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Stability of the permanent rotations of an asymmetric gyrostat in a uniform Newtonian field

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ABSTRACT

The stability of the permanent rotations of a heavy gyrostat is analyzed by means of the Energy-Casimir method. Sufficient and necessary conditions are established for some of the permanent rotations. The geometry of the gyrostat and the value of the gyrostatic moment are relevant in order to get stable permanent rotations. Moreover, the necessary conditions are also sufficient, for some configurations of the gyrostat.

1. Introduction

A gyrostat $\mathcal{G}$ is a mechanical system made of a rigid body $\mathcal{P}$ called the *platform* and other bodies $\mathcal{R}$ called the *rotors*, connected to the platform in such a way that the motion of the rotors does not modify the distribution of mass of the gyrostat $\mathcal{G}$. Due to this double spinning, the platform on the one hand and the rotors on the other, the gyrostat is also known with the name of *dual-spin* body.

In Astrodynamics, gyrostats play an essential role, since they are used for controlling the attitude dynamics of a spacecraft and for stabilizing their rotations. See, for instance, Cochran [8], Hall [15–17], Elipe and coworkers [10–13,20,26], Vera [37], Aslanov [4,5] and also Hughes [19] for further references.

Besides its practical interest, the rotational motion of a gyrostat is very interesting from a mathematical point of view. Indeed, principal moments of inertia and gyrostatic momenta may be considered as parameters in the Euler equations of motion and there is a wide variety of possible equilibria, trajectories and bifurcations even in the simplest case of a gyrostat in free motion, that is to say, under no external forces. The authors have been studying this case for several years, and one of the main results obtained is the proof that when the gyrostat motion is formulated in terms of the angular momentum components, this problem is equivalent to a parametric quadratic Hamiltonian [12], and for those class of quadratic Hamiltonians, the classification of equilibria and bifurcations in different regions of the parametric space are well studied [24,25,27].

A further step in the complexity of the problem, and in the approximation to a real one, is to consider the motion of a gyrostat under the attraction of a Newtonian field. For this problem, some authors have found approximated analytical solutions for particular cases [7,33] and other authors have studied the equilibria and their stability when the gyrostat is in circular orbit [34,35] or in the gravity field of a number of different rigid bodies [21,36]. In this paper we focus on the

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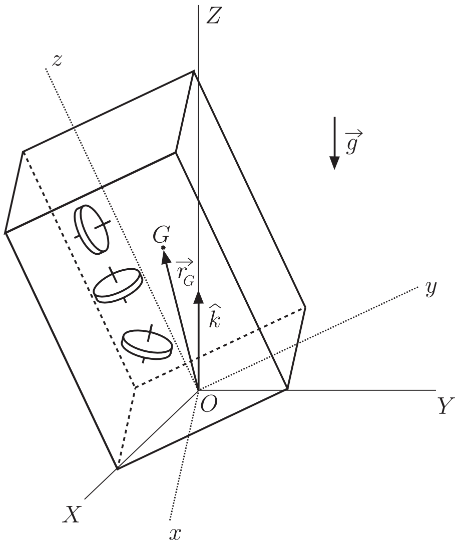

Fig. 1. Asymmetric gyrostat and reference frames.

stability of permanent rotations of a heavy gyrostat with a fixed point, that is to say when the gyrostat is under a uniform gravity field. For this case, both necessary and sufficient conditions of stability have been obtained by means of different methods. In this sense, Rumiantsev [32] and Anchev [1,2] gave sufficient conditions of stability for permanent rotations by constructing appropriate Lyapunov functions. In the particular case the center of mass lies on the first principal axis and the gyrostatic moment is directed along the same axis, Kovalev [22] derived sufficient conditions, that matched those of Rumiantsev, but also applied KAM theory to study the stability when the associated quadratic form of the perturbed Hamiltonian is not sign definite, but the necessary conditions are satisfied.

Previous results can also be derived and improved using the Energy-Casimir method [3,18,30,31] provided the system can be regarded as a Lie-Poisson one. Indeed, this method has been successfully used to study rigid body dynamics [6] and recently applied to study the stability of permanent rotations of a heavy gyrostat [14]. In this paper, the authors obtain, for a special class of permanent rotations, the same results previously derived by classical methods by Kovalev [23]. For the other permanent rotations, they provide sufficient stability conditions. However, these conditions are weak, as they do not depend on the gyrostatic moment. In this paper we obtain new sufficient conditions for all the permanent rotations in the case studied in [14] and also prove that, in some configurations of the moments of inertia, they are also necessary conditions. Besides, on a certain parametric plane, we determine regions for the existence of the equilibria, as well as bifurcation lines, since the stability depends on those parameters.

2. Equations of motion

Let us consider a gyrostat, consisting of a rigid asymmetric platform and three axisymmetric rotors. Each one of these rotors is aligned along one of the principal axis of the platform. The gyrostat is subject to a uniform and constant gravity field. We assume that the gyrostat has a fixed point O. Centered on this point, we consider two orthonormal reference frames (see Fig. 1):

- The inertial fixed reference frame $\mathcal{F}\{O, X, Y, Z\}$. The direction of the Z axis is opposite to the action line of the gravity field.
- The body frame $\mathcal{B}\{O, x, y, z\}$ fixed in the platform. The directions of these axes coincide with the principal axes of the gyrostat.

In the body reference frame $\mathcal{B}$, the tensor of inertia $\mathbb{I}$ of the gyrostat is diagonal, that is, $\mathbb{I} = \mathrm{diag}(I_1, I_2, I_3)$. As we assume an asymmetric gyrostat, $I_1 \neq I_2 \neq I_3$. On the other hand, the total angular momentum of the gyrostat can be written as

$$\boldsymbol{H} = \boldsymbol{\pi} + \boldsymbol{l},$$

where $\boldsymbol{\pi}$ is the angular momentum of the whole gyrostat with rotors at relative rest (in other words, considering the gyrostat as a rigid body), and $\boldsymbol{l}$ is the gyrostatic momentum, that is, the relative angular momentum of the rotors with respect to the platform. Due to the gravity field, the gyrostat is under the action of a gravitational torque $\boldsymbol{N}$ about the fixed point O, given by

$$\boldsymbol{N} = \boldsymbol{r}_G \times m\boldsymbol{g} = -mg\, \boldsymbol{r}_G \times \hat{\boldsymbol{k}},$$

where $\boldsymbol{r}_G$ is the position vector of the center of mass G of the gyrostat, $\hat{\boldsymbol{k}}$ is a unitary vector in the direction of the Z axis, and m is the mass of the gyrostat. Under all these assumptions, and by means of the angular momentum theorem about the fixed point O,

$$\frac{d\boldsymbol{H}}{dt} = \boldsymbol{N},$$

the Euler equations of motion expressed in the body reference frame $\mathcal{B}$ take the form [28]

$$\begin{aligned}
\frac{d\pi_1}{dt} &= \left(\frac{I_2 - I_3}{I_2 I_3}\right)\pi_2\pi_3 + \frac{l_2\pi_3}{I_3} - \frac{l_3\pi_2}{I_2} + mg(z_0k_2 - y_0k_3),\\
\frac{d\pi_2}{dt} &= \left(\frac{I_3 - I_1}{I_1 I_3}\right)\pi_1\pi_3 + \frac{l_3\pi_1}{I_1} - \frac{l_1\pi_3}{I_3} + mg(x_0k_3 - z_0k_1),\\
\frac{d\pi_3}{dt} &= \left(\frac{I_1 - I_2}{I_1 I_2}\right)\pi_1\pi_2 + \frac{l_1\pi_2}{I_2} - \frac{l_2\pi_1}{I_1} + mg(y_0k_1 - x_0k_2),
\end{aligned} \tag{1}$$

where (π_1, π_2, π_3), (l_1, l_2, l_3) and (k_1, k_2, k_3) are the components of the angular momenta vectors $\boldsymbol{\pi}$, $\boldsymbol{l}$ and the unitary vector $\hat{\boldsymbol{k}}$ respectively, expressed in the body reference frame $\mathcal{B}$. In addition, (x_0, y_0, z_0) are the coordinates of the mass center G in the same frame.

On the other hand, the components (k_1, k_2, k_3) also vary in time as they are expressed in the body reference frame $\mathcal{B}$. The time evolution of these components is given by the well known Poisson equations [28]

$$\begin{aligned}
\frac{dk_1}{dt} &= \frac{k_2\pi_3}{I_3} - \frac{k_3\pi_2}{I_2},\\
\frac{dk_2}{dt} &= \frac{k_3\pi_1}{I_1} - \frac{k_1\pi_3}{I_3},\\
\frac{dk_3}{dt} &= \frac{k_1\pi_2}{I_2} - \frac{k_2\pi_1}{I_1}.
\end{aligned} \tag{2}$$

Therefore, equations (1) and (2) are the complete set of equations that rule the rotational dynamics of the asymmetric gyrostat under a uniform and constant gravity field. Although the variables considered are not canonical, the system can be described by means of a Hamiltonian function in the framework of Lie-Poisson systems, in the same way as the classical problem of the motion of a rigid body [6,30]. In this case, the associated Hamiltonian function takes the form (see [14])

$$\mathcal{H} = \frac{1}{2}\left(\frac{\pi_1^2}{I_1} + \frac{\pi_2^2}{I_2} + \frac{\pi_3^2}{I_3}\right) + mg(x_0k_1 + y_0k_2 + z_0k_3), \tag{3}$$

and the corresponding Poisson bracket is given by

$$\{F, G\}(\boldsymbol{\pi}, \hat{\boldsymbol{k}}) = -(\boldsymbol{\pi} + \boldsymbol{l})\cdot(\nabla_\pi F \times \nabla_\pi G) - \hat{\boldsymbol{k}}\cdot(\nabla_\pi F \times \nabla_k G + \nabla_k F \times \nabla_\pi G). \tag{4}$$

Now, it is easy to check that the Eqs. (1–2) of the gyrostat rotational motion can be expressed as

$$\dot{\pi}_i = \{\pi_i, \mathcal{H}\}, \qquad \dot{k}_i = \{k_i, \mathcal{H}\}, \qquad i = 1, 2, 3.$$

Thus, the system is regarded as a Lie-Poisson system and to study the stability of relative equilibria we can make use of the Energy-Casimir method. To this end, the existence of Casimir functions and conserved quantities plays an important role. For this problem, there are two Casimir functions whose Poisson bracket commutes with any smooth function defined in the phase space. These two Casimir functions are

$$C_1 \equiv k_1^2 + k_2^2 + k_3^2 = 1, \tag{5}$$

$$C_2 \equiv (\pi_1 + l_1)k_1 + (\pi_2 + l_2)k_2 + (\pi_3 + l_3)k_3 = p_\psi, \tag{6}$$

being p_ψ a constant.

In what follows we will focus on a special situation, when the center of mass is located on the z axis and only the gyrostatic moment along the z axis is acting. Thus, $x_0 = y_0 = 0,\ l_1 = l_2 = 0$ and the Eqs. (1–2) reduce to

$$\begin{aligned}
\frac{d\pi_1}{dt} &= \frac{I_2 - I_3}{I_2 I_3}\pi_2\pi_3 - \frac{l_3\pi_2}{I_2} + mgz_0k_2,\\
\frac{d\pi_2}{dt} &= \frac{I_3 - I_1}{I_1 I_3}\pi_1\pi_3 + \frac{l_3\pi_1}{I_1} - mgz_0k_1,\\
\frac{d\pi_3}{dt} &= \frac{I_1 - I_2}{I_1 I_2}\pi_1\pi_2,\\
\frac{dk_1}{dt} &= \frac{k_2\pi_3}{I_3} - \frac{k_3\pi_2}{I_2},
\end{aligned}$$

$$\frac{dk_2}{dt} = \frac{k_3\pi_1}{I_1} - \frac{k_1\pi_3}{I_3},$$
$$\frac{dk_3}{dt} = \frac{k_1\pi_2}{I_2} - \frac{k_2\pi_1}{I_1}. \tag{7}$$

3. Equilibrium solutions

Permanent rotations [9,11,28] are of great interest in different fields of application and they are obtained as equilibrium solutions of the system (7). These solutions have been obtained previously by other authors [14]. However, for the sake of completeness, we give the following result.

Theorem 1. *If $x_0 = y_0 = 0,\ l_3 \neq 0$ and $l_1 = l_2 = 0$ there are three families of equilibrium points.*

$$E_1 \equiv (0, 0, I_3\omega, 0, 0, \pm 1), \quad \omega \in \mathbb{R}.$$
$$E_2 \equiv (0, I_2\omega \sin\varphi, I_3\omega\cos\varphi, 0, \sin\varphi, \cos\varphi),$$
$$\text{with}\ \varphi \in (0, 2\pi), \quad \omega \in \mathbb{R}, \quad \text{and} \quad \omega^2(I_3 - I_2)\cos\varphi + \omega l_3 - gmz_0 = 0.$$
$$E_3 \equiv (I_1\omega\sin\varphi, 0, I_3\omega\cos\varphi, \sin\varphi, 0, \cos\varphi),$$
$$\text{with}\ \varphi \in (0, 2\pi), \quad \omega \in \mathbb{R}, \quad \text{and} \quad \omega^2(I_3 - I_1)\cos\varphi + \omega l_3 - gmz_0 = 0.$$

Proof. Equilibria are obtained setting to zero the equations of the motion (7). Thus, it follows from the third equation of the motion that the product $\pi_1\pi_2$ must be zero.

In the first place, we consider that both π_1 and π_2 are zero. Thus, the nontrivial equations of system (7) turn to be

$$\frac{d\pi_1}{dt} = mgz_0k_2, \qquad \frac{d\pi_2}{dt} = -mgz_0k_1, \qquad \frac{dk_1}{dt} = \frac{k_2\pi_3}{I_3}, \qquad \frac{dk_2}{dt} = -\frac{k_1\pi_3}{I_3}.$$

These equations vanish if $k_1 = k_2 = 0$ and π_3 is any real number. By virtue of (5), we obtain two one-parameter families of equilibrium solutions, we name E_1,

$$E_1 \equiv (0, 0, I_3\omega, 0, 0, \pm 1),$$

with $\omega \in \mathbb{R}$.

Now, be $\pi_1 = 0$ and $\pi_2 \neq 0$. Then, the second and the last two equations of the motion are simultaneously equal to zero if $k_1 = 0$. Taking into account (5), we introduce an angle $\varphi \in (0, 2\pi)$ in such a way that

$$k_2 = \sin\varphi, \qquad k_3 = \cos\varphi.$$

Now, the first and the fourth equations of the motion result to be

$$\frac{d\pi_1}{dt} = \frac{I_2 - I_3}{I_2I_3}\pi_2\pi_3 - \frac{l_3\pi_2}{I_2} + mgz_0\sin\varphi, \qquad \frac{dk_1}{dt} = \frac{\pi_3}{I_3}\sin\varphi - \frac{\pi_2}{I_2}\cos\varphi.$$

These two equations are equal to zero if

$$\pi_2 = I_2\omega\sin\varphi, \quad \pi_3 = I_3\omega\cos\varphi$$

and ω is a real number satisfying the equation

$$\omega^2(I_3 - I_2)\cos\varphi + \omega l_3 - gmz_0 = 0.$$

In this way, we obtain the biparametric family of equilibrium solutions named as E_2.

A similar analysis, for the case $\pi_1 \neq 0$ and $\pi_2 = 0$, yields the third family of equilibrium solutions dubbed E_3. □

4. Stability analysis

In this section we will focus on the stability analysis of the equilibrium solutions given in Theorem 1. The stability for the family E_1 has been considered in [37,38] for a symmetric gyrostat. The other two families have been also considered in [14], but the stability conditions given are weak, as they do not depend on the gyrostatic moment.

Taking into account that we are considering a Poisson system, to establish sufficient stability conditions we can use the classical energy-Casimir method [3,18] or a generalized version given in [31], which reads

Theorem 2 (Generalized energy-Casimir method)**.** *Let $(M, \{.,.\}, h)$ be a Poisson system, and $m \in M$ be an equilibrium of the Hamiltonian vector field X_h. If there is a set of conserved quantities $C_1, \ldots, C_n \in C^\infty(M)$ for which*

$$\mathbf{d}(h + C_1 + \cdots + C_n)(m) = 0,$$

and

$$\mathbf{d}^2(h + C_1 + \cdots + C_n)(m)|_{W\times W}$$

is definite for W defined by

$$W = \ker \mathbf{d}C_1(m) \cap \cdots \cap \ker \mathbf{d}C_n(m),$$

then m is stable. If $W = \{0\}$, *m is always stable.*

To begin with, we state the first stability result, concerning the equilibrium point E_1.

Theorem 3. *The equilibrium* E_1 *is stable if the following conditions are satisfied*

$$(I_3 - I_1)\omega^2 + l_3\omega > gmz_0, \quad (I_3 - I_2)\omega^2 + l_3\omega > gmz_0, \quad k_3 = 1.$$
$$(I_3 - I_1)\omega^2 + l_3\omega > -gmz_0, \quad (I_3 - I_2)\omega^2 + l_3\omega > -gmz_0, \quad k_3 = -1.$$

Proof. The proof can be found in [14], Theorem 7. □

This result can be complemented by the necessary conditions of stability, also given in [23]. In this sense, we have the following Theorem.

Theorem 4. *If the equilibrium* E_1 *is stable, then it must be satisfied*

$$[(I_3 - I_1)\omega^2 + l_3\omega - gmz_0][(I_3 - I_2)\omega^2 + l_3\omega - gmz_0] > 0, \quad k_3 = 1.$$
$$[(I_3 - I_1)\omega^2 + l_3\omega + gmz_0][(I_3 - I_2)\omega^2 + l_3\omega + gmz_0] > 0, \quad k_3 = -1.$$

Proof. The proof is straightforward, taking into account that a necessary condition to be stable is to be spectrally stable. Thus, none of the eigenvalues of the linearized system can have positive real part. However, the linearized system around E_1, for the case $k_3 = 1$, is defined by the Jacobian matrix

$$J_{E_1} = \begin{pmatrix} 0 & \dfrac{(I_2 - I_3)\omega - l_3}{I_2} & 0 & 0 & gmz_0 & 0 \\ \dfrac{(I_3 - I_1)\omega + l_3}{I_1} & 0 & 0 & -gmz_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\dfrac{1}{I_2} & 0 & 0 & \omega & 0 \\ \dfrac{1}{I_1} & 0 & 0 & -\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of J_{E_1} are the roots of the characteristic polynomial, which has the following form

$$\lambda^2(\lambda^4 + a\lambda^2 + b).$$

It is clear that there are two eigenvalues equal to 0 and, for the remaining four eigenvalues, it follows that if λ_0 is an eigenvalue, also $-\lambda_0$, $\bar{\lambda}_0$ and $-\bar{\lambda}_0$ are eigenvalues. Thus, a necessary condition for E_1 to be linear stable is that eigenvalues have zero real part, which means that the coefficient b must be greater than 0. However,

$$b = \frac{1}{I_1 I_2}[(I_3 - I_1)\omega^2 + l_3\omega - gmz_0][(I_3 - I_2)\omega^2 + l_3\omega - gmz_0],$$

and the first case is proved. The case $k_3 = -1$ can be proved in the same way. □

Note that, in the case the two expressions in brackets are positive, necessary and sufficient conditions are exactly the same.

Remark 1. It is worth to mention that in the axisymmetric case, $I_2 = I_1$, the two stability conditions, for $k_3 = 1$ (similarly for $k_3 = -1$), reduce to one

$$(I_3 - I_1)\omega^2 + l_3\omega > gmz_0. \tag{8}$$

However, this is a different stability condition of the classical one [18,37]

$$(I_3\omega + l_3)^2 \geq 4gmz_0 I_1. \tag{9}$$

This is due to the fact that, in the augmented Hamiltonian used to prove the stability in Theorem 3, the conservation of the third component of the angular momentum, appearing in the symmetric case, cannot be considered. Thus, the stability condition is more restrictive, in the sense that if (8) is satisfied, also (9) is satisfied, but not necessarily in the reverse way. Indeed,

$$\frac{(I_3\omega + l_3)^2}{4I_1} \geq (I_3 - I_1)\omega^2 + l_3\omega,$$

provided that

$$(I_3\omega + l_3)^2 - 4I_1((I_3 - I_1)\omega^2 + l_3\omega) = (I_3\omega + l_3 - 2I_1\omega)^2 \geq 0.$$

To emphasize this situation, we consider a gyrostat with the following values for the parameters

$$I_1 = I_2 = 1, \quad I_3 = 2, \quad gmz_0 = 2, \quad \omega = 1, \quad l_3 = 0.9.$$

It is easy to check that

$$(I_3 - I_1)\omega^2 + l_3\omega = 1.9 < gmz_0 = 2$$

and

$$(I_3\omega + l_3)^2 = 8.41 > 4mgz_0 I_1 = 8.$$

That is to say, stability condition (9) is satisfied, but not condition (8). Numerical integration of a trajectory starting close to the equilibrium position shows that it remains close to it, but performing a small precession motion.

Remark 2. It is also worth noting that in the limit case, $l_3 = 0$, we are left with stability conditions of a triaxial heavy top. Indeed, for $z_0 > 0$ we have an upright sleeping top and, from Theorem 3, stability holds if

$$\omega^2 > \frac{gmz_0}{I_3 - I_m},$$

being $I_m = \max\{I_1, I_2\}$ and I_3 the biggest moment of inertia. In the case $z_0 < 0$, that is a hanging sleeping top, stability takes place if I_3 is the biggest moment of inertia or

$$\omega^2 < \frac{gm|z_0|}{I_m - I_3},$$

and I_3 the smallest moment of inertia.These stability conditions match those established in [29].

Now, we state a result about the stability of the second and third families of permanent rotations, E_2 and E_3, which turns to be more general than those given in [14].

Theorem 5. *The equilibrium E_2 is stable if the following conditions are satisfied*

$$I_1 > I_3, \quad l_2^2 > (I_2 - I_1)(I_1\omega^2 - 4l_3\omega\cos\varphi + 3(I_1 - I_2)\omega^2\cos^2\varphi).$$

In particular, if $\cos\varphi = 0$, E_2 is stable if

$$I_1 > I_3, \quad l_3^4 > I_1(I_2 - I_1)m^2g^2z_0^2.$$

Analogously, the equilibrium E_3 is stable if the following conditions are satisfied

$$I_1 > I_2, \quad l_3^2 > (I_3 - I_1)(I_1\omega^2 - 4l_3\omega\cos\varphi + 3(I_1 - I_3)\omega^2\cos^2\varphi).$$

In particular, if $\cos\varphi = 0$, E_3 is stable if

$$I_1 > I_2, \quad l_3^4 > I_1(I_3 - I_1)m^2g^2z_0^2.$$

Proof. We will perform the proof for the equilibrium E_3, as the other case is exactly the same, interchanging the role played by the moments of inertia I_2 and I_3. Following Theorem 2, we introduce the augmented Hamiltonian

$$H = \frac{1}{2}\left(\frac{\pi_1^2}{I_1} + \frac{\pi_2^2}{I_2} + \frac{\pi_3^2}{I_3}\right) + mgz_0k_3 + \lambda(\pi_1k_1 + \pi_2k_2 + (l_3 + \pi_3)k_3) + \mu(k_1^2 + k_2^2 + k_3^2), \tag{10}$$

where to the Hamiltonian function (3) we have added a linear combination of the two Casimir functions (5) and (6). It is easy to check that equilibrium E_3 is a critical point of H if the parameters λ and μ are given by

$$\lambda = -\omega, \qquad \mu = \frac{I_1\omega^2}{2}.$$

Let us now determine the space

$$W = \ker \mathbf{d}C_1(E_3) \cap \ker \mathbf{d}C_2(E_3),$$

where C_1 and C_2 are the Casimir functions already given by (5) and (6) and introduced in the augmented Hamiltonian (10). On the one hand, we have

$$\mathbf{d}C_1(E_3) = 2\sin\varphi\, dk_1 + 2\cos\varphi\, dk_3$$

and, on the other hand,

$$\mathbf{d}C_2(E_3) = \sin\varphi\, d\pi_1 + \cos\varphi\, d\pi_3 + I_1\omega\sin\varphi\, dk_1 + (l_3 + I_3\omega\cos\varphi)\, dk_3.$$

Equating to zero the above expressions, we obtain the relations

$$d\pi_3 = -d\pi_1 \tan\varphi + \left(\frac{l_3}{\cos\varphi} + (I_3 - I_1)\omega\right)\tan\varphi\, dk_1, \quad dk_3 = -dk_1 \tan\varphi.$$

Thus, $W = \ker \mathbf{d}C_1(E_3) \cap \ker \mathbf{d}C_2(E_3)$ is generated by the vectors

$$\hat{\mathbf{e}}_1 \cos\varphi - \hat{\mathbf{e}}_3 \sin\varphi,\ \hat{\mathbf{e}}_2,\ \hat{\mathbf{e}}_3[l_3 + (I_3 - I_1)\omega\cos\varphi]\sin\varphi + \hat{\mathbf{e}}_4 \cos^2\varphi - \hat{\mathbf{e}}_6 \sin\varphi\cos\varphi,\ \hat{\mathbf{e}}_5.$$

Let $\boldsymbol{v}$ be a six dimensional vector in W, then

$$\begin{aligned}\boldsymbol{v} = (&x_1\cos\varphi, x_2,\\ &-(x_1 - l_3x_3 + (I_1 - I_3)\omega x_3\cos\varphi)\sin\varphi, x_3\cos^2\varphi, x_4, -x_3\cos\varphi\sin\varphi),\end{aligned}$$

where $x_i \in \mathbb{R},\ i = 1, \dots, 4$. Then, the Hessian matrix of the augmented Hamiltonian in the reduced space W is computed straightforwardly from the quadratic form in the variables x_i,

$$\boldsymbol{v}^{\mathrm{T}} \cdot \begin{bmatrix} \frac{\partial^2 H}{\partial \pi_1^2} & \frac{\partial^2 H}{\partial \pi_1 \pi_2} & \frac{\partial^2 H}{\partial \pi_1 \pi_3} & \frac{\partial^2 H}{\partial \pi_1 k_1} & \frac{\partial^2 H}{\partial \pi_1 k_2} & \frac{\partial^2 H}{\partial \pi_1 k_3} \\ \frac{\partial^2 H}{\partial \pi_1 \pi_2} & \frac{\partial^2 H}{\partial \pi_2^2} & \frac{\partial^2 H}{\partial \pi_2 \pi_3} & \frac{\partial^2 H}{\partial \pi_2 k_1} & \frac{\partial^2 H}{\partial \pi_2 k_2} & \frac{\partial^2 H}{\partial \pi_2 k_3} \\ \frac{\partial^2 H}{\partial \pi_1 \pi_3} & \frac{\partial^2 H}{\partial \pi_2 \pi_3} & \frac{\partial^2 H}{\partial \pi_3^2} & \frac{\partial^2 H}{\partial \pi_3 k_1} & \frac{\partial^2 H}{\partial \pi_3 k_2} & \frac{\partial^2 H}{\partial \pi_3 k_3} \\ \frac{\partial^2 H}{\partial \pi_1 k_1} & \frac{\partial^2 H}{\partial \pi_2 k_1} & \frac{\partial^2 H}{\partial \pi_3 k_1} & \frac{\partial^2 H}{\partial k_1^2} & \frac{\partial^2 H}{\partial k_1 k_2} & \frac{\partial^2 H}{\partial k_1 k_3} \\ \frac{\partial^2 H}{\partial \pi_1 k_2} & \frac{\partial^2 H}{\partial \pi_2 k_2} & \frac{\partial^2 H}{\partial \pi_3 k_2} & \frac{\partial^2 H}{\partial k_1 k_2} & \frac{\partial^2 H}{\partial k_2^2} & \frac{\partial^2 H}{\partial k_2 k_3} \\ \frac{\partial^2 H}{\partial \pi_1 k_3} & \frac{\partial^2 H}{\partial \pi_2 k_3} & \frac{\partial^2 H}{\partial \pi_3 k_3} & \frac{\partial^2 H}{\partial k_1 k_3} & \frac{\partial^2 H}{\partial k_2 k_3} & \frac{\partial^2 H}{\partial k_3^2} \end{bmatrix} \cdot \boldsymbol{v},$$

where the full Hessian matrix is evaluated at E_3. In this way, we arrive to

$$\mathrm{Hess}|_{W\times W} = \begin{bmatrix} \frac{\cos^2\varphi}{I_1} + \frac{\sin^2\varphi}{I_3} & 0 & H_{13} & 0 \\ 0 & \frac{1}{I_2} & 0 & -\omega \\ H_{13} & 0 & \frac{\varkappa}{I_3} & 0 \\ 0 & -\omega & 0 & I_1\omega^2 \end{bmatrix},$$

where

$$H_{13} = \frac{-I_3\omega\cos^3\varphi - l_3\sin^2\varphi + (I_1 - 2I_3)\omega\cos\varphi\sin^2\varphi}{I_3}$$

and

$$\varkappa = I_1I_3\omega^2\cos^4\varphi + l_3^2\sin^2\varphi - 2(I_1 - 2I_3)l_3\omega\cos\varphi\sin^2\varphi + (I_1^2 - 3I_1I_3 + 3I_3^2)\omega^2\cos^2\varphi\sin^2\varphi.$$

Now, we apply the Sylvester criterion to determine the definiteness of the matrix, computing the principal minors. They are given by

$$\begin{aligned} \Delta_1 &= \frac{\cos^2\varphi}{I_1} + \frac{\sin^2\varphi}{I_3}, \quad \Delta_2 = \frac{\Delta_1}{I_2}, \\ \Delta_3 &= \frac{l_3^2 - (I_3 - I_1)(I_1\omega^2 - 4l_3\omega\cos\varphi + 3(I_1 - I_3)\omega^2\cos^2\varphi)}{4I_1I_2I_3}\sin^2 2\varphi, \\ \Delta_4 &= (I_1 - I_2)\omega^2\Delta_3. \end{aligned}$$

It is clear that Δ_1 and Δ_2 are always positive, despite the value of the parameter $\varphi \in (0, 2\pi)$. The other two minors are positive if the following inequalities are satisfied

$$l_3^2 - (I_3 - I_1)(I_1\omega^2 - 4l_3\omega\cos\varphi + 3(I_1 - I_3)\omega^2\cos^2\varphi) > 0, \quad I_1 > I_2. \tag{11}$$

In the special case $\cos\varphi = 0$, we can obtain ω from the relation satisfied by ω and φ for the existence of the equilibrium E_3. That is,

$$\omega^2(I_3 - I_1)\cos\varphi + \omega l_3 - gmz_0 = 0.$$

As $\cos\varphi = 0$ it follows

$$\omega = \frac{gmz_0}{l_3}$$

and (11) reduces to

$$l_3^4 > I_1(I_3 - I_1)m^2g^2z_0^2, \qquad I_1 > I_2.$$

□

Remark 3. It is worth noticing that if $I_1 > I_3$ the first condition in (11) is always satisfied.

Indeed, taking into account the existence relation for the family E_3

$$\omega^2(I_3 - I_1)\cos\varphi + \omega l_3 - gmz_0 = 0 \tag{12}$$

we obtain that

$$l_3 = (I_1 - I_3)\omega + \frac{gmz_0}{\omega}.$$

Substituting this relation in (11) we arrive to the equivalent inequality

$$I_1(I_1 - I_3)\omega^4 - 2gmz_0 I_1 - I_3)\omega^2\cos\varphi + g^2m^2z_0^2 > 0.$$

However, the left hand side of the inequality is a biquadratic polynomial in ω, with roots

$$\omega^2 = gmz_0\frac{(I_1 - I_3)\cos\varphi \pm \sqrt{(I_3 - I_1)(I_1\sin^2\varphi + I_3\cos^2\varphi)}}{I_1(I_1 - I_3)}.$$

It is clear that there are no real roots if $I_1 > I_3$. Taking into account that the coefficient of the leading term is $I_1(I_1 - I_3) > 0$, the first inequality is satisfied. In this way, we obtain the weak stability conditions

$$I_1 > I_2, \quad I_1 > I_3.$$

Thus, if I_1 is the biggest moment of inertia, it does not matter the value of the gyrostatic moment l_3, the equilibrium position E_3 is always stable. This is precisely the conclusion in [14]. However, the first inequality in (11) is more general and we can obtain stability in different situations, when I_1 is not the biggest moment of inertia.

It is also remarkable that, in the case $I_1 > I_2$, the sufficient condition established in Theorem 5 is also a necessary condition. Indeed, we have the following result.

Theorem 6. *A necessary condition for the equilibrium E_3 to be stable is*

$$(I_1 - I_2)(l_3^2 - (I_3 - I_1)(I_1\omega^2 - 4l_3\omega\cos\varphi + 3(I_1 - I_3)\omega^2\cos^2\varphi)) > 0.$$

Proof. Spectral stability is necessary to have Lyapunov stability. In this way, as the system is Hamiltonian, eigenvalues of the linearized system come in quadruplets of the form $\pm a \pm bi$ and linear stability takes place if the real part of the eigenvalues are equal to zero.

The eigenvalues are the roots of the polynomial equation

$$\det(J_{E_3} - \lambda I_6) = 0, \tag{13}$$

where I_6 is the 6×6 identity matrix and J_{E_3} is the Jacobian matrix of the linearized system at E_3. This matrix results to be

$$J_{E_3} = \begin{bmatrix} 0 & J_{12} & 0 & 0 & mgz_0 & 0 \\ J_{21} & 0 & J_{23} & -mgz_0 & 0 & 0 \\ 0 & J_{32} & 0 & 0 & 0 & 0 \\ 0 & -\dfrac{\cos\varphi}{I_2} & 0 & 0 & \omega\cos\varphi & 0 \\ \dfrac{\cos\varphi}{I_1} & 0 & -\dfrac{\sin\varphi}{I_3} & -\omega\cos\varphi & 0 & \omega\sin\varphi \\ 0 & \dfrac{\sin\varphi}{I_2} & 0 & 0 & -\omega\sin\varphi & 0 \end{bmatrix},$$

where

$$J_{12} = \frac{-l_3 + (I_2 - I_3)\omega\cos\varphi}{I_2}, \quad J_{21} = \frac{l_3 - (I_1 - I_3)\omega\cos\varphi}{I_1},$$

$$J_{23} = -\frac{(I_1 - I_3)\omega\sin\varphi}{I_3}, \quad J_{32} = \frac{(I_1 - I_2)\omega\sin\varphi}{I_2}.$$

The polynomial equation (13) turns to be of the form

$$\lambda^2(\lambda^4 + b\lambda^2 + c) = 0, \tag{14}$$

with b and c real numbers. It is clear that if all the roots of equation (14) have zero real part, then $c > 0$. However,

$$c = \frac{\omega^2 \sin^2\varphi}{I_1 I_2 I_3}(I_1 - I_2)[l_3^2 - (I_3 - I_1)(I_1\omega^2 - 4l_3\omega\cos\varphi + 3(I_1 - I_3)\omega^2\cos^2\varphi)],$$

and the result follows immediately, by taking into account that the moments of inertia are always positive. □

Theorems 5 and 6 give us a complete characterization of the stability properties of E_3 if $I_1 > I_2$. Now, we are in position to obtain a picture of the stability regions in terms of the relevant parameters of the problem: φ and l_3, the gyrostatic moment. The rest of the parameters stands for the geometry of the gyrostat and the position of the center of mass. For a prescribed geometry, the equilibrium E_3 exists if ω is a real number. Solving (12) to obtain ω, we find that E_3 exists if

$$l_3^2 - 4\, gmz_0(I_1 - I_3)\cos\varphi \geq 0. \tag{15}$$

When the inequality is transformed in an equality, it defines a curve in the plane (φ, l_3) dividing it into two regions, one of them for the region of existence of E_3 and the other one for the region where E_3 does not exist. We note that the existence region depends on the sign of $(I_1 - I_3)z_0$. In the same way, there is a curve that separates the stability and instability regions. To describe these regions is sufficient to study the case $I_1 < I_3$ because, if $I_1 > I_3$, the stability conditions are always satisfied (Remark 3) and, therefore, the stability region is the same as the existence region. For the case $I_1 < I_3$, we have to proceed carefully as, once l_3 and φ are fixed, two different values of ω are obtained

$$\omega_\pm = \frac{-l_3 \pm \sqrt{l_3^2 + 4(I_3 - I_1)gmz_0\cos\varphi}}{2(I_3 - I_1)\cos\varphi} \tag{16}$$

and, therefore, also two equilibrium points we name E_{3+} and E_{3-}, corresponding to the values ω_+ and ω_- respectively. Substituting these two expressions into the first inequality in (11) we arrive to the limiting curve

$$\begin{aligned} &I_1(l_3^4 + g^2m^2z_0^2 I_1(I_1 - I_3)) - 6\,gmz_0 I_1 l_3^2(I_1 - I_3)\cos\varphi + 6\,g^2m^2z_0^2 I_1(I_1 - I_3)^2\cos^2\varphi \\ &\quad - 2\,gmz_0 l_3^2(I_1 - I_3)^2\cos^3\varphi + 9\,g^2m^2z_0^2(I_1 - I_3)^3\cos^4\varphi = 0, \end{aligned} \tag{17}$$

that separates the stability and instability regions.

It is worth noting that Eqs. (15) and (17) do not depend on I_2, but only on I_1 and I_3. Moreover, $(I_1 - I_3)$ appears as a relevant quantity. For this reason we introduce the quantities

$$a = I_3 - I_1, \qquad b = \frac{I_3 - I_1}{I_1},$$

as the parameters to describe the geometry of the gyrostat. In terms of a and b, the curves delimiting (15) and (17) become respectively

$$\begin{aligned} f_1(l_3, \varphi; a, b, gmz_0) &\equiv l_3^2 + 4agmz_0\cos\varphi = 0, \\ f_2(l_3, \varphi; a, b, gmz_0) &\equiv -b\, l_3^4 + a^2g^2m^2z_0^2 - 6\,abgmz_0 l_3^2\cos\varphi - 6\,a^2bg^2m^2z_0^2\cos^2\varphi \\ &\qquad + 2\,ab^2\, gmz_0 l_3^2\cos^3\varphi + 9\,a^2b^2g^2m^2z_0^2\cos^4\varphi = 0. \end{aligned} \tag{18}$$

The two curves are double symmetric since for $i = 1, 2$,

$$\begin{aligned} f_i(l_3, \varphi; a, b, gmz_0) &= f_i(-l_3, \varphi; a, b, gmz_0), \\ f_i(l_3, \pi + \varphi; a, b, gmz_0) &= f_i(l_3, \pi - \varphi; a, b, gmz_0). \end{aligned}$$

We also note that $ab > 0$ and that, for $a < 0$ and $b < 0$, the curve defined by f_2 does not exist because, as it was proven in Remark 3, when $I_1 > I_3$ equilibrium E_3 is stable and, consequently, $f_2 \neq 0$.

We stress that, in fact, f_2 is a two branched curve, one of its branches coming from ω_+ and the other one from ω_- and these branches are different depending on the sign of z_0. For the case $z_0 > 0$, the two branches intersect each other at the points

$$\left(\varphi = \arccos\frac{1}{\sqrt{3b}}, l_3 = 0\right), \qquad \left(\varphi = 2\pi - \arccos\frac{1}{\sqrt{3b}}, l_3 = 0\right),$$

provided $b \geq 1/3$. For $b < 1/3$ the two branches do not cross. The aspect of the branches for different values of b is depicted in Fig. 2. The red branch stands for the value ω_+ and the blue one for the value of ω_-. When the red branch is crossed the corresponding equilibrium point E_{3+} changes its stability character. In this way, it is easy to check that E_{3+} is stable above the red branch and unstable below it. For E_{3-} the situation is the opposite, the equilibrium point is stable below the blue branch and unstable above it.

The previous considerations show that for each equilibrium point of the family E_3 there is a critical value of the gyrostatic moment in such a way that if it is crossed, the stability changes. Some members of the family E_{3+} require positive values of

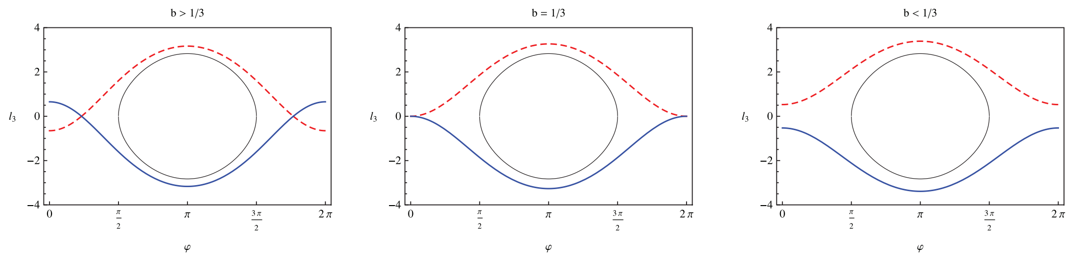

Fig. 2. The two branches of the curve defined $f_2 = 0$ for different values of b and $z_0 > 0$. The red dashed branch comes from ω_+, and the blue one from ω_-. The area inside the black curve corresponds to the region where the family of equilibrium points E_3 does not exist. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article).

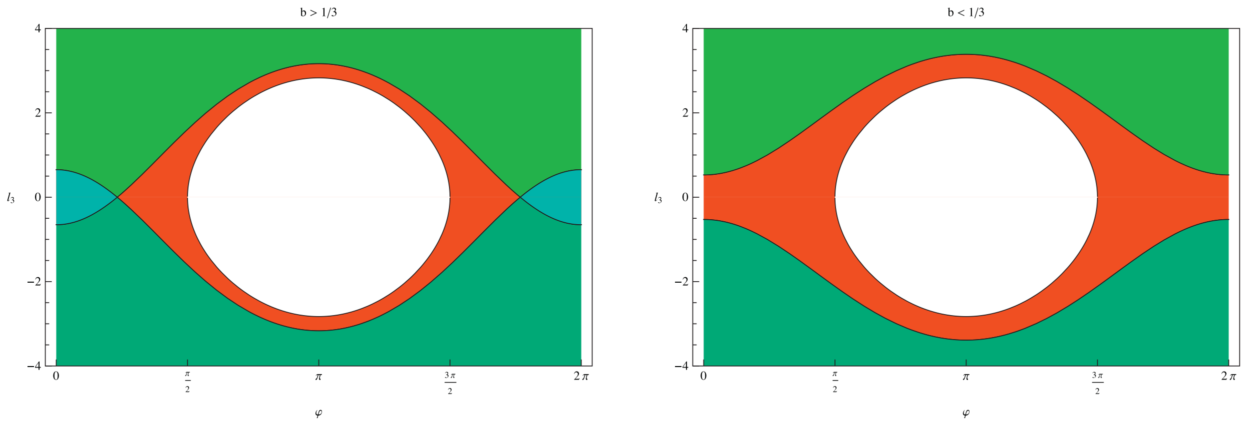

Fig. 3. Stability regions for the two equilibrium points associated to a pair (φ, l_3). The green areas (light and dark) indicate stability for one of the equilibrium points and instability for the other. Red and blue zones stand for unstable and stable regions for both equilibria, respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article).

the gyrostatic moment to be stable and the other ones, E_{3-}, negative values. Nevertheless, the geometry of the gyrostat is also important. Indeed, if $b > 1/3$, small absolute values of l_3 give rise to stable points if the angle φ verifies $\cos\varphi > 1/\sqrt{3b}$. On the other hand, if $b > 1/3$ and

$$l_3^2 < agmz_0\left(-3 + b + \frac{(1+b)^{3/2}}{\sqrt{b}}\right),$$

every member of the family E_3 is unstable, regardless the value of the angle φ. Fig. 3 summarizes this. There, the stability regions for the two equilibrium points associated to a pair (φ, l_3) are shown. The light green area indicates stability for the equilibrium E_{3+} and instability for the other. The dark green area stands for the stable region for E_{3-} and instability for the other. The red zone indicates instability for both equilibrium points and the blue one stability for the two.

A similar analysis can be made for $z_0 < 0$ obtaining similar results. Indeed, we obtain exactly the same but interchanging the intervals $[0, \pi/2] \cup [3\pi/2, 2\pi]$ and $[\pi/2, 3\pi/2]$. Now the branches associated to the two values $\omega_\pm$ intersect at the points

$$\left(\varphi = \arccos\frac{-1}{\sqrt{3b}}, l_3 = 0\right), \qquad \left(\varphi = 2\pi - \arccos\frac{-1}{\sqrt{3b}}, l_3 = 0\right),$$

provided $b \geq 1/3$, otherwise they do not intersect. Fig. 4 shows the two branches for different values of b. As in the case $z_0 > 0$, when the red branch is crossed from above to below the equilibrium point E_{3+} changes its character from stable to unstable. When the blue branch is crossed from above to below, the equilibrium point E_{3-} changes from unstable to stable.

A different approach to the stability regions can be made if we consider as the relevant parameters φ and ω. Now, fixed a pair (φ, ω), $\omega \neq 0$, there is only one l_3 defining an equilibrium point. Thus, we do not have to face the analysis of the branches and also we do not have to take care about the region where the equilibrium exists. In this way, proceeding as above, we obtain l_3 from (12), then introduce its expression into the first inequality in (11) and we obtain that the stability area is delimited by the curves defined by $f_3 = 0$, where f_3 is the function

$$f_3 \equiv -a^2\omega^4 + bg^2m^2z_0^2 + 2abgm\omega^2 z_0\cos\varphi.$$

This is a two branched curve, and the branches are different depending on the sign of z_0 and stability takes place in the bounded region between the two branches. To account for the effect of l_3, we can fix its value and depict the corresponding

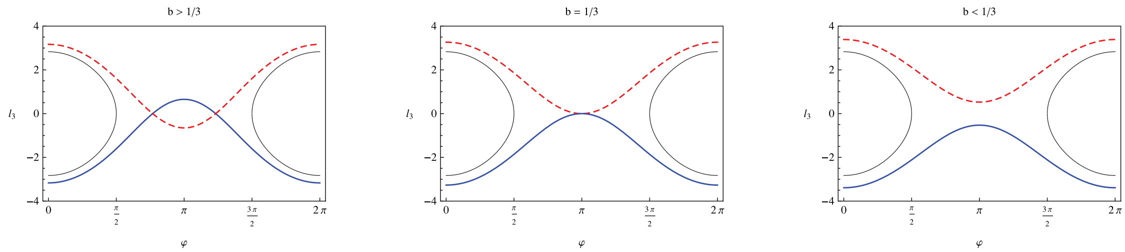

Fig. 4. The two branches of the curve defined $f_2 = 0$ for different values of b and $z_0 < 0$. The red dashed branch comes from ω_+, and the blue one from ω_-. The area inside the black curve corresponds to the region where the family of equilibrium points E_3 does not exist. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article).

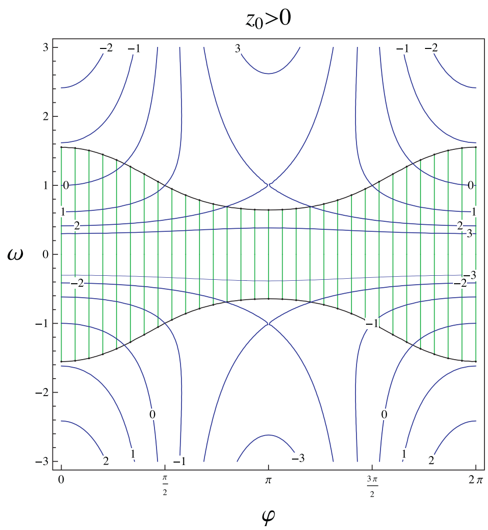

Fig. 5. Different curves for the gyrostatic moment onto the stability plane in (φ, ω) for $a = b = 1$ and $z_0 > 0$. The part of the curve inside the shaded region gives rise to a stable equilibrium point, whereas the equilibrium is unstable in the outer part.

line defined in the (φ, ω) plane by the existence condition (12). Different curves are shown in Fig. 5, where it can be seen what pairs of (φ, ω) are stable for a prescribed value of l_3. It is worth noticing the pseudo symmetry for the positive and negative values of l_3. Indeed, the curves for $l_3 > 0$ and $l_3 < 0$ are symmetric with respect the axis $\omega = 0$. That means that if a pair (φ_0, ω_0) is stabilized by l_3, the pair $(\varphi_0, -\omega_0)$ is stabilized by $-l_3$. Furthermore, it can also be observed the behavior described previously in Fig. 3, that is, how for small values of $|l_3|$ and small φ_0 we can find two values of ω for which the corresponding equilibrium is stable.

Remark 4. It is interesting to note that if $I_2 > I_1$ the above described stability regions seem to change. That is to say, stability turns to be instability and vice versa. However, a detailed study, using KAM theory must be performed, as some instabilities can appear due to the presence of resonances.

5. Conclusions

The stability of the permanent rotations of a heavy gyrostat with a fixed point, whose center of mass is located in one of the principal axis and the gyrostatic moment acts along the same axis, has been studied by means of the Energy-Casimir method. First of all, we have established the existence of three families of permanent rotations. For the first family, we have obtained sufficient and necessary stability conditions, and we have proved that they are the same in half the region where the necessary conditions are satisfied. Moreover, we have stressed that these conditions do not approach, in the limit, to the classical conditions for a axisymmetric gyrostat, due to the appearance of a conserved quantity for the symmetric case. The other two families can be treated at once, as they are, in some sense, symmetric. For these two families the vector defining the permanent rotation lies in the plane of two of the principal axes and the third one is orthogonal to it. For these families it is proved that the stability conditions are independent of the value of the moment of inertia corresponding to the orthogonal principal axis. Besides, if the other two moments of inertia verify that the one corresponding to the principal axis where the center of mass lies is not the biggest, then the permanent rotations are always stable, does not matter the value of the gyrostatic moment. On the other hand, if it is the biggest, then a minimum gyrostatic moment is necessary to stabilize the rotations. This fact is analyzed in detail in a suitable parametric plane.

Acknowledgments

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References

[1] A. Anchev, On the stability of permanent rotations of a heavy gyrostat, Pmm-J. Appl. Math. Mech. 26 (1962) 22–28.
[2] A. Anchev, Permanent rotations of a heavy gyrostat having a stationary point, Pmm-J. Appl. Math. Mech. 31 (1967) 49–58.
[3] V.I. Arnold, On an a priori estimate in the theory of hydrodynamical stability, (russian), Izv. Vyssh. Uchebn. Zaved. Mat. Nauk. 54 (1966) 3–5. (English) Am. Math. Soc. Transl. 79 (1969) 267–260.
[4] V.S. Aslanov, Integrable cases of the problem of the free motion of a gyrostat, Pmm-J. Appl. Math. Mech. 78 (2014) 445–453.
[5] V.S. Aslanov, V.V. Yudintsev, Dynamics and chaos control of asymmetric gyrostat satellites, Cosmic Res. 52 (2014) 216–228.
[6] A.M. Bloch, J.E. Marsden, Stabilization of rigid body dynamics by the energy-casimir method, Syst. Control Lett. 14 (1990) 341–346.
[7] J.A. Cavas, A. Vigueras, An integrable case of a rotational motion analogous to that of Lagrange and Poisson for a gyrostat in a newtonian force field, Celest. Mech. Dyn. Astron. 60 (1994) 317–330.
[8] J. E.Cochran, P.H. Shu, S.D. Rew, Attitude motion of asymmetric dual-spin spacecraft, J. Guid. Control Dynam. 5 (1982) 37–42.
[9] A. Deprit, A. Elipe, Complete reduction of the Euler-Poinsot problem, J. Astronaut. Sci. 41 (1993) 603–628.
[10] A. Elipe, M. Arribas, A. Riaguas, Complete analysis of bifurcations in the axial gyrostat problem, J. Phys. A: Math. Gen. 30 (1997) 587–601.
[11] A. Elipe, V. Lanchares, Phase flow of an axially symmetrical gyrostat with one constant rotor, J. Math. Phys. 38 (1997) 3533–3544.
[12] A. Elipe, V. Lanchares, Two equivalent problems: gyrostats in free motion and parametric quadratic Hamiltonians", Mech. Res. Commun. 24 (1997) 583–590.
[13] A. Elipe, V. Lanchares, Exact solution of a triaxial gyrostat with one rotor, Celest. Mech. Dyn. Astron. 101 (2008) 49–68.
[14] M.T. de Bustos Muñoz, J.L.G. Guirao, J.A.V. López, A.V. Campuzano, On sufficient conditions of stability of the permanent rotations of a heavy triaxial gyrostat, Qual. Theory Dyn. Syst. 14 (2015) 265280.
[15] C.D. Hall, Spinup dynamics of biaxial gyrostats, J. Astronaut. Sci. 43 (1995) 263–275.
[16] C.D. Hall, Spinup dynamics of gyrostats, J. Guid. Control Dynam. 18 (1995) 1177–1183.
[17] C.D. Hall, R.H. Rand, Spinup dynamics of axial dual-spin spacecraft, J. Guid. Control Dynam. 17 (1994) 30–37.
[18] D. Holm, J.E. Marsden, T.S. Ratiu, A. Weinstein, Nonlinear stability of fluid and plasma equilibria, Phys. Rep. 123 (1985) 1–116.
[19] P.C. Hughes, Spacecraft Attitude Dynamics, John Wiley & Sons, New York, 1986.
[20] M. Iñarrea, V. Lanchares, Chaos in the reorientation process of a dual-spin spacecraft with time dependent moments of inertia, Int. J. Bifurc. Chaos 10 (2000) 997–1018.
[21] T.J. Kalvouridis, Stationary solutions of a small gyrostat in the Newtonian field of two bodies with equal masses, Nonlinear Dyn. 61 (2010) 373–381.
[22] A.M. Kovalev, Stability of steady rotations of a heavy gyrostat about its principal axis, Pmm-J. Appl. Math. Mech. 44 (1981) 709–712.
[23] A.M. Kovalev, Stability of stationary motions of mechanical systems with a rigid body as the basic element, Nonlinear Dyn. Syst. Theory. 1 (2001) 81–96.
[24] V. Lanchares, A. Elipe, Bifurcations in biparametric quadratic potentials, CHAOS: Interdiscip. J. Nonlinear Sci. 5 (1995) 367–373.
[25] V. Lanchares, A. Elipe, Bifurcations in biparametric quadratic potentials. II, CHAOS: Interdiscip. J. Nonlinear Sci. 5 (1995) 531–535.
[26] V. Lanchares, M. Iñarrea, J.P. Salas, Spin rotor stabilization of a dual-spin spacecraft with time dependent moments of inertia, Int. J. Bifurc. Chaos 8 (1998) 609–617.
[27] V. Lanchares, M. Iñarrea, J.P. Salas, J.D. Sierra, A. Elipe, Surfaces of bifurcation in a triparametric quadratic Hamiltonian, Phys. Rev. E 52 (1995) 5540–5548.
[28] E. Leimanis, The General Problem of the Motion of Coupled Rigid Bodies About a Fixed Point, Springer-Verlag, Berlin, 1965.
[29] D. Lewis, T. Ratiu, J.C. Simo, J.E. Marsden, The heavy top: a geometric treatment, Nonlinearity 5 (1992) 1–48.
[30] J.E. Marsden, Lectures on Mechanics, Cambridge University Press, Cambridge, 1992.
[31] J.P. Ortega, T.S. Ratiu, Non-linear stability of singular relative periodic orbits in Hamiltonian systems with symmetry, J. Geom. Phys. 32 (1999) 160–188.
[32] V. Rumiantsev, On the stability of motion of gyrostats, Pmm-J. Appl. Math. Mech. 25 (1961) 9–19.
[33] M.E. Sansaturio, A. Vigueras, Translatory-rotatory motion of a gyrostat in a Newtonian force field, Celest. Mech. 41 (1988) 297–311.
[34] V.A. Sarychev, Dynamics of an axisymmetric gyrostat satellite under the action of a gravitational moment, Cosmic Res. 48 (2010) 18893.
[35] V.A. Sarychev, Dynamics of an axisymmetric gyrostat satellite. equilibrium positions and their stability, Pmm-J. Appl. Math. Mech. 78 (2014) 249257.
[36] V. Tsogas, T.J. Kalvouridis, A.G. Mavraganis, Equilibrium states of a gyrostat satellite moving in the gravitational field of an annular configuration of n big bodies, Acta Mech. 175 (2005) 181–195.
[37] J.A. Vera, A. Vigueras, Estabilidad de ciertos equilibrios de un giróstato simétrico bajo un potencial con simetría axial $\mathbf{U}(\mathbf{k}_3)$, in: Métodos de dinámica orbital y rotacional (Proc. IV Jornadas de Trabajo en Mecánica Celeste), Servicio de Publicaciones Universidad de Murcia, 2002, pp. 175–181.
[38] J.A. Vera, A. Vigueras, Soluciones de equilibrio en un problema generalizado del de Lagrange-Poisson: condiciones necesarias y suficientes de estabilidad, Monografías de la Real Academia de Ciencias de Zaragoza 22 (2003) 141–150.