CLOSED SIMPLICIAL MODEL STRUCTURES FOR EXTERIOR AND PROPER HOMOTOPY THEORY.

J. M. GARCÍA-CALCINES, M. GARCÍA-PINILLOS
AND L.J. HERNÁNDEZ-PARICIO

Abstract. The notion of exterior space consists of a topological space together with a certain nonempty family of open subsets that is thought of as a 'system of open neighborhoods at infinity' while an exterior map is a continuous map which is 'continuous at infinity'. The category of spaces and proper maps is a subcategory of the category of exterior spaces.

In this paper we show that the category of exterior spaces has a family of closed simplicial model structures, in the sense of Quillen, depending on a pair $\{T, T'\}$ of suitable exterior spaces. For this goal, for a given exterior space T , we construct the exterior T-homotopy groups of an exterior space under T. Using different spaces T we have as particular cases the main proper homotopy groups: the Brown-Grossman, Čerin-Steenrod, p-cylindrical, Baues-Quintero and Farrell-Taylor-Wagoner groups, as well as the standard (Hurewicz) homotopy groups.

The existence of this model structure in the category of exterior spaces has interesting applications. For instance, using different pairs $\{T, T'\}$, it is possible to study the standard homotopy type, the homotopy type at infinity and the global proper homotopy type.

Introduction

As it is well known, one of the main applications of proper homotopy theory is the study of non-compact spaces. One of the first proper homotopy invariants was the notion of ideal point of a surface, introduced in 1923 by B. Kérékjárto, in order to give the classification of non-compact surfaces. In 1931, H. Freudenthal [12] generalized this notion by defining the end point of a space. Later, in 1965, L. Siebenmann [27] analyzed the obstruction to finding a boundary for an open manifold in dimension greater than five. In 1970 he proposed that, for the study of non-compact spaces, the homotopy hypothesis should be given in the category of spaces and proper maps [28].

E.M. Brown [7] gave a notion of n-th proper homotopy group associated with a non-compact space and a Freudenthal end, represented by a base-ray $\alpha : [0, +\infty) \to X$. He constructed this group by taking the set of all proper homotopy classes relative to $[0, +\infty)$ of germs of proper maps from $\underline{S}^n$ to X, where $\underline{S}^n$ is obtained by attaching an n-dimensional sphere to every integer in $[0, +\infty)$. W. Grossman [18] also constructed homotopy groups of pro-spaces using the corresponding analogues in the new context.

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In order to prove a proper Whitehead theorem, in 1973, F.T. Farrell, L.R. Taylor and J.B. Wagoner [13] introduced the Δ-homotopy groups of a locally compact CW-complex X with a set of base points. Their result can be applied in the finite dimensional case without the σ-compactness condition.

Z. Čerin [8], in 1980, gave a different notion of proper homotopy group (Čerin-Steenrod group) as the set of proper homotopy classes of base-ray preserving proper maps from $S^n \times [0, +\infty)$ to X. He also studied some relationships with the Brown's proper homotopy groups and the Quigley's shape groups [23]. Later, in 1988, L.J. Hernández and T. Porter [19] gave a modified notion of these groups and a global version of the Brown-Grossman groups.

We can also cite the cylindrical p-homotopy groups, given by R. Ayala, E. Domínguez and A. Quintero [2], [3]. These groups are proper homotopy invariants associated to a given finite set of proper ends, represented by a suitable tree. Later, H. J. Baues and A. Quintero [5], using trees which need not have a finite number of ends, constructed a more general version of proper homotopy groups.

As we have mentioned, the category of spaces and proper maps, $\mathbf{P}$, is a nice framework for proper homotopy theory. However, we cannot develop some homotopic constructions in this category, such as loop spaces or homotopy fibers, since there are few limits and colimits. A useful technique which avoids this problem is to embed the proper category into a complete and co-complete category and to use homotopy theories that assume the existence of limits and colimits. For example, we have the Edwards-Hastings embedding [10] of the proper homotopy category of locally compact, σ-compact Hausdorff spaces into the homotopy category of pro-spaces. One of the disadvantages of this embedding is that one has to restrict to locally compact σ-compact spaces. On the other hand, the homotopy constructions produce pro-spaces that many times cannot be geometrically interpreted as a space.

An alternative embedding can be found in [16]. The notion of *exterior space* is introduced in such a way that the category of exterior spaces, $\mathbf{E}$, is complete and co-complete and the proper category can be considered as a full subcategory. Roughly speaking, an exterior space is a topological space with a 'neighborhood system at infinity' which we call *externology*, while an exterior map is a continuous map which is 'continuous at infinity'. An important role is played by the family ε_{cc}^X of the complements of closed-compact subsets of X, also called the *co-compact externology* of X. This gives rise to the mentioned full embedding $e : \mathbf{P} \to \mathbf{E}$.

In this paper, we firstly introduce the notion of *exterior homotopy T-group*, where T is a suitable space provided with its co-compact externology. If X is an exterior space and $\rho : T \to X$ is an exterior map, we construct the *q-th exterior homotopy T-group*, $\pi_q^T(X, \rho)$, by taking the set of exterior homotopy classes relative to T of exterior maps from $T \bar{\times} S^q$ to X, where $T \bar{\times} S^q$ is the product space $T \times S^q$ provided with its co-compact externology.

One important advantage of this approach is that we have a unified theory for many different homotopy groups, depending on the choice of the exterior space T we obtain the following:

- If $T = \mathbb{N}$ is the discrete space of natural numbers with the co-finite externology then we have a global version of Brown-Grossman groups.
- When $T = \mathbb{R}^+$ is the half line with the co-compact externology, we obtain the Čerin-Steenrod groups.

- In the case $T = P$, the one-point space with its co-compact externology, the standard Hurewicz groups are obtained.
- Using contractible one dimensional CW-complexes one has the p-cylindrical and the Baues-Quintero homotopy groups.
- Taking T as a set of base points we find the Δ-groups constructed by F.T. Farrel, L.R. Taylor and J.B. Wagoner.

In 1967, D. Quillen (see [24], [25]) introduced the notion of closed model category, which consists of a category $\mathbf{C}$ together with three distinguished classes of morphisms called fibrations, cofibrations and weak equivalences satisfying certain axioms that provide sufficient conditions to develop a homotopy theory. It is one of the best known approaches of axiomatic homotopy theory and reduces the study of various invariants to checking Quillen's axioms. The closed model axioms have interesting basic consequences; for example, an expanded notion of homotopy and a Whitehead theorem. The associated homotopy category is defined as being the result of formally inverting the weak equivalences of the closed model category.

We proved in [16] that $\mathbf{E}$ has a closed simplicial model category taking, as weak equivalences, the morphisms which induce isomorphisms with respect to the Brown-Grossman homotopy groups. However, when we consider the full embedding $e : \mathbf{P} \to \mathbf{E}$, this structure cannot distinguish between two different compact spaces. To avoid this problem, we construct in this paper, a new closed simplicial model structure associated to every pair $\{T, T'\}$ of adequate exterior spaces. For instance, taking $T = \mathbb{N}$ and $T' = P$, we have a new strong notion of weak equivalence which is able to distinguish the homotopy type of compact spaces.

One of the more important result of this paper is the existence of the following closed model structure associted with a pair of Hausdorff, locallly compact, σ-compact spaces T, T' with the co-compact externology:

Theorem. The category $\mathbf{E}$, together with the classes of exterior $\{T, T'\}$-fibrations, exterior $\{T, T'\}$-cofibrations and weak exterior $\{T, T'\}$-equivalences and its simplicial structure, is a closed simplicial model category.

We have also compared the different localization categories for some pairs of spaces. For instance we have analysed the relations with the standard homotopy category and the localization category given in [16] .

As an application of the existence of this model structure associated to a pair $\{T, T'\}$ we obtain a version of the Whitehead theorem in the category of exterior spaces which involves the exterior T-homotopy groups and the exterior T'-homotopy groups. In order to give this theorem, we introduce the notion of $\{T, T'\}$-complex. This is an exterior space with a filtration $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset ... \subset X_n \subset ... \subset X$ such that X is the colimit of the filtration and for $n \geq 0$, X_n is obtained from X_{n-1} by attaching T-cells and T'-cells, which can be non-compact cells. Depending on the choice of T and T' we obtain as particular cases the notion of CW-complex (provided with its topology as externology), $\mathbb{N}$-*complex* [16] and *bi-complex* [15].

We have obtained as corollaries some versions of the Whitehead theorem. When $T = P$ and $T' = \emptyset$ we have the standard Whitehead theorem. If $T = \mathbb{N}$ and $T' = \emptyset$ then we have a proper Whitehead theorem for finite dimensional CW-complexes having on each dimension k either no k-cells or an infinite countable number of k-cells. Finally, if $T = \mathbb{N}$ and $T' = P$ we obtain a proper Whitehead theorem for finite dimensional strongly locally finite CW-complexes.

1. Preliminaries

1.1. **Closed simplicial model category structure.** Axiomatic homotopy theory is the development of the basic constructions of homotopy theory in an abstract setting, so that they may be applied to other categories. The best known approach is that of Quillen who introduces the notion of a (closed) model category.

Given a commutative solid arrow diagram in a category $\mathbf{C}$:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "v"'] \arrow[ur, dotted] & Y, \end{tikzcd}$$

it is said that i has the left lifting property (LLP) with respect to p and p is said to have the right lifting property (RLP) with respect to i if there exists a map $h : B \to X$ such that $hi = u$ and $ph = v$.

A closed model category is a category $\mathbf{C}$ endowed with three distinguished classes of morphisms called *cofibrations*, *fibrations* and *weak equivalences*, satisfying CM1-CM5 axioms (see [25]). An equivalent but different formulation was given in [24].

Given a closed model category $\mathbf{C}$, the *homotopy category* $\mathbf{Ho(C)}$, is obtained from $\mathbf{C}$ by formally inverting all the weak equivalences (see [14] and [24]).

The initial object of $\mathbf{C}$ is denoted by $\emptyset$, and the final object by $*$. An object X is said to be *cofibrant*, if the unique morphism $\emptyset \to X$ is a cofibration; dually X is called *fibrant* if $X \to *$ is a fibration. We denote by $\mathbf{C_{cof}}$ and $\mathbf{C_{fib}}$ the full subcategories of $\mathbf{C}$ determined by cofibrant objects and fibrant objects, respectively.

Example 1. *We consider the category* $\mathbf{SS}$ *of simplicial sets. It is well known that it is a closed model category with the following structure: a simplicial map* $f : X \to Y$ *is a fibration (resp. trivial fibration) if it has the RLP with respect to* $V(n,k) \hookrightarrow \Delta[n]$*, for* $0 \leq k \leq n$ *and* $n > 0$ *(resp. to* $\dot{\Delta}[n] \hookrightarrow \Delta[n]$*, for* $n \geq 0$*), where* $V(n,k)$ *is the simplicial subset generated by the i-faces,* $i \neq k$*, of the standard n-simplex* $\Delta[n]$*;* $\dot{\Delta}[n]$ *is generated by all the faces of* $\Delta[n]$*. A simplicial map* $i : A \to B$ *is a cofibration (resp. trivial cofibration) if it has the LLP with respect to trivial fibrations (resp. fibrations). Finally, a weak equivalence is a simplicial map that can factored as a trivial cofibration followed by a trivial fibration.*

In order to introduce the notion of simplicial category, if X and K are simplicial sets we shall denote by X^K the simplicial set given by $(X^K)_q = \mathrm{Hom}_{\mathbf{SS}}(K \times \Delta[q], X)$.

A *simplicial category* is a category $\mathbf{C}$ endowed with a functor $\underline{\mathrm{Hom}}_{\mathbf{C}} : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{S}S$, satisfying the conditions given in [24], pages 1.1 and 1.2; in particular we have that $\underline{\mathrm{Hom}}_{\mathbf{C}}(X,Y)_0 \cong Hom(X,Y)$. Associated with a simplicial category $\mathbf{C}$, we have the category $\pi_{\mathbf{0}}(\mathbf{C})$, which has the same objects as $\mathbf{C}$ and the hom-set defined by $Hom_{\pi_{\mathbf{0}}(\mathbf{C})}(X,Y) = \pi_0 \underline{Hom}_{\mathbf{C}}(X,Y)$, where $\pi_0 \underline{Hom}_{\mathbf{C}}(X,Y)$ is the set of connected components of the simplicial set $\underline{Hom}_{\mathbf{C}}(X,Y)$.

A *closed simplicial model category* is a closed model category $\mathbf{C}$ which is also a simplicial category and satisfies SM0 and SM7 axioms ([24], page 2.2).

Example 2. *Let* $\mathbf{Top}$ *denote the category of topological spaces and continuous maps. A map* $f : X \to Y$ *in* $\mathbf{Top}$ *will be called a fibration if it is a fiber map in the sense of Serre, and a weak equivalence if it is a weak homotopy equivalence* *(i.e. $\pi_q(X,x) \xrightarrow{\cong} \pi_q(Y,f(x))$, for all $x \in X$ and $q \geq 0$). A map will be called cofibration it it has the LLP with respect to all trivial fibrations. On the other hand, if X and Y are spaces, consider the function complex $\underline{\mathrm{Hom}}_{\mathbf{Top}}(X,Y)$ given by $\underline{\mathrm{Hom}}_{\mathbf{Top}}(X,Y)_n = \mathrm{Hom}_{\mathbf{Top}}(X\times|\Delta[n]|,Y)$ with natural simplicial operations, where $|.|$ denotes geometric realization. If $f \in \underline{\mathrm{Hom}}_{\mathbf{Top}}(X,Y)_n$ and $g \in \underline{\mathrm{Hom}}_{\mathbf{Top}}(Y,Z)_n$, let $g \circ f$ be the composite*

$$X \times |\Delta[n]| \xrightarrow{id_X \times \Delta} X \times |\Delta[n]| \times |\Delta[n]| \xrightarrow{f \times id_{|\Delta[n]|}} Y \times |\Delta[n]| \xrightarrow{g} Z.$$

Considering $X \otimes K = X \times |K|$ and $X^K = X^{|K|}$, Quillen proved that **Top***, with this structure, is a closed simplicial model category.*

Since the notion of closed model category was introduced by Quillen, these models have been used and studied by many authors. For a survey, a monumental pre-print and a book on these structures we refer the reader to [9] , [20] , and [17], respectively.

1.2. **The category of exterior spaces.** A continuous map $f : X \to Y$ is said to be *proper* if $f^{-1}(K)$ is a compact subset of X, for every closed compact subset K of Y. The category of spaces and proper maps is very useful for the study of non-compact spaces (as well as the corresponding proper homotopy category) but it has not enough limits and colimits. In order to obtain a solution of this problem it was defined in [16] the notion of *exterior space*. An *exterior space* $(X, \varepsilon \subset \tau)$ consists of a space (X,τ) together with a nonempty collection ε of open subsets, called *externology*, satisfying:

E1: If E_1, $E_2 \in \varepsilon$ then $E_1 \cap E_2 \in \varepsilon$,
E2: if $E \in \varepsilon$, $U \in \tau$ and $E \subset U$ then $U \in \varepsilon$.

An open E which is in ε is said to be an *exterior-open* subset, or in short, an *e-open* subset. A map $f : (X, \varepsilon \subset \tau) \to (X', \varepsilon' \subset \tau')$ is said to be *exterior* if it is continuous and $f^{-1}(E) \in \varepsilon$, for all $E \in \varepsilon'$.

The category of exterior spaces and maps, **E**, is complete and co-complete. One interesting property of this category is the existence of a full embedding $e : \mathbf{P} \to \mathbf{E}$, which carries a space X to the exterior space X_e provided with the topology of X and the externology ε^X_{cc}, the complements of closed-compact subsets of X (also called the *co-compact* externology of X. A proper map $f : X \to Y$ is carried to the exterior map $f_e : X_e \to Y_e$ given by $f_e = f$. In this way, we can think that the category of exterior spaces 'contains as a full subcategory' the category of spaces and proper maps, **P** .

We consider the following three functorial constructions.

- If X, Z are exterior spaces, we set $Z^X = \mathrm{Hom}_{\mathbf{E}}(X,Z)$ with the topology generated by the subsets of the form:

$$(K,U) = \{\alpha \in Z^X : \alpha(K) \subset U\},$$

$$(L,E) = \{\alpha \in Z^X : \alpha(L) \subset E\},$$

where $K \subset X$ is a compact subset, $U \subset Z$ is an open subset, $L \subset X$ is an e-compact subset (i.e. $L - E$ is compact, for all E e-open subset) and $E \subset Z$ an e-open subset.

- If X is an exterior space and Y a topological space let $X \times Y$ be the product space. We consider on $X \times Y$ the externology given by those open subsets $E \in \varepsilon_{X\times Y}$ such that for each $y \in Y$ there exists an open neighbourhood of y, U_y, and $E_y \in \varepsilon_X$ such that $E_y \times U_y \subset E$. This exterior space is denoted by $X \bar{\times} Y$.

  When Y is a compact space then E is an e-open subset if and only if it is an open subset and there exists $G \in \varepsilon_X$ such that $G \times Y \subset E$. If Y is a compact space and $\varepsilon_X = \varepsilon_{cc}^X$ then $\varepsilon_{X \bar{\times} Y}$ is the co-compact externology of $X \times Y$.
- Let Y be a topological space and Z be an exterior space. We consider on $Z^Y = \mathrm{Hom}_{\mathbf{Top}}(Y, Z)$ the compact-open topology and the externology given by the open subsets E of Z^Y such that E contains a subset of the form (K, G), where K is a compact subset of Y and G is an e-open subset of Z.

Theorem 1. *Let X, Z be exterior spaces and Y a topological space, then*

(i) If X is a Hausdorff, locally compact space and $\varepsilon_X = \varepsilon_{cc}^X$ there is a natural bijection

$$\mathrm{Hom}_{\mathbf{E}}(X \bar{\times} Y, Z) \cong \mathrm{Hom}_{\mathbf{Top}}(Y, Z^X).$$

(ii) If Y is a locally compact space there is a natural bijection

$$\mathrm{Hom}_{\mathbf{E}}(X \bar{\times} Y, Z) \cong \mathrm{Hom}_{\mathbf{E}}(X, Z^Y).$$

Remark 1. *Let $U : \mathbf{E} \to \mathbf{Top}$ be the forgetful functor and $V : \mathbf{Top} \to \mathbf{E}$ be the functor which carries a space X to the exterior space provided with its own topology as externology, $\varepsilon_X = \tau_X$. One can easily prove that $V \cong P \bar{\times} \_$ and $U \cong (\_)^P$, where P denotes the one point space provided with the co-compact externology (note that $P \ncong *$). Hence, by (i), U and V are adjoint functors.*

For a proof of theorem above and other properties of exterior spaces, we refer the reader to [16] .

2. T-homotopy groups

Along this section T will be a fixed Hausdorff, locally compact space provided with the co-compact externology. Let $\mathbf{E}^T$ be the category of exterior spaces under T. The objects in this category, $\rho : T \to X$, will be denoted by (X, ρ), and the morphisms by $f : (X, \rho_X) \to (Y, \rho_Y)$.

Definition 1. Let $f, g : (X, \rho_X) \to (Y, \rho_Y))$ be morphisms in $\mathbf{E}^T$. f is said to be *e-homotopic* to g *relative to* T, if there is an exterior map $F : X \bar{\times} I \to Y$ such that $F(x, 0) = f(x)$, $F(x, 1) = g(x)$ and $F(\rho_X(\lambda), t) = \rho_Y(\lambda)$, for all $x \in X$, $\lambda \in T$ and $t \in I$.

The set of exterior homotopy classes relative to T will be denoted by

$$[(X, \rho_X), (Y, \rho_Y)]^T.$$

When $T = \emptyset$ we obtain the notion of non-relative e-homotopy. From theorem 1, part (i), we obtain the natural bijections:

$$\mathrm{Hom}_{\mathbf{E}}(T \bar{\times} S^q, X) \cong \mathrm{Hom}_{\mathbf{Top}}(S^q, X^T),$$

$$\mathrm{Hom}_{\mathbf{E}}(T \bar{\times} (S^q \times I), X) \cong \mathrm{Hom}_{\mathbf{Top}}(S^q \times I, X^T),$$

where S^q denotes the q-dimensional pointed sphere (for $q = -1$ take $S^{-1} = \emptyset$). They induce, for any (X, ρ) in $\mathbf{E^T}$, another natural bijection

$$[(T\bar{\times}S^q, id_T\bar{\times}*), (X, \rho)]^T \cong [(S^q, *), (X^T, \rho)],$$

where the second member is the standard set of pointed homotopy classes and $id_T\bar{\times}* : T \to T\bar{\times}S^q$ is given by $i(d_T\bar{\times}*)(\lambda) = (\lambda, *)$. Therefore, $[(T\bar{\times}S^q, id_T\bar{\times}*), (X, \rho)]^T$ has the structure of a group for $q \geq 1$ which is abelian for $q \geq 2$. In the case $q = 0$ one has a pointed set.

Definition 2. Let (X, ρ) be an exterior space under T. For $q \geq 0$ the *q-th exterior homotopy T-group functor* of (X, ρ) is given by

$$\pi_q^T(X, \rho) = [(T\bar{\times}S^q, id_T\bar{\times}*), (X, \rho)]^T.$$

Depending on the choice of the exterior space T we obtain different homotopy groups:

(1) $T = \mathbb{N}$. In this case $\pi_q^{\mathbb{N}}(X, \rho)$ is called the *q-th Brown-Grossman exterior homotopy group* of (X, ρ), and it is also denoted by $\pi_q^B(X, \rho)$. If X is an object in $\mathbf{P}^{\mathbb{N}} \subset \mathbf{E}^{\mathbb{N}}$, then the homotopy groups, $\pi_q^B(X, \rho)$, are the global version of Brown's proper homotopy groups [19]. The differences from Brown's groups are that we are using proper maps instead of germs of proper maps and we consider a base sequence instead of a base-ray.

(2) $T = \mathbb{R}_+ = [0, +\infty)$. $\pi_q^{\mathbb{R}_+}(X, \rho)$ is called the *q-th Čerin-Steenrod exterior homotopy group* of (X, ρ). It is also denoted by $\pi_q^S(X, \rho)$. As in (1), it represents an extension to the category of exterior spaces of the Čerin-Steenrod homotopy groups given for the proper category.

(3) $T = P$. Taking into account that $X^P \cong U(X)$, where $U : \mathbf{E} \to \mathbf{Top}$ is the forgetful functor, we obtain that q-th exterior homotopy groups of a based exterior space (X, x_0) , $\pi_q^{\mathbb{P}}(X, x_0)$, is the q-th Hurewicz homotopy groups $\pi_q(U(X), x)$, $q \geq 0$.

(4) Given a natural number r, $T[r]$ is the space obtained by identifying the origins of r copies of the half line $[0, +\infty)$. We consider $T = T[r, k] = T[r] \times \mathbb{R}^k$. Then we have the exterior homotopy $T[r, k]$-groups, $\pi_q^{T[r,k]}(X, u)$, where $u : T[r, k] \to X$ is an exterior map. If X is in $\mathbf{P^{T[r,k]}} \subset \mathbf{E^{T[r,k]}}$, then these groups are exactly the cylindrical p-homotopy groups, see [2] and [3].

(5) Let T be a contractible locally finite 1-dimensional simplicial complex, T^0 be its 0-skeleton and let E be a countable set. H.J. Baues and A. Quintero [5], using a finite-to-one function $\epsilon : E \to T^0$, construct the spherical object S_ϵ^n by attaching n-dimensional spheres S^n to the vertexes of T. For a space (X, α) under T, they consider the set of relative homotopy classes $[S_\epsilon^n, X]^T$. It is not difficult to see that, taking the composite

$$\rho_\epsilon : E \xrightarrow{\epsilon} T^0 \subset T \xrightarrow{\alpha} X$$

there is a canonical isomorphism $[S_\epsilon^n, X]^T \cong \pi_n^E(X, \rho_\epsilon)$, where in E we consider the discrete topology and the co-finite externology and in X the co-compact externology.

(6) Let X be a locally finite CW-complex and let in $: D \subset X$ be the inclusion of a subset D of X which satisfies that for any compact subspace K of X each non-compact component of $X - K$ contains some point of D and any subset $D' \subset D$ satisfying this condition has the cardinality of D. In this case the Δ-homotopy

group $\Delta(X, D; \pi_k)$ introduced by F.T. Farrell, L.R. Taylor and J.B. Wagoner [13] is canonically isomorphic to $\pi_k^D(X, \mathrm{in})$, where D is provided with the discrete topology and the co-finite externology and X has the co-compact externology.

3. The {T,T'}-closed simplicial model category structure for $\mathbf{E}$

In our earlier paper [16], we proved that $\mathbf{E}$ have a closed simplicial model category structure involving the Brown-Grossman exterior homotopy groups. However, this structure can not distinguish compact spaces when we consider the full embedding $e : \mathbf{P} \to \mathbf{E}$ since there is not any exterior map $\mathbb{N} \to X_e$ when X is compact. A modification of the existing structure avoids this problem. However, this modification is a particular case of a more general setting: a closed simplicial model structure associated with two exterior spaces T, T' that, for some particular pairs, gives interesting closed model categories.

Along this section, T, T' will denote two Hausdorff, locally compact, σ-compact spaces provided with their co-compact externology.

Definition 3. Let $f : X \to Y$ be an exterior map. We say that f is a *weak exterior* $\{T, T'\}$*-equivalence* (respectively, *exterior* $\{T, T'\}$*-fibration*) if $f^T : X^T \to Y^T$ and $f^{T'} : X^{T'} \to Y^{T'}$ are weak equivalences (respectively, fibrations) in **Top**.

It is easy to check that f is a weak exterior $\{T, T'\}$-equivalence if and only if

1. For T, f satisfies one of the following cases:
   (a) If $X^T = \emptyset$ then $Y^T = \emptyset$,
   (b) if $X^T \neq \emptyset$ then $\pi_q^T(f) : \pi_q^T((X, \rho)) \to \pi_q^T((Y, f\rho))$ is an isomorphism for all $\rho \in X^T$, $q \geq 0$,
2. The same holds for T'.

On the other hand, f is an exterior $\{T, T'\}$-fibration if and only if f has the RLP with respect to

$$\{\delta_0^T : T \bar{\times} D^q \to T \bar{\times} (D^q \times I), \delta_0^{T'} : T' \bar{\times} D^q \to T' \bar{\times} (D^q \times I)\}_{q \geq 0},$$

where $\delta_0^T(\lambda, x) = (\lambda, x, 0)$.

An exterior map which is both an exterior $\{T, T'\}$-fibration and a weak exterior $\{T, T'\}$-equivalence is said to be an *exterior trivial* $\{T, T'\}$*-fibration.*

Definition 4. Let $f : X \to Y$ be an exterior map. We say that f is an *exterior* $\{T, T'\}$*-cofibration* if it has the LLP with respect to all exterior trivial $\{T, T'\}$-fibrations.

Similarly, an exterior map which is an exterior $\{T, T'\}$-cofibration and a weak exterior $\{T, T'\}$-equivalence is said to be an *exterior trivial* $\{T, T'\}$*-cofibration.* An exterior space X such that $X \to *$ is an exterior $\{T, T'\}$-fibration (resp. $\emptyset \to X$ is an exterior $\{T, T'\}$-cofibration) is called $\{T, T'\}$*-fibrant* (resp. $\{T, T'\}$*-cofibrant*).

Remarks 1. *From the above definitions it is straightforward to see that, for these classes of exterior maps, CM2 is satisfied. On the other hand, taking into account that the notions of exterior* $\{T, T'\}$*-(co)fibrations are given by lifting properties, they are closed by retracts. Furthermore, a retract of an isomorphism is an isomorphism, and* π_q^T *is a functor, for all* $q \geq 0$, *so CM3 is also satisfied.*

Combining the closed model category structure of **Top** and the adjunctions $T \bar{\times} \_ \dashv (\_)^T$, $T' \bar{\times} \_ \dashv (\_)^{T'}$, one has the following:

Corollary 1. *Let $f : X \to Y$ be an exterior space. Then f is an exterior trivial $\{T, T'\}$-fibration if and only if f has the LLP with respect to*

$$\{T\bar{\times}S^{q-1} \hookrightarrow T\bar{\times}D^q, T'\bar{\times}S^{q-1} \hookrightarrow T'\bar{\times}D^q\}_{q\geq 0},$$

where $S^{-1} = \emptyset$.

Hence, from this corollary, $T\bar{\times}S^{q-1} \hookrightarrow T\bar{\times}D^q$ and $T'\bar{\times}S^{q-1} \hookrightarrow T'\bar{\times}D^q$ are exterior $\{T, T'\}$-cofibrations.

One can easily check that $\mathbf{E}$ with the following structure is a simplicial category. The functor $\underline{\mathrm{Hom}}_{\mathbf{E}}$ is defined by $\underline{\mathrm{Hom}}_{\mathbf{E}}(X, Y)_n = \mathrm{Hom}_{\mathbf{E}}(X\bar{\times}|\Delta[n]|, Y)$; the composite $g \circ_n f$ is given by

$$X\bar{\times}|\Delta[n]| \xrightarrow{id_X\bar{\times}\Delta} X\bar{\times}(|\Delta[n]| \times |\Delta[n]|) \xrightarrow{f\bar{\times}id_{|\Delta[n]|}} Y\bar{\times}|\Delta[n]| \xrightarrow{g} Z,$$

and the homeomorphism $|\Delta[0]| \cong *$ induces $\mathrm{Hom}_{\mathbf{E}}(X, Y) \cong \underline{\mathrm{Hom}}_{\mathbf{E}}(X, Y)_0$, a natural isomorphism. For any exterior space and finite simplicial set K we consider $X \otimes K = X\bar{\times}|K|$ and $X^K = X^{|K|}$. Then SM0 axiom is satisfied.

Now, suppose a sequence in $\mathbf{E}$:

$$X_0 \xrightarrow{i_1} X_1 \xrightarrow{i_2} X_2 \longrightarrow \cdots \longrightarrow X_n \longrightarrow \cdots,$$

such that X_n is obtained from X_{n-1} by a push-out in $\mathbf{E}$ of the form

$$\begin{tikzcd} \coprod_{\lambda\in\Lambda} Z_\lambda \arrow[r] \arrow[d, "\coprod_{\lambda\in\Lambda}\varphi_\lambda"'] & X_{n-1} \arrow[d, "i_n"] \\ \coprod_{\lambda\in\Lambda} Z'_\lambda \arrow[r] & X_n, \end{tikzcd}$$

where each φ_λ is injective, closed and e-closed and the points of Z_λ, Z'_λ are closed and e-closed. Under these conditions each $i_k : X_k \hookrightarrow X_{k+1}$ is injective, closed, e-closed and the points of $X_k - X_0$ are closed and e-closed.

We take $\tilde{X} = \mathrm{colim}\ X_n$. Then,

Proposition 1. *Let Z be a Hausdorff, σ-compact, locally compact space provided with its co-compact externology. If $f : Z \to \tilde{X}$ is an exterior map, then f factors through X_n, for n sufficiently large.*

We are now ready to prove the main result of this section.

Theorem 2. *The category $\mathbf{E}$, together with the classes of exterior $\{T, T'\}$-fibrations, exterior $\{T, T'\}$-cofibrations and weak exterior $\{T, T'\}$-equivalences and its simplicial structure, is a closed simplicial model category.*

Proof. We only have to check CM4, CM5 and SM7 axioms.

CM5: Let $f : X \to Y$ be an exterior map. We start showing the factorization of f as an exterior $\{T, T'\}$- cofibration followed by an exterior trivial $\{T, T'\}$-fibration. Using inductive arguments we construct a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[dr, "f"'] & X_1 \arrow[r, "i_2"] \arrow[d, "p_1"] & X_2 \arrow[r] \arrow[dl, "p_2"] & \cdots \arrow[r] & X_n \arrow[r] \arrow[dlll, "p_n"] & \cdots \\ & Y, & & & & \end{tikzcd}$$

as follows: take $X_0 = X$, $p_0 = f$ and suppose that X_{n-1} and p_{n-1} are obtained. Then consider the set Λ of all commutative diagrams:

$$\begin{tikzcd} \mathfrak{S}_T^{q_\lambda -1} \arrow[r, "u_\lambda"] \arrow[d] & X_{n-1} \arrow[d, "p_{n-1}"] \\ \mathfrak{D}_T^{q_\lambda} \arrow[r, "v_\lambda"'] & Y, \end{tikzcd} \qquad (q_\lambda \geq 0)$$

where $\mathfrak{S}_T^q, \mathfrak{D}_T^q$ denote $T \bar{\times} S^q, T \bar{\times} D^q$ respectively; consider also the set Γ of all commutative diagrams:

$$\begin{tikzcd} \mathfrak{S}_{T'}^{q_\gamma -1} \arrow[r, "u_\gamma"] \arrow[d] & X_{n-1} \arrow[d, "p_{n-1}"] \\ \mathfrak{D}_{T'}^{q_\gamma} \arrow[r, "v_\gamma"'] & Y. \end{tikzcd} \qquad (q_\gamma \geq 0)$$

Then $i_n : X_{n-1} \to X_n$ is obtained by the following push-out in $\mathbf{E}$:

$$\begin{tikzcd} (\coprod_{\lambda\in\Lambda} \mathfrak{S}_T^{q_\lambda -1}) \coprod (\coprod_{\gamma\in\Gamma} \mathfrak{S}_{T'}^{q_\gamma -1}) \arrow[rr, "(\coprod_{\lambda\in\Lambda} u_\lambda) \coprod (\coprod_{\gamma\in\Gamma} u_\gamma)"] \arrow[d] & & X_{n-1} \arrow[d, "i_n"] \\ (\coprod_{\lambda\in\Lambda} \mathfrak{D}_T^{q_\lambda -1}) \coprod (\coprod_{\gamma\in\Gamma} \mathfrak{D}_{T'}^{q_\gamma -1}) \arrow[rr, "(\coprod_{\lambda\in\Lambda} w_\lambda) \coprod (\coprod_{\gamma\in\Gamma} w_\gamma)"'] & & X_n. \end{tikzcd}$$

$p_n : X_n \to Y$ is the sum of p_{n-1} and all maps $v_\lambda, \lambda \in \Lambda$, $v_\gamma, \gamma \in \Gamma$ so p_n extends p_{n-1}. We consider $\tilde{X} = \text{colim } X_n$ and $p = \text{colim } p_n$.

Let $k_n : X_n \to \tilde{X}$ denote the natural inclusion of X_n into $\tilde{X}$. Then $f = pi$, where $i = k_0$.

It is easy to check that the class of exterior $\{T, T'\}$-cofibrations is closed under co-products and co-base extensions. Therefore, each $i_n : X_{n-1} \to X_n$ is an exterior $\{T, T'\}$-cofibration. Hence, the fact that $i : X \to \tilde{X}$ is an exterior $\{T, T'\}$-cofibration can be easily deduced from inductive arguments and the universal property of the colimit.

Now, take any commutative diagram:

$$\begin{tikzcd} \mathfrak{S}_T^{q_\lambda -1} \arrow[r, "\alpha_\lambda"] \arrow[d] & \widetilde{X} \arrow[d, "p"] \\ \mathfrak{D}_T^{q_\lambda -1} \arrow[r, "v_\lambda"'] & Y. \end{tikzcd} \qquad (q_\lambda \geq 0)$$

By proposition 1, $\alpha_\lambda : \mathfrak{S}_T^{q_\lambda -1} \to \widetilde{X}$ factors through X_m for m sufficiently large, $\alpha_\lambda = k_m u_\lambda$. Taking into account the construction of X_{m+1}, there is an exterior map $w_\lambda : \mathfrak{D}_T{}^{q_\lambda} \to X_{m+1}$ such that $p_{m+1} w_\lambda = v_\lambda$ and $w_\lambda|_{\mathfrak{S}_T{}^{q_\lambda -1}} = i_{m+1} u_\lambda$. Then $h = k_{m+1} w_\lambda$ is the desired lifting for the diagram. Analogously, p has the RLP with respect to any $\mathfrak{S}_{T'}^{q_\gamma -1} \hookrightarrow \mathfrak{D}_{T'}^{q_\gamma}$, so p is an exterior trivial $\{T, T'\}$-fibration.

The factorization $f = qj$, where j is an exterior trivial $\{T,T'\}$-cofibration and q is an exterior $\{T,T'\}$-fibration is similarly obtained by constructing a diagram:

$$\begin{tikzcd}
X \arrow[r, "j_1"] \arrow[dr, "f"'] & X_1 \arrow[r, "j_2"] \arrow[d, "q_1"] & X_2 \arrow[r] \arrow[dl, "q_2"] & \cdots \arrow[r] & X_n \arrow[r] \arrow[dlll, "q_n"] & \cdots \\
 & Y. & & & &
\end{tikzcd}$$

Take $X_0 = X$, $q_0 = f$ and suppose constructed X_{n-1} and q_{n-1}. Then consider the set Λ of all commutative diagrams:

$$\begin{tikzcd}
\mathfrak{D}_T^{q_\lambda} \arrow[r, "u_\lambda"] \arrow[d, "\partial_0^\lambda"'] & X_{n-1} \arrow[d, "q_{n-1}"] \\
\mathfrak{D}_T^{q_\lambda}\bar{\times} I \arrow[r, "v_\lambda"'] & Y,
\end{tikzcd} \qquad (q_\lambda \geq 0)$$

and also consider the set Γ of all commutative diagrams:

$$\begin{tikzcd}
\mathfrak{D}_{T'}^{q_\gamma} \arrow[r, "u_\gamma"] \arrow[d, "\partial_0^\gamma"'] & X_{n-1} \arrow[d, "q_{n-1}"] \\
\mathfrak{D}_{T'}^{q_\gamma}\bar{\times} I \arrow[r, "v_\gamma"'] & Y.
\end{tikzcd} \qquad (q_\gamma \geq 0)$$

The push-out of $(\coprod_{\lambda\in\Lambda} u_\lambda)\coprod(\coprod_{\gamma\in\Gamma} u_\gamma)$ and $(\coprod_{\lambda\in\Lambda} \delta_0^\lambda)\coprod(\coprod_{\gamma\in\Gamma} \delta_0^\gamma)$ gives rise to the exterior map $j_n : X_{n-1} \to X_n$. Furthermore, $q_n : X_n \to Y$ is obtained by the push-out property by the sum of q_{n-1} and all maps v_λ, v_γ. Then $\widetilde{X} = \operatorname{colim}\ X_n$, $q = \operatorname{colim}\ q_n$ and $j = X \to \widetilde{X}$ is the natural inclusion. Obviously, we have that $f = qj$. As, in factorization (i), it is not difficult to see that j is an exterior $\{T,T'\}$-cofibration and q is an exterior $\{T,T'\}$-fibration.

Since ∂_0^λ, ∂_0^γ are strong deformation retracts, $j_n : X_{n-1} \to X_n$ is a strong deformation retract too. Therefore, one has bijections $\pi_q^T(j) : \pi_q^T((X,\rho)) \to \pi_q^T((\widetilde{X}, j\rho))$, $\pi_q^{T'}(j) : \pi_q^{T'}((X,\rho)) \to \pi_q^T((\widetilde{X}, j\rho))$ when $X^T \neq \emptyset \neq X^{T'}$, since the exterior maps $\mathfrak{S}_T^q \to \widetilde{X}$, $\mathfrak{S}_T^q\bar{\times}I \to \widetilde{X}$ and $\mathfrak{S}_{T'}^q \to \widetilde{X}$, $\mathfrak{S}_{T'}^q\bar{\times}I \to \widetilde{X}$ verify the hypothesis of proposition 1. Then j is a weak exterior $\{T,T'\}$-equivalence. Furthermore, it is easy to check that j has the LLP with respect to all exterior $\{T,T'\}$-fibrations.

CM4: The only nontrivial part of this axiom consists of showing the existence of lifting in any commutative diagram of the form:

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r, "v"'] & Y,
\end{tikzcd}$$

where i is an exterior trivial $\{T,T'\}$-cofibration and p is an exterior $\{T,T'\}$-fibration. We consider a factorization of i, $i = qj$, where $j : A \to Z$ is an exterior trivial $\{T,T'\}$-cofibration which has the LLP with respect to all exterior $\{T,T'\}$-fibrations and q is an exterior $\{T,T'\}$-fibration. Since, by CM2 axiom, q is also a weak exterior

$\{T, T'\}$-fibration, there is a lifting:

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & Z \arrow[d, "q"] \\ B \arrow[r, "id_B"'] \arrow[ur, dotted, "r"] & B, \end{tikzcd}$$

so i is a retract of j. Therefore i has the LLP with respect to all exterior $\{T, T'\}$-fibrations.

SM7: We will show that $\mathbf{E}$ verifies SM7(a). Let $p : X \to Y$ be an exterior $\{T, T'\}$-fibration (resp. exterior trivial $\{T, T'\}$-fibration): Since $(.)^T$, $(.)^{T'}$ are right adjoint functors, it follows that they preserve pull-backs. On the other hand, $(X^Y)^Z \cong (X^Z)^Y$ in **Top**, for every exterior space X, every locally compact space Y and every Hausdorff, locally compact space provided with its co-compact externology. Then one has that p verifies SM7(a) if and only if the fibrations p^T, $p^{T'}$ in **Top** verify SM7(a). However, the last statement follows from the fact that **Top** has a closed simplicial model category structure.

□

Remarks 2. (i) *We observe that in the particular case of* $T = \mathbb{N}$ *and* $T' = \emptyset$ *we obtain the closed simplicial model structure for* $\mathbf{E}$ *given in* [16].

(ii) *In the proof of the theorem the spaces* T *and* T' *are supposed to satisfy the* σ*-compactness condition. This condition can be removed by taking transfinite sequences in the factorizations given in the proof above. Taking* $T' = P$ *and* T *a discrete space with the co-finite externology we can develop closed model categories using* T*'s with higher and higher cardinality.*

(iii) *In some cases the structured associated with the pair* $\{T, T'\}$ *is equal to the structure induced by* $\{T \cup T', \emptyset\}$ *, for example if we have the additional condition that* $X^T = \emptyset$ *if and only if* $X^{T'} = \emptyset$ *.*

4. Comparison between the T-structure and the $\{T, T'\}$-structure.

If $T' = \emptyset$ we will denote the pair $\{T, \emptyset\}$ by T. As it has been proved, the category $\mathbf{E}$ has a family of closed simplicial model category structures, depending on a pair $\{T, T'\}$. In the particular case of $T = \mathbb{N}$ and $T' = P$ we have that

(i) f is an exterior $\{\mathbb{N}, P\}$-fibration (resp. weak exterior $\{\mathbb{N}, P\}$-equivalence) if and only if f is an exterior $\mathbb{N}$-fibration (resp. weak exterior $\mathbb{N}$-equivalence) and $U(f)$ is a Serre fibration (resp. weak equivalence) in **Top**.

(ii) f is an exterior $\{\mathbb{N}, P\}$-cofibration if and only if it has the LLP with respect to all exterior trivial $\{\mathbb{N}, P\}$-fibrations.

On the other hand, we have other several families of maps, for example, if $T = \emptyset$ and $T' = P$ or if $T = \mathbb{N}$ and $T' = \emptyset$. The aim of this section is to give a comparison between the T-structure and the $\{T, T'\}$-structure in $\mathbf{E}$ (i.e. the localized categories induced by the correspondent closed model category structures).

First of all, note that all objects in $\mathbf{E}$ are $\{T, T'\}$-fibrant. We will denote the homotopy category obtained from $\mathbf{E}$ by formally inverting all the weak exterior $\{T, T'\}$-equivalences by $\mathbf{Ho_{\{T,T'\}}(E)}$.

Observe the trivial situation of adjointness:

$$\mathbf{E} \underset{id}{\overset{id}{\rightleftarrows}} \mathbf{E}.$$

It is clear that $id : \mathbf{E} \to \mathbf{E}$ carries exterior $\{T, T'\}$-fibrations to exterior T-fibrations and it carries weak exterior $\{T, T'\}$-equivalences to weak exterior T-equivalences. Therefore, $id : \mathbf{E} \to \mathbf{E}$ also carries exterior T-cofibrations to exterior $\{T, T'\}$-cofibrations.

Proposition 2. *Let $f : X \to Y$ be a weak exterior T-equivalence between T-cofibrant exterior spaces. Then, f is a weak exterior $\{T, T'\}$-equivalence.*

Proof. We note that all the exterior spaces are T-fibrant, therefore for T-cofibrant exterior spaces, right homotopies, left homotopies and exterior homotopies induce the same relation in mapping sets between T-cofibrant spaces. Now if $f : X \to Y$ is a weak T-equivalence between T-cofibrant exterior spaces, by the Whitehead Theorem for T-cofibrant and T-fibrant objects, one has that f is an exterior homotopy equivalence; this implies that f is a weak exterior $\{T, T'\}$-equivalence.

□

Using well-known results given in [24] we have as a consequence the following.

Corollary 2. *There is an adjunction*

$$\underline{\underline{L}}(id) : \mathbf{Ho_T}(\mathbf{E}) \rightleftarrows \mathbf{Ho_{\{T,T'\}}}(\mathbf{E}) : \underline{\underline{R}}(id),$$

where $\underline{\underline{L}}(id)$ and $\underline{\underline{R}}(id)$ denote the total left-derived functor and the total right-derived functor of $id : \mathbf{E} \to \mathbf{E}$, respectively.

Theorem 3. *$\underline{\underline{L}}(id) : \mathbf{Ho_T}(\mathbf{E}) \to \mathbf{Ho_{\{T,T'\}}}(\mathbf{E})$ is a full and faithful functor.*

Proof. If X is an object in $\mathbf{Ho_T}(\mathbf{E})$, then $\underline{\underline{L}}(id)(X) = Q_T(X)$, where $Q_T(X)$ is the T-cofibrant approximation of X; let $p_X : Q_T(X) \to X$ denote the associated exterior trivial T-fibration, which is an isomorphism in $\mathbf{Ho_T}(\mathbf{E})$. On the other hand, if Y is an object in the category $\mathbf{Ho_{\{T,T'\}}}(\mathbf{E})$, since Y is $\{T, T'\}$-fibrant, we have that $\underline{\underline{R}}(id)(Y) = Y$.

Then for any object X in $\mathbf{Ho_T}(\mathbf{E})$ one has tha canonical isomorphism

$$\underline{\underline{R}}(id)\underline{\underline{L}}(id)(X) = \underline{\underline{R}}(id)Q_T(X) = Q_T(X) \cong X$$

This implies that the category $\mathbf{Ho_T}(\mathbf{E})$ is isomorphic to a reflexive subcategory of $\mathbf{Ho_{\{T,T'\}}}(\mathbf{E})$.

□

We remark that if $\gamma_T : \mathbf{E} \to \mathbf{Ho_T}(\mathbf{E})$ is the localization functor, then the composite $\underline{\underline{L}}(id) \circ \gamma_T : \mathbf{E} \to \mathbf{Ho_{\{T,T'\}}}(\mathbf{E})$ in general is not isomorphic to the canonical localization functor $\gamma_{T,T'} : \mathbf{E} \to \mathbf{Ho_{\{T,T'\}}}(\mathbf{E})$.

Corollary 3. *$\underline{\underline{L}}(id) : \mathbf{Ho}_{\mathbb{N}}(\mathbf{E}) \to \mathbf{Ho}_{\{\mathbb{N},\mathbf{P}\}}(\mathbf{E})$ is a full and faithful functor.*

Remark 2. *Suppose $T = \mathbb{N}$ and $T' = P$. If $f : K \to L$ be a map between compact CW complexes of different homotopy type. If we apply the functor $\underline{\underline{L}}(id) \circ \gamma_T$ to $f_e : X_e \to Y_e$ we obtain an isomorphism because $\gamma_T(f_e)$ is an isomorphism but if we apply the functor $\gamma_{T,T'} f_e$ we will obtain a map which is not an isomorphism. Hence if X and Y are compact CW-complexes of different type, the exterior spaces*

X_e and Y_e have different type in $\mathbf{Ho}_{\{\mathbf{T},\mathbf{T'}\}}(\mathbf{E})$. Nevertheless, the composite of localization functors $\mathbf{E} \to \mathbf{Ho}_{\{\mathbb{N},\mathbf{P}\}}(\mathbf{E}) \to \mathbf{Ho}_{\mathbb{N}}(\mathbf{E})$ is, up to isomorphism, the same as the localization $\mathbf{E} \to \mathbf{Ho}_{\mathbb{N}}(\mathbf{E})$.

Another interesting comparison is that between $\mathbf{Ho}(\mathbf{Top})$ and $\mathbf{Ho}_{\{\mathbf{T},\mathbf{P}\}}(\mathbf{E})$. Recall the functor $V : \mathbf{Top} \to \mathbf{E}$ given is subsection 1.2 .

Proposition 3. *The functor $V : \mathbf{Top} \to \mathbf{E}$ carries weak equivalences in $\mathbf{Top_{cof}}$ to weak exterior $\{T, P\}$-equivalences.*

Proof. By the Whitehead theorem and taking into account that every object in $\mathbf{Top}$ is fibrant, we have that, given $f : X \to Y$ a weak equivalence in $\mathbf{Top_{cof}}$, f is a homotopy equivalence. Since $V(Z \times I) = V(Z)\bar{\times}I$, for all Z in $\mathbf{Top}$, then $V(f)$ is an exterior homotopy equivalence so f is a weak exterior T-equivalence and a weak exterior P-equivalence, then it follows that f is a weak exterior $\{T, P\}$-equivalence. □

Corollary 4. *There is an adjunction*

$$\underline{\underline{L}}(V) : \mathbf{Ho}(\mathbf{Top}) \rightleftarrows \mathbf{Ho}_{\{\mathbf{T},\mathbf{P}\}}\mathbf{E} : \underline{\underline{R}}(U).$$

Theorem 4. *$\underline{\underline{L}}(V) : \mathbf{Ho}(\mathbf{Top}) \to \mathbf{Ho}_{\{\mathbf{T},\mathbf{P}\}}\mathbf{E}$ is a full and faithful functor.*

Proof. It is left as an exercise. □

5. Applications

A classical theorem of J.H.C. Whitehead [29] establishes that a continuous map between CW-complexes $f : X \to Y$ is a homotopy equivalence if and only if f is a weak homotopy equivalence. The aim of this section is to give a version of the Whitehead theorem in the category of exterior spaces, involving special complexes and different homotopy groups, and give, as corollaries, the correspondent one in $\mathbf{P}$.

We begin by stating the notion of $\{T, T'\}$-complex, where T and T' are again two Hausdorff, locally compact, σ-compact spaces provided with their co-compact externology.

Definition 5. A *$\{T, T'\}$-complex* consists of an exterior space X with a filtration $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset ... \subset X_n \subset ... \subset X$ such that X is the colimit of the filtration and for $n \geq 0$, X_n is obtained from X_{n-1} by a push-out in $\mathbf{E}$ of the form

$$\begin{tikzcd}
(\coprod_{\alpha\in A}(\mathfrak{S}^{n-1}_T)_\alpha) \coprod (\coprod_{\beta\in B}(\mathfrak{S}^{n-1}_{T'})_\beta) \arrow[r] \arrow[d] & X_{n-1} \arrow[d, "i_n"] \\
(\coprod_{\alpha\in A}(\mathfrak{D}^{n}_T)_\alpha \coprod (\coprod_{\beta\in B}(\mathfrak{D}^{n}_{T'})_\beta) \arrow[r] & X_n.
\end{tikzcd}$$

Examples 1. *(1) If $T = P$, $T' = \emptyset$, then we have the notion of P-complex which coincides with the notion of CW-complex X provided with its topology as externology.*

(2) $T = \mathbb{N}$ and $T' = \emptyset$. In this case we obtain the notion of $\mathbb{N}$-complex studied in [16].

(3) If $T = \mathbb{N}$ and $T' = P$ then we have the notion of bi-complex, introduced in [15].

We can do further combinations, involving $\emptyset$, P, $\mathbb{N}$ *and* $\mathbb{R}_+$.

Note that every $\{T,T'\}$-complex is a $\{T,T'\}$-cofibrant exterior space. It is clear that, for $\{T,T'\}$-cofibrant exterior spaces, right homotopies, left homotopies and exterior homotopies induce the same relation between exterior maps. From Quillen [24] one has that the homotopy category $\pi_{\mathbf{0}}(\mathbf{E}_{\{\mathbf{T},\mathbf{T}'\}-\mathbf{cof}})$ is equivalent to $\mathbf{Ho}_{\{\mathbf{T},\mathbf{T}'\}}(\mathbf{E})$. Then, one has

Theorem 5. *Let X, Y be $\{T,T'\}$-complexes and let $f : X \to Y$ be an exterior map. Then f is a homotopy exterior equivalence if and only if f is a weak exterior $\{T,T'\}$- equivalence.*

If X is a CW-complex then $X \equiv V(X)$ has the structure of a P-complex structure, so it is a P-cofibrant exterior space. Taking into account that f is a weak equivalence in **Top** if and only if $f = V(f)$ is a weak exterior P-equivalence and the fact that $V(f) = f$ is an exterior homotopy equivalence in **E** if and only if f is a homotopy equivalence in **Top** we obtain as a corollary the standard version of the Whitehead theorem.

Corollary 5. *Let $f : X \to Y$ be a continuous map between CW-complexes. Then f is a homotopy equivalence if and only if f is a weak equivalence.*

Now, suppose that X is a locally finite CW-complex with finite dimension d, and for each $0 \leq k \leq d$ either X has no k-cells or X has an infinite countable number of k-cells. Under these conditions X_e admits the structure of a finite $\mathbb{N}$-complex. Therefore

Corollary 6. *Let $f : X \to Y$ be a proper map between CW-complexes satisfying the good conditions described above. Then f is a proper homotopy equivalence if and only if $f = f_e$ is a weak exterior $\mathbb{N}$-equivalence.*

Finally, if X is a finite dimensional strongly locally finite CW-complex then X_e admits the structure of a finite $\{\mathbb{N},P\}$-complex.

Corollary 7. *Let $f : X \to Y$ be a proper map between finite dimensional strongly locally finite CW-complexes. Then f is a proper homotopy equivalence if and only if $f = f_e$ is a weak exterior $\{\mathbb{N},P\}$-equivalence.*

Remark 3. *Using the closed model structure suggested at the end of 3 section we can obtain as corollary the proper Whitehead theorem given by F.T. Farrel, L.R. Taylor and J.B. Wagoner for finite dimensional locally finite CW-complexes.*

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J.M. García-Calcines
Departamento de Matemática Fundamental. Universidad de La Laguna.
38271 La Laguna, España. **jmgarcal@ull.es**
M. García-Pinillos
Departamento de Matemáticas y Computación. Universidad de La Rioja.
26004 Logroño, España. **tvirgos@roble.pntic.mec.es**
L.J. Hernández-Paricio
Departamento de Matemáticas y Computación. Universidad de La Rioja.
26004 Logroño España. **luis-javier.hernandez@dmc.unirioja.es**