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Lyapunov stability for a generalized Hénon–Heiles system in a rotating reference frame

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ABSTRACT

In this paper we focus on a generalized Hénon–Heiles system in a rotating reference frame, in such a way that Lagrangian-like equilibrium points appear. Our goal is to study their nonlinear stability properties to better understand the dynamics around these points. We show the conditions on the free parameters to have stability and we prove the super-stable character of the origin for the classical case; it is a stable equilibrium point regardless of the frequency value of the rotating frame.

1. Introduction

Hénon–Heiles system is probably one of the most studied dynamical systems, because it can be used to model different physical problems and also to highlight different properties inherent to most of two degrees of freedom nonlinear Hamiltonian systems. It arose as a simple model to find additional conservation laws in galactic potentials with axial symmetry [10]. However, it was shown that many other problems can be reduced to the Hénon–Heiles Hamiltonian or, at least, to a very similar one, in what it is called the Hénon–Heiles family or generalized Hénon–Heiles systems. For instance, models for ion traps [12], reaction processes [11], three particle systems [14], black holes [20] and others (see [3] for more examples) can be described by means of a proper generalized Hénon–Heiles Hamiltonian. In the context of galactic dynamics, to study stellar orbits, the rotation of the galaxy must be taken into account [21] so that it makes sense to consider a generalized Hénon–Heiles system in a rotating frame expressed by the Hamiltonian

$$\mathcal{H} = \frac{1}{2}(X^2 + Y^2) - \omega(xY - yX) + \frac{1}{2}(x^2 + y^2) + ayx^2 + by^3, \tag{1}$$

where ω is the angular velocity and a and b are parameters. Hamiltonian (1) is also of great interest in the study of the dynamics of Rydberg atoms subject to different external fields. For instance, a magnetic field or a circularly polarized micro-wave field lead to the presence of the term $\omega(xY - yX)$ in the Hamiltonian function and, thus, to a generalized Hénon–Heiles system [18]. Indeed, in [18] a particular case of Hamiltonian (1), when the parameters a and b are in the same ratio as in the classical Hénon–Heiles system, is considered. The chaotic ionization dynamics of atoms is studied, showing the appearance of a fractal Weyl law behavior.

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We will focus on equilibrium solutions and their stability properties, as many qualitative aspects of the dynamics can be inferred. Indeed, trapped and escape dynamics are regulated by the presence of critical points with special properties of stability [4]. Moreover, the existence of stable equilibria is crucial in constructing self-consistent galaxy models from a given potential [21] and also to demonstrate the existence of nondispersive, coherent states for Rydberg atoms in the presence of external fields [13]. For the Hamiltonian (1) we find different type of equilibria, but the most relevant fact is the appearance, for appropriate values of the parameters, of Lagrangian-like equilibrium points, similar to L_4 and L_5 in the restricted three body problem. It is known that these equilibria are not always stable, but they can be either stable or unstable depending on the mass parameter [16,19]. So, the same can happen for the Lagrangian-like equilibrium points of the system defined by (1). The goal of the paper is to determine the values of the parameters for which stability takes place.

The discussion of stability starts by performing a linear analysis in a small neighborhood of the equilibrium solutions. A necessary condition for stability is that all eigenvalues have zero real part. If this is the case, if the corresponding linear Hamiltonian function is positive or negative defined, Dirichlet criterion (also Morse lemma and Lyapunov theorem) ensures nonlinear stability [5,19]. However, if the linear Hamiltonian is not defined, it is not possible to ensure Lyapunov stability. Indeed, there are well known examples of equilibrium points of a Hamiltonian system that are stable in the linear sense, but unstable in the Lyapunov one [17,19]. We will consider the case of distinct pure imaginary eigenvalues, such that we can apply classical results from KAM theory.

First of all it is necessary to transform the Hamiltonian function into normal form in a neighborhood of the equilibrium, by means of a successive changes of variables entailing a cumbersome process, because of the many symbolic algebraic manipulations [16]. Once the Hamiltonian has been brought to normal form, Arnold's theorem [1] guarantees the stability of equilibria if a certain non degeneracy condition is satisfied. Nevertheless, this theorem does not apply in the presence of resonances of order less than or equal to four. If these resonances appear, results given by Markeev [15], Cabral and Meyer [6] and Elipe et al. [7] must be applied. Also these results can be applied for higher order resonances, when degenerate situations appear. In this way, it is interesting to find those values of the parameters corresponding to degenerate cases because, although Arnold's theorem ensures stability in most cases when dealing with higher order resonances, sometimes they can lead to instability.

The paper is organized as follows. First, in Section 2, we find equilibria of the system. Next, in Section 3, we study the linear stability and, in Section 4, the Lyapunov stability is considered. The last two sections are devoted to the analysis of the stability in presence of resonances. Finally some concluding remarks are presented.

2. Equilibria

It is well known that, for the classical Hénon–Heiles problem, there are four equilibria whose coordinates (x, y, X, Y) are given by $(0,0,0,0)$, $(0,1,0,0)$ and $(\pm\sqrt{3}/2, -1/2, 0, 0)$. The origin is a minimum of the potential, and the other three equilibria are saddle points, located at the vertices of an equilateral triangle with barycenter at the origin, accounting for the D_3 symmetry of the problem. However, in the generalized Hénon–Heiles system, the symmetry can be destroyed by the parameters a and b, yielding to a new scenario of equilibrium points depending on the values of a, b and ω. In particular, we obtain the following result.

Theorem 2.1. *Let us consider the Hamiltonian system defined by (1), then there are at most four equilibrium points. Moreover*

(i) If $\omega^2 = 1$ and $ab \neq 0$ or $\omega^2 \neq 1$ and $a = b = 0$, $E_1 \equiv (0,0,0,0)$ is the unique equilibrium point.
(ii) If $\omega^2 \neq 1$ and $b \neq 0$, there are two equilibria: E_1 and

$$E_2 \equiv \left(0, \frac{\omega^2 - 1}{3b}, -\omega\frac{\omega^2 - 1}{3b}, 0\right).$$

(iii) If $\omega^2 \neq 1, b \neq 0$ and $a(2a - 3b) > 0$, there are four equilibria: E_1, E_2 and

$$E_{3,4} \equiv \left(\pm\frac{|\omega^2-1|}{2}\sqrt{\frac{2a-3b}{a^3}}, \frac{\omega^2-1}{2a}, -\omega\frac{\omega^2-1}{2a}, \pm\omega\frac{|\omega^2-1|}{2}\sqrt{\frac{2a-3b}{a^3}}\right).$$

Proof. Equilibria of the system (1) are the solutions of the corresponding Hamilton equations equated to zero. These are

$$\begin{cases} \dot{x} &= X + \omega y, \\ \dot{y} &= Y - \omega x, \\ \dot{X} &= -x - 2axy + \omega Y, \\ \dot{Y} &= -ax^2 - \omega X - y - 3by^2. \end{cases} \tag{2}$$

From these equations we obtain $X = -\omega y$ and $Y = \omega x$. Thus, x and y satisfy the system

$$\begin{cases} ((\omega^2 - 1) - 2ay)x &= 0, \\ -ax^2 + (\omega^2 - 1)y - 3by^2 &= 0. \end{cases} \tag{3}$$

The first equation in (3) is verified if $x = 0$ or $y = (\omega^2 - 1)/(2a)$. By substitution of these values into (2), the result follows straightforwardly. □

It is worth noting that, for $\omega^2 \neq 1$, there is a limiting case when b tends to zero. Indeed, if b tends to zero, the X and y coordinates of the equilibrium point E_2 blow up to infinity and, as the parameter b changes its sign, the same do X and y. This can be viewed as if the point E_2 escapes to infinity along the y axis and, as soon as b changes its sign, E_2 appears at the opposite side of the y axis. We will see later that $b = 0$, together with $2a - 3b = 0$ and $a = 0$, constitute the bifurcation lines in the parameter plane. If one of these lines is crossed the number or the nature of equilibria changes.

We also note that, for $\omega^2 = 1$ and $ab = 0$, a set of non isolated equilibria appears. If both a and b are zero, the dynamics reduces to that of a linear system and can be easily figured out. However, if a and b are not zero at the same time, the dynamics is more intricate as the nonlinear terms modify the behavior of the system.

The nature of equilibrium points can be characterized from two different points of view. On the one hand, we can establish its linear stability properties from the Jacobian matrix of the system (2) evaluated at the equilibria. On the other hand, we can see the equilibria as the critical points of the effective potential

$$\Phi_{eff} = \mathcal{H} - \frac{1}{2}(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}\left(x^2(2ay - \omega^2 + 1) + y^2(2by - \omega^2 + 1)\right) \tag{4}$$

and establish their nature. We will start with the second approach, as it also provides information about the trapping and escape dynamics. Consequently, we can establish the following result

Theorem 2.2. *For the general Hénon–Heiles system (1):*

(i) E_1 is a minimum of the effective potential if $\omega^2 < 1$ and a maximum if $\omega^2 > 1$.
(ii) E_2 is a minimum of the effective potential if $b(2a - 3b) > 0$ and $\omega^2 > 1$; a maximum if $b(2a - 3b) > 0$ and $\omega^2 < 1$ and a saddle point if $b(2a - 3b) \leqslant 0$.
(iii) $E_{3,4}$ are always saddle points of the effective potential, when they exist.

Proof. The result is deduced from the Hessian matrix of the effective potential,

$$\mathbf{H} = \begin{pmatrix} 1 - \omega^2 + 2ay & 2ax \\ 2ax & 1 - \omega^2 + 6by \end{pmatrix},$$

evaluated at critical points. For instance, for E_2, $\mathbf{H}$ results to be

$$\mathbf{H} = \begin{pmatrix} \frac{(2a-3b)(\omega^2-1)}{3b} & 0 \\ 0 & \omega^2 - 1 \end{pmatrix},$$

and therefore the character of the critical point E_2 depends on the signs of $b(2a - 3b)$ and $\omega^2 - 1$. If $b(2a - 3b) \leqslant 0$, the critical point is a saddle. On the other hand, if the inequality holds in the opposite direction, E_2 is a maximum if $\omega^2 - 1 < 0$ and a minimum if $\omega^2 - 1 > 0$. A similar discussion can be made for the rest of critical points. □

It is worth noting that, in the case $\omega^2 = 1$, no information can be deduced for the unique critical point if $ab \neq 0$, as the Hessian matrix is the null matrix. This situation, as well as the case when a dense set of critical points exists, deserves a special treatment and it is out of the scope of this paper. Thus, hereinafter we will focus on the case $\omega^2 \neq 1$.

From Theorems 2.1 and 2.2 it follows that the parameter plane can be divided into different regions where the number of critical points or their character changes. Two cases must be considered depending on the value of $\omega : \omega^2 > 1$ and $\omega^2 < 1$. Fig. 1 shows the different regions in the parameter plane and Fig. 2 exhibits the effective potential and its projection onto the xy plane for the configurations attained in each region. It can be seen that the role of maxima and minima is interchanged when ω^2 crosses the limiting value 1.

Fig. 1 reveals the presence of a symmetry respect to the parameters a and b. Indeed, we have

$$\mathcal{H}(x, y, X, Y; a, b, \omega) = \mathcal{H}(-x, -y, -X, -Y; -a, -b, \omega).$$

In addition, there is another symmetry respect to ω, provided that

$$(x, y, X, Y; a, b, \omega) \longrightarrow (x, y, -X, -Y; a, b, -\omega).$$

These two symmetries allow us to restrict our analysis to the cases $a > 0$ or $b > 0$ and $\omega > 0$. In this way, from here on, we will assume $\omega > 0$ and $a > 0$.

On the other hand, Fig. 2 shows the trapping regions located in the neighborhood of the minima and the escape channels through the saddle points if the energy of the system is great enough. This is an interesting issue in the context of ionization dynamics where these kind of systems appear. A careful analysis of phase space around the saddle points gives insight about

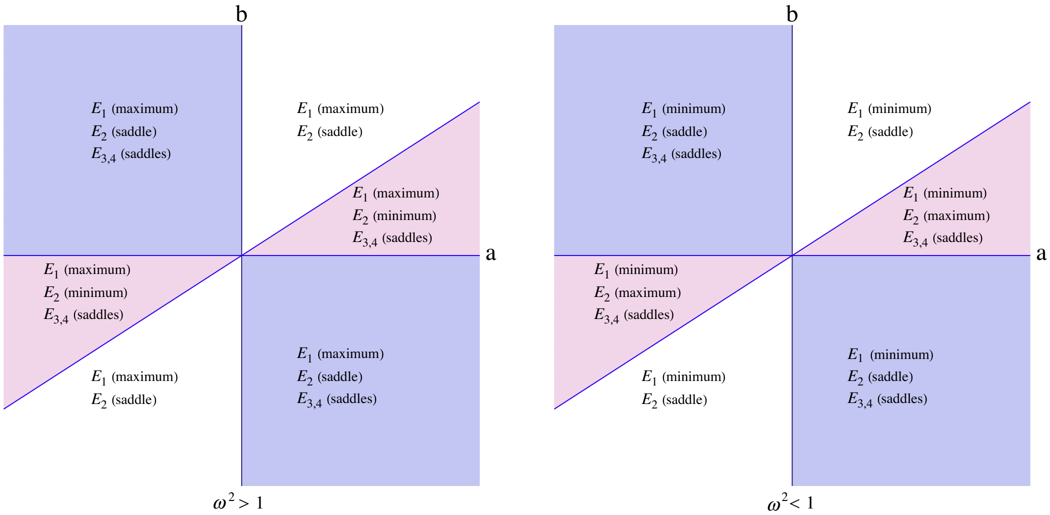

Fig. 1. Parameter plane with the bifurcation lines for the critical points of the effective potential: $a = 0, b = 0$ and $2a - 3b = 0$.

the ionization mechanism [2,9]. However, the dynamics around the maxima needs more insight. Indeed, it is known that the effect of the rotating term can act as a stabilizer and orbits with energy above the maximum can remain not only bounded, but confined in a small neighborhood of the critical point [8,13]. In this case, an important question is to determine the stability properties of the corresponding maximum and a necessary condition to have Lyapunov stability is linear stability. This is the subject of the next section.

3. Linear stability

The linear stability of an equilibrium point (x_0, y_0, X_0, Y_0) is derived from the eigenvalues of the Jacobian matrix associated to the equations of the motion (2), which is written as

$$J(x_0, y_0, X_0, Y_0) = \begin{pmatrix} 0 & \omega & 1 & 0 \\ -\omega & 0 & 0 & 1 \\ -(1+2ay_0) & -2ax_0 & 0 & \omega \\ -2ax_0 & -(1+6by_0) & -\omega & 0 \end{pmatrix}.$$

It is easy to see that eigenvalues of $J(x_0, y_0, X_0, Y_0)$ can be expressed as

$$\pm\sqrt{-(M \pm \sqrt{N})}, \tag{5}$$

where

$$\begin{aligned} M &= \omega^2 + 1 + y_0(a+3b), \\ N &= 4(a^2x_0^2 + \omega^2) + 4\omega^2 y_0(a+3b) + y_0^2(a-3b)^2. \end{aligned} \tag{6}$$

It is worth noting that the eigenvalues do not depend on the momenta X_0 and Y_0, so we can refer to the equilibrium points by their coordinates x_0 and y_0. Now, we are in position to give the necessary conditions for linear stability. Indeed, the equilibrium (x_0, y_0) is linear stable if and only if eigenvalues are pure imaginary and the Jacobian matrix is semisimple. Thence, we can establish the following result about the linear stability of maxima E_1 and E_2 in the generalized Hénon–Heiles problem.

Theorem 3.1. *E_1 is a linear stable maximum if and only if $\omega > 1$. E_2 is a linear stable maximum if and only if $\frac{1}{\sqrt{5}} < \omega < 1, b > 0$ and*

$$\frac{1 - \omega^4 + 2\omega(1-\omega^2)^{3/2}}{3(5\omega^4 - 2\omega^2 + 1)}\, a < b < \frac{2}{3}a. \tag{7}$$

Proof. The equilibrium E_1 is a maximum when $\omega > 1$. Eigenvalues of the Jacobian matrix are given by

$$\lambda_{1,2} = \pm(\omega+1)i, \quad \lambda_{3,4} = \pm(\omega-1)i.$$

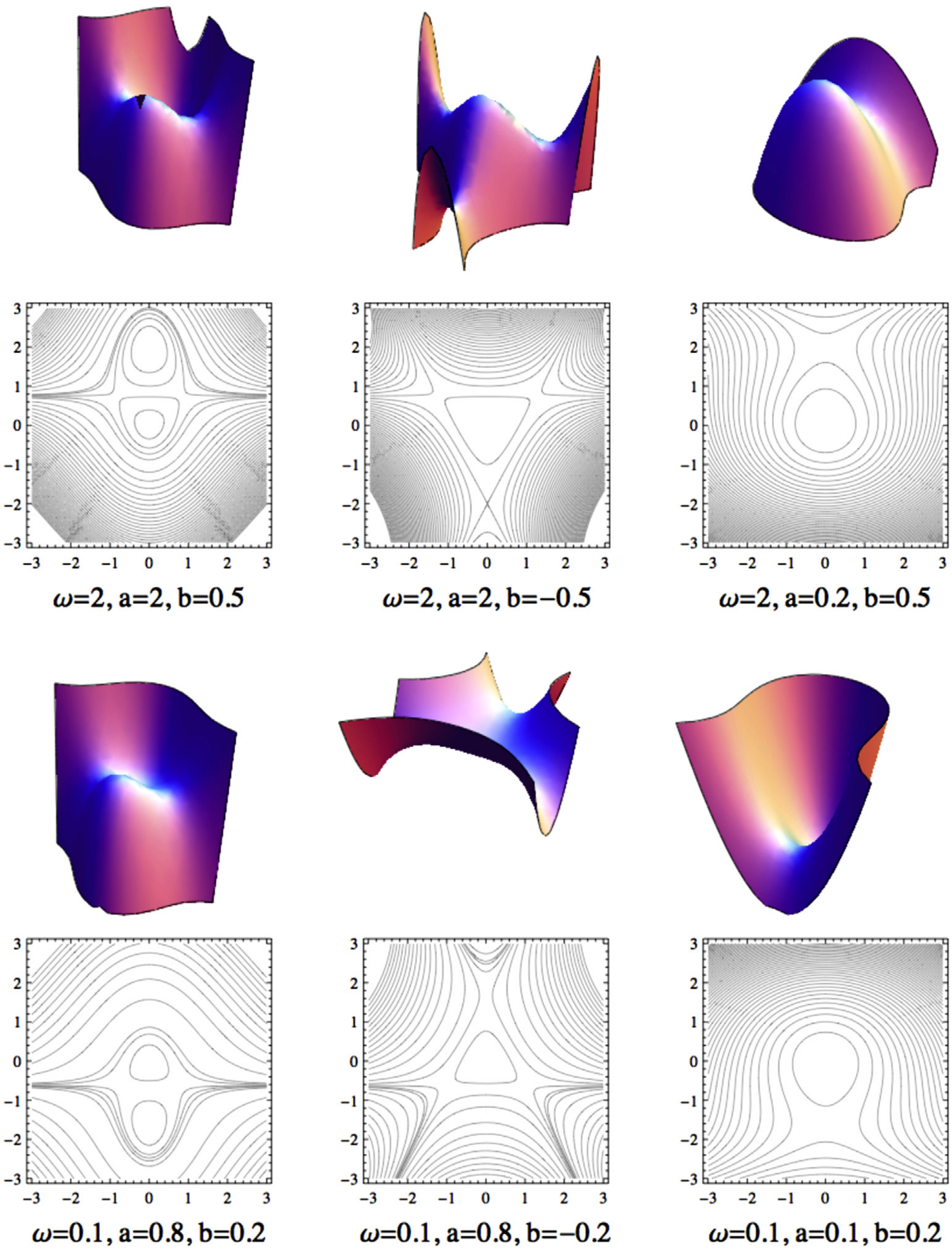

Fig. 2. The effective potential and its projection for $\omega^2 > 1$ and $\omega^2 < 1$ in the different regions of the parameter plane.

Taking into account that $\omega > 1$, we obtain four different pure imaginary eigenvalues and, therefore, E_1 is linear stable.

The equilibrium E_2 is a maximum when $b(2a - 3b) > 0$ and $\omega < 1$. Provided that, for $E_2, x_0 = 0$ and $y_0 = \frac{\omega^2-1}{3b}$, we obtain for the expressions in (6)

$$M = \omega^2 + 1 + \tfrac{(\omega^2-1)(a+3b)}{3b},$$
$$N = 4\omega^2\left(1 + \tfrac{(\omega^2-1)(a+3b)}{3b}\right) + \left(\tfrac{(\omega^2-1)(a-3b)}{3b}\right)^2.$$

It can be seen that if $N = 0$ and $M > 0$ we have multiple pure imaginary eigenvalues. However, it can be checked that the corresponding Jacobian matrix is not semisimple. Therefore, it follows that the equilibrium E_2 is linear stable if and only if $N > 0, M > 0$ and $M^2 - N > 0$. The last condition is always satisfied because

$$M^2 - N = \frac{(2a-3b)(\omega^2-1)^2}{3b},$$

and $b(2a-3b) > 0$. Thus, we are left with the two inequalities $M > 0$ and $N > 0$. The first one is satisfied if and only if

$$b > a\frac{1-\omega^2}{6\omega^2}. \tag{8}$$

However, it must be $b(2a-3b) > 0$ and this implies that (8) is satisfied if and only if $1/\sqrt{5} < \omega < 1$.

On the other hand, the inequality $N > 0$ holds if a and b do not belong to the region in between the two straight lines

$$b = a\frac{1-\omega^4 \pm 2\omega(1-\omega^2)^{3/2}}{3(5\omega^4 - 2\omega^2 + 1)}.$$

An analysis of the slope of these lines and that given by (8), in the case $1/\sqrt{5} < \omega < 1$, yields the result stated in the Theorem. □

It is interesting to note that the region of linear stability for the equilibrium E_2 is delimited, in the plane ab, by two straight lines, one of them with constant slope $2/3$ and another one with a variable slope that is a function of ω, given by

$$m(\omega) = \frac{1-\omega^4 + 2\omega(1-\omega^2)^{3/2}}{3(5\omega^4 - 2\omega^2 + 1)}. \tag{9}$$

As the value of ω increases, the slope $m(\omega)$ decreases and, therefore, the size of the region of linear stability increases (see Fig. 3). In the limit $\omega = 1/\sqrt{5}$ the slope $m(\omega)$ is equal to $2/3$ and the region of linear stability is empty. On the contrary, if $\omega = 1$ the region of linear stability cover the whole region where E_2 is a maximum.

4. Lyapunov stability

Linear stability is not enough to ensure Lyapunov stability, when the equilibrium point is a maximum of the effective potential. To solve this question it is necessary to apply KAM theory and this implies to bring the Hamiltonian to its Birkhoff normal form in a vicinity of the equilibrium point, in canonical action angle variables $(I_1, I_2, \theta_1, \theta_2)$. To achieve this, a series of canonical change of variables must be performed [16] and the normal form reads as

$$\mathcal{H} = \omega_1 I_1 - \omega_2 I_2 + a_{20} I_1^2 + a_{11} I_1 I_2 + a_{02} I_2^2 + O(I^{5/2}). \tag{10}$$

Here, ω_1 and ω_2 are the moduli of the eigenvalues of the linearized system at the equilibrium, a_{20}, a_{11} and a_{02} are real numbers independent of the variables and the coefficients of $O(I^{5/2})$ are finite Fourier series in the angles θ_1 and θ_2. Moreover, it is assumed that ω_1 and ω_2 do not satisfy a resonance condition of order less or equal than four, that is, there not exist integers n_1 and n_2 such that

$$n_1\omega_1 + n_2\omega_2 = 0, \qquad |n_1| + |n_2| \leqslant 4,$$

where $|n_1| + |n_2|$ is the order of the resonance. Once the Hamiltonian is in normal form, Arnold's Theorem [1] gives conditions for Lyapunov stability if some non degeneracy conditions are satisfied. Indeed, the corresponding equilibrium position is Lyapunov stable if

$$\mathcal{D} = a_{20}\,\omega_2^2 + a_{11}\,\omega_1\omega_2 + a_{02}\,\omega_1^2 \neq 0. \tag{11}$$

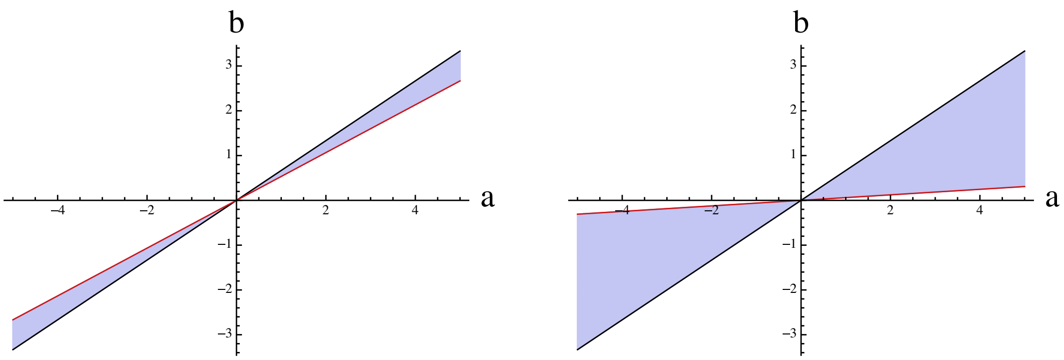

Fig. 3. Colored regions correspond to the region of linear stability for the maximum E_2, in the plane ab, for two different values of ω with $1/\sqrt{5} < \omega < 1$ (left $\omega = 0.6$, right $\omega = 0.9$). The lines $b = \frac{2a}{3}$ and $b = m(\omega)a$ are painted in black and red color respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

We do not enter here in the cumbersome process of the computation of the normal form and we omit the final expression, due to its complexity. We will directly analyze, for E_1 and E_2, the relations the parameters a, b and ω must satisfy, in order that the non degeneracy condition (11) is fulfilled under the linear stability assumptions established in the previous section.

Let us begin analyzing the stability for E_1 in the case of a maximum, when $\omega > 1$. We find that $\mathcal{D}$ vanishes if a and b are located on the straight lines

$$b = m_1(\omega)a, \qquad b = m_2(\omega)a, \tag{12}$$

where the slopes $m_1(\omega)$ and $m_2(\omega)$ are given by

$$m_{1,2}(\omega) = \frac{r_1(\omega) \pm 2\sqrt{r_2(\omega)}}{45 - 233\omega^2 + 1035\omega^4 - 207\omega^6}, \tag{13}$$

and

$$r_1(\omega) = 27 + 73\omega^2 - 291\omega^4 + 63\omega^6,$$
$$r_2(\omega) = (9 - \omega^2)(9\omega^2 - 1)(9\omega^4 + 7)(5\omega^4 - 24\omega^2 + 3).$$

Note that the two lines (12) exist if $r_2(\omega) \geqslant 0$. This happens for ω belonging to the interval $[\kappa, 3]$, where

$$\kappa = \sqrt{\frac{12 + \sqrt{129}}{5}}.$$

Outside this interval the equilibrium point E_1 is Lyapunov stable, provided a resonance condition of order less or equal than four is not satisfied. These resonances take place when $\omega = 3$, for a third order resonance, and when $\omega = 2$, for a fourth order resonance. We can summarize the previous discussion in the following result

Theorem 4.1. *The equilibrium point E_1 is Lyapunov stable if*

$$\omega \in (1,2) \cup (2,\kappa) \cup (3,\infty), \quad \text{or} \quad \omega \in [\kappa, 3) \text{ and } b \neq m_{1,2}(\omega)a.$$

It is worth noting that, for the classical Hénon–Heiles system, E_1 is Lyapunov stable, unless a resonance of third or fourth order takes place. Indeed, $a + 3b = 0$ and a, b are located on a straight line with slope $-1/3$. However, the slopes $m_1(\omega)$ and $m_2(\omega)$ reach this value when $\omega = 3$, just in the case of a third order resonance. This case will be considered in Section 5.

Now, we perform a similar analysis for E_2, when the conditions of Theorems 2.2 and 3.1 are satisfied. In this case, $\mathcal{D}$ vanishes if one of the following equations hold

$$b = 0, \qquad p(a, b; \omega) = \sum_{i+j=6} a^i b^j \alpha_{ij} = 0, \tag{14}$$

where the coefficients α_{ij} are given by

$$\alpha_{60} = 64 - 256\omega^2 + 384\omega^4 - 256\omega^6 + 64\omega^8,$$
$$\alpha_{51} = -546 + 4104\omega^2 - 9036\omega^4 + 7944\omega^6 - 2466\omega^8,$$
$$\alpha_{42} = -2277 - 32364\omega^2 + 57186\omega^4 - 8172\omega^6 - 14373\omega^8,$$
$$\alpha_{33} = 33642 + 175932\omega^2 - 174744\omega^4 + 67284\omega^6 - 102114\omega^8,$$
$$\alpha_{24} = -117450 - 426870\omega^2 + 419418\omega^4 - 260010\omega^6 - 465264\omega^8,$$
$$\alpha_{15} = 170100 + 419904\omega^2 - 775656\omega^4 + 1353024\omega^6 + 450036\omega^8,$$
$$\alpha_{06} = -91125 - 94770\omega^2 + 338256\omega^4 - 867510\omega^6 - 217971\omega^8.$$

Note that $b = 0$ matches with one of the lines delimiting the region where E_2 is a maximum. Thus, we only have to analyze the equation $p(a, b; \omega) = 0$. Let us note that $p(a, b; \omega)$ is a homogeneous polynomial in a and b, then its graph is a collection of straight lines through the origin in the plane ab, for each value of ω. The real roots of the polynomial in $b, p(1, b; \omega)$, determine the slope of the straight lines, which are of the form

$$b = \mu_k(\omega)a, \tag{15}$$

$\mu_k(\omega)$ being a root of the polynomial $p(1, b; \omega)$ and k an index running from 1 to the number of real roots ($k \leqslant 6$). However, the number of real roots is not fixed, as it depends on ω. Indeed, from the resultant of $p(1, b; \omega)$, we deduce that the number of real roots changes from four to two when ω reaches the values 0.559284 and 0.998856, approximately. The question now is to establish how many lines of the form (15) lie in the region of linear stability. Accordingly, we have to select those $\mu_k(\omega)$ satisfying

$$m(\omega) < \mu_k(\omega) < 2/3, \tag{16}$$

where $m(\omega)$ is given by (9). It can be proven that there is only one $\mu_k(\omega) = \mu(\omega)$ fulfilling the above condition if $1/\sqrt{5} < \omega < 1$, regardless if the number of real roots is two or four. This can be viewed in Fig. 4 where it is depicted in

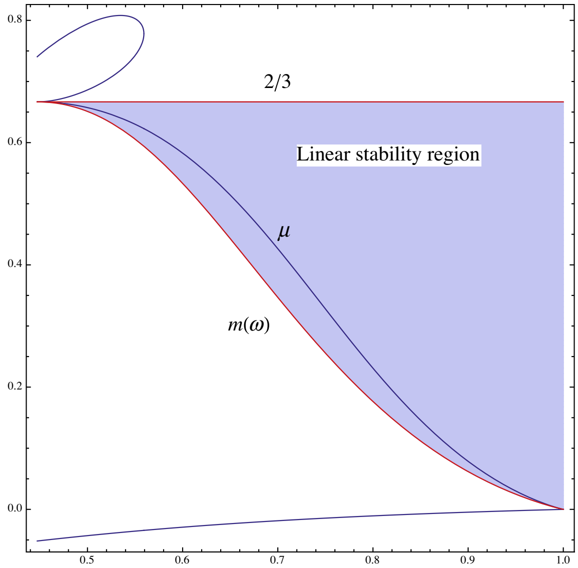

Fig. 4. Roots of the polynomial $p(1,b;\omega)$, in blue, and the slopes of the lines delimiting the linear stability region for E_2, namely $m(\omega)$ and $2/3$. Only one of the roots of $p(1,b;\omega)$ satisfies the condition $m(\omega) < \mu < 2/3$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

red the slopes of lines delimiting the stability region, as a function of ω, and in blue the slopes of the straight lines arising from the solutions of $p(1,b;\omega) = 0$. It is clear that only one solution verifies $m(\omega) < \mu(\omega) < 2/3$.

As a consequence, Arnold's theorem cannot be applied if $b = \mu(\omega)a$, with $\mu(\omega)$ the unique root of $p(1,b;\omega)$ satisfying condition (16). In addition, it can neither be applied if a resonance of order less or equal than four is satisfied. A resonance of third order takes place if the eigenvalues associated to E_2, given by (5) and (6), verify

$$\sqrt{M+\sqrt{N}} = 2\sqrt{M-\sqrt{N}}.$$

This equation is fulfilled if

$$b = \frac{25 - 18\omega^2 - 7\omega^4 \pm 5\sqrt{(9+55\omega^2)(1-\omega^2)^3}}{3(25 - 50\omega^2 + 89\omega^4)}a. \tag{17}$$

Analogously, a resonance of fourth order occurs if

$$b = \frac{25 - 32\omega^2 + 7\omega^4 \pm 10\sqrt{(4+5\omega^2)(1-\omega^2)^3}}{3(25 - 50\omega^2 + 61\omega^4)}a. \tag{18}$$

An analysis of the slopes of the straight lines above shows that only the lines with the plus sign lie inside the region of linear stability. Thus, we can summarize the Lyapunov stability of E_2 in the following result

Theorem 4.2. *The equilibrium point E_2 is Lyapunov stable if it is linear stable and*

$$b \neq \frac{25 - 18\omega^2 - 7\omega^4 + 5\sqrt{(9+55\omega^2)(1-\omega^2)^3}}{3(25 - 50\omega^2 + 89\omega^4)}a,$$

$$b \neq \frac{25 - 32\omega^2 + 7\omega^4 + 10\sqrt{(4+5\omega^2)(1-\omega^2)^3}}{3(25 - 50\omega^2 + 61\omega^4)}a,$$

and $b \neq \mu(\omega)a$, with $\mu(\omega)$ the unique root of $p(1,b;\omega)$ satisfying (16).

5. Third and fourth order resonances

Theorems 4.1 and 4.2 fail to give stability conditions for E_1 and E_2 when the corresponding eigenvalues satisfy a resonance of third or fourth order. In such cases, Birkhoff's normal form is no longer as in (10). Indeed, for a third order resonance $(\omega_1 = 2\omega_2)$ the normal form, around an equilibrium position, reads as

$$\mathcal{H} = 2\omega_2 I_1 - \omega_2 I_2 + \mathcal{H}_3(I_1, I_2, \theta_1 + 2\theta_2) + O(I^2)$$

where $\mathcal{H}_3(I_1, I_2, \theta_1 + 2\theta_2)$ is a homogeneous polynomial of third degree in $I_1^{1/2}$ and $I_2^{1/2}$, whose coefficients are finite Fourier series in the angle $\theta_1 + 2\theta_2$.

For the case of a fourth order resonance ($\omega_1 = 3\omega_2$), the normal form can be expressed as

$$\mathcal{H} = 3\omega_2 I_1 - \omega_2 I_2 + \mathcal{H}_4(I_1, I_2, \theta_1 + 3\theta_2) + O(I^{5/2}),$$

$\mathcal{H}_4(I_1, I_2, \theta_1 + 3\theta_2)$ being a homogeneous polynomial of second degree in I_1 and I_2, whose coefficients are finite Fourier series in the angle $\theta_1 + 3\theta_2$.

Markeev [15] established stability conditions for these resonances, that were generalized by Cabral and Meyer [6] and Elipe et al. [7], including degenerate resonant cases of higher order. We can summarize the main result as follows.

Theorem 5.1. *Let be $\theta = \theta_1 + 2\theta_2$ in the case of a third order resonance and $\theta = \theta_1 + 3\theta_2$ in the case of a fourth order resonance. Then, the equilibrium point is Lyapunov stable if the function*

$$\Psi(\theta) = \mathcal{H}_{3,4}(\omega_2, \omega_1, \theta) \tag{19}$$

does not vanish. On the contrary, the equilibrium point is Lyapunov unstable if there exists θ_0 such that $\Psi(\theta_0) = 0$ and $\Psi'(\theta_0) \neq 0$.

Our goal is to apply Theorem 5.1 to determine the stability of E_1 and E_2 in the resonant cases. After computing the normal form for E_1, the function $\Psi(\theta)$ in the case of a third order resonance is, excepting for a constant factor,

$$\Psi(\theta) = (a + 3b)\cos\theta.$$

Thus, we can state the following stability result

Theorem 5.2. *In the presence of a 1:2 resonance, the equilibrium point E_1 is Lyapunov stable if and only if $a + 3b = 0$.*

Proof. The proof is straightforward. If $a + 3b \neq 0$ the function $\Psi(\theta)$ has simple zeroes and, by Theorem 5.1, the equilibrium point is Lyapunov unstable. On the other hand, if $a + 3b = 0$ the analysis of stability must be pushed to the next order of the normal form. By doing so, Cabral & Meyer's theorem [6] can be applied, and the stability of E_1 follows. □

It is worth noticing that, in general, a third order resonance is Lyapunov unstable. Stability is only achieved in a particular case to which the classical Hénon–Heiles system belongs. This situation is reflected in the different behavior of the orbits around the equilibrium point. In particular, in Fig. 5 it can be seen how the orbits look like near E_1 in the stable and unstable cases. In the stable case the orbits remain in a small neighborhood of E_1, while in the unstable case, even though they are bounded, they spread away from any arbitrary small vicinity of E_1.

When a fourth order resonance takes place, the function $\Psi(\theta)$ in Theorem 5.1, for E_1, takes the form

$$\Psi(\theta) = \alpha + \beta\cos\theta + \gamma\sin\theta, \tag{20}$$

where the following relations hold

$$\alpha^2 = \frac{\left(121a^2 + 122ab + 485b^2\right)^2}{6350400}, \qquad \beta^2 + \gamma^2 = \frac{(a + 3b)^2(3a + 7b)^2}{1200}. \tag{21}$$

Now, it is possible to establish another stability result.

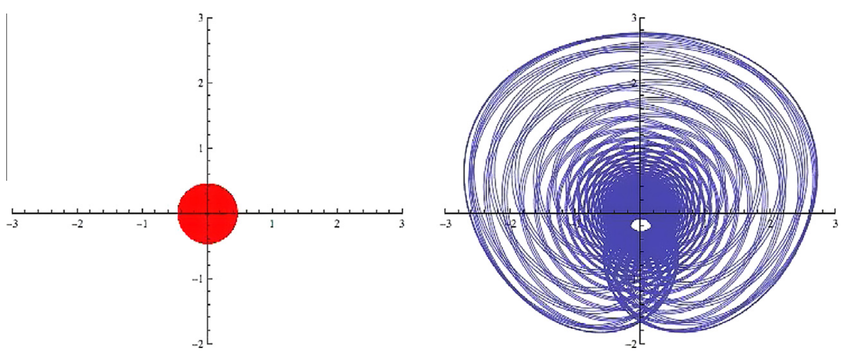

Fig. 5. Orbits around E_1 projected onto the plane xy for the stable case (left) and for the unstable one (right) in the presence of a 1:2 resonance. The orbits have been obtained by numerical integration.

Theorem 5.3. *Let m_1 and m_2 be given by*

$$m_{1,2} = \frac{61 - 336\sqrt{3} \pm 2\sqrt{14\left(2265\sqrt{3} - 887\right)}}{882\sqrt{3} - 485},$$

and let us assume that a 1:3 resonance takes place. Then

(i) *E_1 is Lyapunov stable if $m_2 a < b < m_1 a$.*
(ii) *E_1 is Lyapunov unstable if $b < m_2 a$, or $b > m_1 a$.*

Proof. From Eq. (20) and Theorem 5.1, it follows that E_1 is a Lyapunov stable equilibrium if the inequality

$$\alpha^2 > \beta^2 + \gamma^2$$

holds. If the inequality is satisfied in the opposite direction, E_1 is Lyapunov unstable. The limiting case that divides stability and instability occurs when the equality takes place. From (21), this happens if and only if $b = m_{1,2}\, a$ and the result follows. □

As it happened in the case of the 1:2 resonance, for a and b corresponding to the classical Hénon–Heiles system, the equilibrium point E_1 is Lyapunov stable. In Fig. 6 the region of stability in the parameter plane ab is depicted, where it can be seen that the case of the classical Hénon–Heiles (dashed line) lies inside the region of Lyapunov stability. Taking this into account, as well as the results in Theorems 4.1 and 5.2, we can conclude that E_1 is a superstable equilibrium for the classical case, as it is Lyapunov stable, regardless of the value of ω.

Now, we proceed to analyze the Lyapunov stability of E_2 in the presence of the resonances 1:2 and 1:3. We recall that E_2 is a maximum of the effective potential when $\omega < 1$ and $2a - 3b > 0$. Moreover, it is linearly stable when $1/\sqrt{5} < \omega < 1, b > 0$ and

$$m(\omega)a < b < \frac{2}{3}a,$$

where $m(\omega)$ is given by (9).

After computing the corresponding normal form for the 1:2 resonance, we obtain for the function $\Psi(\theta)$

$$\Psi(\theta) = \frac{\left(2\sqrt{3b(2a-3b)} + 3(a+b)\right)\sqrt{2 + \sqrt{\frac{2a-3b}{3b}}}}{\left(6\sqrt{3}(1-\omega^2)\sqrt{\frac{2a-3b}{b}}\right)^{3/4}} \cos\theta.$$

It is easy to check that the numerator of $\Psi(\theta)$ cannot vanish for any value of a and b. Thus we arrive to the following result

Theorem 5.4. *In the presence of a 1:2 resonance, the equilibrium point E_2 is always Lyapunov unstable.*

Proof. The function $\Psi(\theta)$ has simple roots and, by Theorem 5.1, the equilibrium point is Lyapunov unstable. □

For the case of the resonance 1:3, the function $\Psi(\theta)$ has the expression given in Eq. (20). Then, the stability properties of E_2 are deduced from the sign of $\gamma_3 = \alpha^2 - (\beta^2 + \gamma^2)$. After some cumbersome algebra we obtain

$$\gamma_3 = -\frac{b\left(54\sqrt{\frac{3(2a-3b)}{b}}\, b\, p(a,b) + 5\, q(a,b)\right)}{15052800(1-w^2)^4(2a-3b)^3}, \tag{22}$$

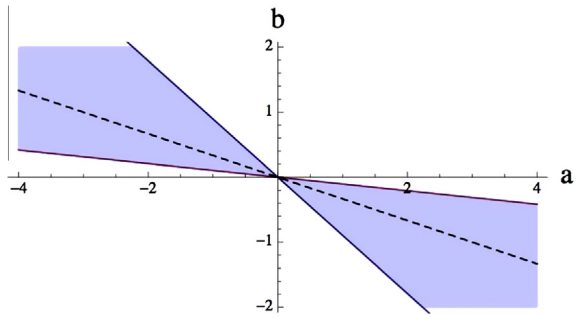

Fig. 6. Lyapunov stability region for the equilibrium point E_1 in the case of a 1:3 resonance. The dashed line corresponds to the classical Hénon–Heiles system, inside the stability region.

where

$$p(a,b) = 709569a^5 + 9426632a^4b + 3367462a^3b^2 - 34507476a^2b^3 - 3804975ab^4 + 22050900b^5,$$

and

$$q(a,b) = 1323135a^6 + 99066294a^5b + 212989291a^4b^2 - 555254916a^3b^3 - 1023722739a^2b^4 + 1848639294ab^5 - 520503975b^6.$$

Taking into account (22) we can state the following result

Theorem 5.5. *In the presence of a 1:3 resonance, the equilibrium E_2 is Lyapunov stable if*

$$\frac{1}{\sqrt{5}} < \omega < 0.5438736060079052\ldots.$$

On the contrary, E_2 is Lyapunov unstable if

$$0.5438736060079052\ldots < \omega < 1.$$

Proof. We note that the sign of γ_3 is the sign of the numerator of the expression given in Eq. (22). Moreover, the numerator changes its sign when it is equal to zero. This happens if

$$s(a,b) = 54^2 3\, b\,(2a - 3b)\, p(a,b)^2 - 25\, q(a,b)^2 = 0,$$

which is a homogeneous equation of twelfth degree in a and b. As a consequence, the solutions are straight lines of the form $b = r_k\, a$, where r_k is a real root of the polynomial $s(1,b)$. Computing the roots of $s(1,b)$, we find two real roots whose values are approximately

$$r_1 \approx 0.038506249611936064, \quad r_2 \approx 0.6427508461205317.$$

However, only the second one is a solution of $\gamma_3 = 0$. Taking into account the value of r_2 and the resonance condition (18), we obtain the value $\omega = 0.5438736060079052\ldots$, dividing the stable and unstable cases. In Fig. 7 it can be seen the change of sign of γ_3 in the interval $(1/\sqrt{5}, 1)$, when ω crosses the critical value. □

6. Higher order resonances

Higher order resonances must be taken into account in the case that the non degeneracy condition (11) is not satisfied. The first resonances to be analyzed are those of order less than or equal to six. In this way, we find two resonances of fifth order, 1:4 and 2:3, and the resonance of sixth order, 1:5.

For the case of the equilibrium point E_1 it is easy to compute the values of ω for such resonances, provided that $\omega_1 = \omega + 1$ and $\omega_2 = \omega - 1$. Thus, we obtain for the resonances 1:4, 2:3 and 1:5 values of ω equal to 5/3, 5 and 3/2 respectively. All of these values are located outside the interval $[\kappa, 3)$ given in Teorem 4.1 and, consequently, E_1 is stable for these resonant cases. Moreover, if we push forward the normal form up to sixth order for the degenerate cases of Theorem 4.1, that is to say

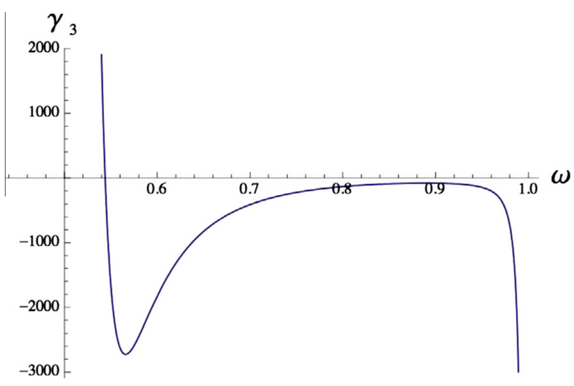

Fig. 7. The plot of γ_3, the coefficient in the normal form that determines the stability of the equilibrium point, as a function of ω. A positive sign implies stability for E_2, while a negative one implies instability. The sign changes when $\omega = 0.543873606 \pm 10^{-11}$.

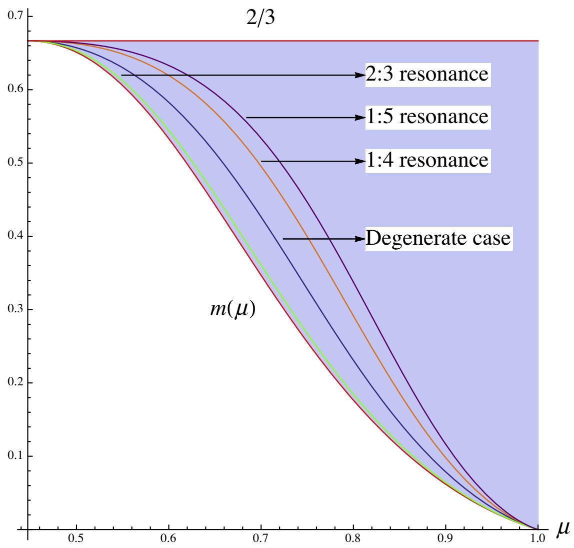

Fig. 8. The slope of the resonance lines 1:4, 2:3 and 1:5 and the slope of the degenerate case arising from Theorem 4.2. The four curves coincide at $\omega = 1/\sqrt{5}$ and $\omega = 1$, but not at any other point.

$$\omega \in [\kappa, 3), \quad b = m_{1,2}(\omega)a,$$

where $m_{1,2}(\omega)$ are given by (13), we are able to prove that E_1 is Lyapunov stable. As a consequence, the only cases of instability are those reported in Theorems 5.2 and 5.3.

The same can be done for the equilibrium point E_2. We can obtain the resonance curves for the three cases 1:4, 2:3 and 1:5. After some manipulations we arrive at

$$\text{Resonance } 1:4 \quad b = \frac{289 - 450\omega^2 + 161\omega^4 + 17\sqrt{(1-\omega^2)^3(31\omega^2 + 225)}}{867 - 1734\omega^2 + 1635\omega^4}a.$$

$$\text{Resonance } 2:3 \quad b = \frac{169 - 50\omega^2 - 119\omega^4 + 13\sqrt{(1-\omega^2)^3(551\omega^2 + 25)}}{507 - 1014\omega^2 + 2235\omega^4}a.$$

$$\text{Resonance } 1:5 \quad b = \frac{169 - 288\omega^2 + 119\omega^4 + 26\sqrt{(1-\omega^2)^3(36 - 11\omega^2)}}{507 - 1014\omega^2 + 807\omega^4}a.$$

The equilibrium E_2 will be stable for one of these three resonances if Theorem 4.2 is satisfied. That is, the slopes of the lines defining the resonances are not roots of the polynomial $p(1, b; \omega)$, given in (14), in the interval $(m(\omega), 2/3)$, for $\omega \in (1/\sqrt{5}, 1)$. Substituting the value of the slopes in $p(1, b; \omega)$ we arrive at a polynomial equation in ω with no roots in $(m(\omega), 2/3)$ if $\omega \in (1/\sqrt{5}, 1)$. In fact, the slopes match the polynomial equation at the limiting values $\omega = 1$ and $\omega = 1/\sqrt{5}$. This can be seen in Fig. 8, where the slopes of the resonance lines and the slope of the degenerate case, $b = \mu(\omega)a$, are depicted. As a consequence, E_2 is Lyapunov stable if a resonance of order fifth or sixth takes place.

7. Conclusions

In this paper we have studied the Lyapunov stability of the equilibrium points of a generalized Hénon–Heiles system in a rotating frame. We have shown the existence of equilibria similar to those appearing in the restricted three body problem, corresponding to maxima of the effective potential. For these points an exhaustive analysis covering almost all possible values of the parameters has been made. In particular, if the rotating frequency is greater than unity, we have proven that the origin is always a Lyapunov stable equilibrium, if the parameters a and b satisfy the relation of the classical case, that is $b = -a/3$. In this sense the origin can be regarded as a superstable equilibrium for the classical system.

Acknowledgements

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