β when there is no drag perturbation ($\alpha = 0$). When the drag is added to the system dynamics, this rotation becomes a stable limit cycle. In contrast to rotations G and H, which persist as the drag parameter is increased, this rotation C finally suffers a cyclic-fold bifurcation [Nayfeh & Balachandran, 1995], when it coalesces with the unstable periodic rotation D. In this bifurcation, both motions C and D obliterate each other, and therefore one of the sinks of the damped system disappears, as is observed

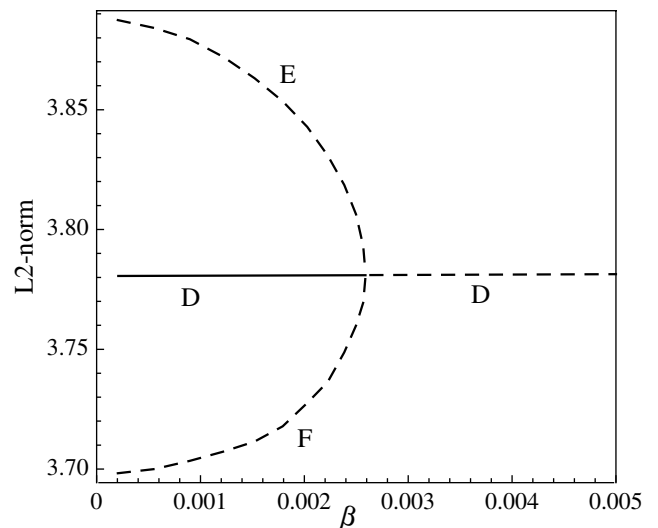


Fig. 8. Diagram of the symmetry breaking bifurcation involving the periodic rotations D, E and F as a function of β , for $K = 1, \Omega = \pi/2, e = 0.02$ and $\alpha = 0$. Solid and dashed lines stand for stable and unstable motions respectively.

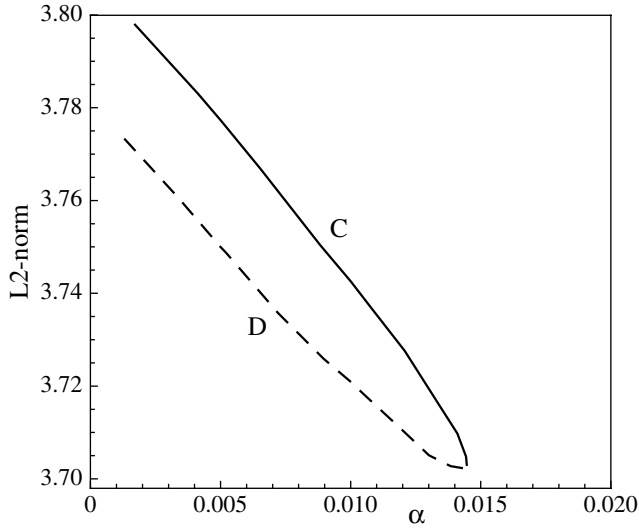


Fig. 9. Diagram of the cyclic-fold bifurcation involving the periodic rotations C and D as a function of α , for $K = 1$, $\Omega = \pi/2$ and $e = \beta = 0.02$. Solid and dashed lines stand for stable and unstable motions respectively.

in Fig. 6(c). In Fig. 9, we show the corresponding diagram of this bifurcation, where the L2-norm of the periodic motions C and D is, in this case, plotted versus the drag parameter α for $e = \beta = 0.02$. The bifurcation point takes place for $\alpha \approx 1.44 \cdot 10^{-2}$.

All these results provided by the bifurcation analysis are in very good agreement with the evolution of the Poincaré surfaces of section shown in Figs. 5 and 6. In the same way, these results explain the disappearances of periodic motions (center points, saddle points and sinks) detected in the evolution of those surfaces of section.

6. Conclusions

The pitch attitude motion dynamics of an asymmetric magnetic spacecraft in an almost circular polar orbit under the action of a gravity gradient torque has been investigated. The system is also subject to the influence of three different perturbations: (i) the small eccentricity of the elliptic orbit, (ii) a small magnetic torque due to the interaction between the Earth's magnetic field and the magnetic moment of the spacecraft, and (iii) a small aerodynamic viscous drag generated by the action of the Earth's atmosphere.

By means of the Melnikov method, we have analytically proved that the perturbations generate transient heteroclinic chaotic behaviors in the pitch motion of the spacecraft. Additionally, this method has provided us with an analytical criterion for the existence of chaotic attitude motions in terms of

the system parameters. We have found that system dynamics suffers a transition from chaos to order when the viscous drag is increased, as the transient chaotic motions disappear for big drags.

Moreover, we have also numerically studied the pitch dynamics of the spacecraft making use of several tools based on computer simulation of the attitude motions, including time history, Poincaré surfaces of section and bifurcation analysis of periodic motions. This numerical research has confirmed with very good agreement the analytical results provided by the Melnikov method. In spite of the chaos generated by both, the small eccentricity of the spacecraft orbit and the magnetic perturbation, we have found, by means of Poincaré surfaces of section that some periodic pitch motions persist in the perturbed system with the same period as the orbital motion of the spacecraft. In addition, when the viscous drag goes into action, despite its damping effect, these persistent periodic motions still continue as limit cycles. This persistency may be understood as a consequence of a balance between the addition and dissipation of energy produced by the three perturbations. Some of these persistent periodic motions could be considered as target motions for a suitable control method to remove chaotic behaviors from the attitude dynamics of the spacecraft.

Finally, the numerical continuation of these periodic pitch motions, as functions of the perturbations parameters, has revealed to us that some of the periodic rotations suffer two different kinds of bifurcations. On the one hand, when the magnetic interaction is increased without drag, one of the stable periodic rotations becomes unstable through a symmetry breaking bifurcation. On the other hand, when the aerodynamic drag becomes greater, one of the limit cycles disappears in a cyclic-fold bifurcation.

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