S-TYPES OF GLOBAL TOWERS OF SPACES AND EXTERIOR SPACES

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Abstract. The closed model category of exterior spaces, that contains the proper category, is a useful tool for the study of non compact spaces and manifolds. The notion of exterior weak $\mathbb{N}$-S-equivalences is given by exterior maps which induce isomorphisms on the k-th $\mathbb{N}$-exterior homotopy groups $\pi_k^{\mathbb{N}}$ for $k \in S$, where S is a set of non negative integers. The category of exterior spaces with a base ray localized by exterior weak $\mathbb{N}$-S-equivalences is called the category of exterior $\mathbb{N}$-S-types. The existence of closed model structures in the category of exterior spaces permits to establish equivalences between homotopy categories obtained by dividing by exterior homotopy relations, and categories of fractions (localized categories) given by the inversion of classes of week equivalences. The family of neighbourhoods 'at infinity' of an exterior space can be interpreted as a global prospace and under the condition of first countable at infinity we can consider a global tower instead of a prospace. The objective of this paper is to use localized categories to find the connection between S-types of exterior spaces and S-types of global towers of spaces.

The main result of this paper establishes an equivalence between the category of S-types of rayed first countable exterior spaces and the category of S-types of global towers of pointed spaces. As a consequence of this result, categories of global towers of algebraic models localized up to weak equivalences can be used to give some algebraic models of S-types.

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1. Introduction

In some cases, for the study of a non compact space it is advisable to consider as neighbourhoods at infinity the complements of closed-compact subsets. The Proper Homotopy Theory arises when we consider spaces and maps which are continuous at infinity. In order to have a category with limits and colimits it is interesting to extend the proper category to obtain a complete and cocomplete category. The category of exterior spaces satisfies these properties, contains the proper category and has limits and colimits. The study of non compact spaces and more generally exterior spaces has interesting applications, for example, Siebenmann [?] or Brown and Tucker [?] used proper invariants of non compact spaces to obtain some properties and classifications of open manifolds. More recently, the use of exterior spaces has permitted to develop a Whitehead-Ganea approach for proper Lusternik-Schnirelmann category, [?]. We can also use exterior spaces to find applications in the study of compact-metric spaces. A compact metric space can be embedded in the Hilbert cube, its open neighbourhoods provide the Hilbert cube with the structure of an exterior space. In this way, the homotopy invariants of exterior spaces become invariants of metric-compact spaces, see [?].

To develop the Algebraic Topology at infinity (or in the category of exterior spaces) it is useful to consider some analogues of the standard Hurewicz homotopy groups. If instead of n-spheres we use sequences of n-spheres converging to infinity, then we obtain the Brown-Grossman proper homotopy groups $\pi_q^{\mathbb{N}}$, see [?, ?]. On the other hand, if we move an n-sphere continuously towards infinity, we get infinite semitubes which represent elements of the Steenrod proper homotopy groups, see [?, ?, ?]. The analog of the previous groups have been considered for the category of exterior spaces, see [?, ?]. In this homotopy theory the role of a base point is played by a base ray; that is, an exterior map from the exterior space of non negative real numbers $\mathbb{R}_+$ to an exterior space X. The homotopy progroups, given by the homotopy groups of the neighborhoods at infinity together with the bonding morphisms induced by the base ray, are also important invariants in this theory, see [?, ?, ?].

One useful tool for the study of a homotopy category is the notion of closed model category introduced by Quillen [?, ?]. The existence of a closed model structure in a category induces a category equivalence between the homotopy category of cofibrant-fibrant objects and the category of fractions induced by inverting weak equivalences. In our case, we have that the category of rayed exterior spaces and global tower of pointed spaces admits closed model structures, see [**?**, **?**, **?**].

In this paper, if we take a set S of non negative integers, we have the class $\Sigma_{\mathbb{N}}^{S}$ of weak S-$\mathbb{N}$-equivalences in the category of rayed exterior spaces, given by exterior maps which induce isomorphism on the exterior homotopy groups $\pi_q^{\mathbb{N}}$, and, in the category of global towers of pointed spaces, the class Σ^S of global morphisms which induce isomorphism on the global towers of homotopy groups.

Our main result establishes that the category of rayed exterior spaces which are first countable at infinity localized by exterior weak S-$\mathbb{N}$-equivalences is equivalent to the category of global towers of pointed spaces localized by weak S-equivalences:

Theorem 1. Suppose that S is a set of non negative integer numbers. Then $\tilde{\varepsilon}\colon \mathbf{E}_{fc}^{\mathbb{R}_+} \to \mathrm{tow}^+\mathbf{Top}^*$ induces an equivalence of categories

$$\mathbf{E}_{fc}^{\mathbb{R}_+}[\Sigma_{\mathbb{N}}^{S}]^{-1} \simeq \mathrm{tow}^+\mathbf{Top}^*[\Sigma^{\mathrm{S}}]^{-1}.$$

That is, the category of exterior $\mathbb{N}$-S-types is equivalent to the category of S-types of global towers of pointed spaces.

This result has important particular cases:
(i) S is the set of all non negative integer numbers,
(ii) $S = \{0, \cdots, n\}$, $n \geq 0$,
(iii) $S = \{n, n+1, n+2, \cdots\}$, $n > 0$,
(iv) $S = \{n, n+1, \cdots, m\} = [n, m]$.

In the first case (i), we can compare our result to the Edwards-Hastings embedding of the proper homotopy category of rayed σ-compact spaces given in [**?**]. There are differences of two kinds: First, our result is formulated in terms of equivalence of categories instead of embeddings, and, second, the localized category $\mathbf{P}_{\sigma}^{\mathbb{R}_+}[\Sigma_{\mathbb{N}}^{S}]^{-1} \subset \mathbf{E}_{fc}^{\mathbb{R}_+}[\Sigma_{\mathbb{N}}^{S}]^{-1}$ is not canonically isomorphic to the proper homotopy category considered by Edwards-Hastings and similarly for the corresponding localizations of global towers of pointed spaces (we have considered larger classes of weak equivalences). We also remark that in general a weak S-$\mathbb{N}$-equivalence in $\mathbf{E}_{fc}^{\mathbb{R}_+}$ is not a proper homotopy equivalence; however, there is a version of the Whitehead theorem for exterior spaces which is satisfied in the subcategory of rayed cofibrant exterior spaces.

In the second case (ii), we are comparing a category of n-types of exterior spaces to n-types of global towers of pointed spaces. The notion of n-type introduced by Whitehead [**?**, **?**] has a clear geometric meaning and can also be established in the context of non compact spaces and proper maps, see Geoghegan [**?**]. A program for the study of proper n-types was initiated by Hernández and Porter in [**?**], where some of the Whitehead's results about n-types were generalized to the context of pro-pointed spaces and some applications were given for proper n-types and shape theory. One advantage of our formulation is that the new result gives a category equivalence instead of the category embeddding given in [**?**].

In case (iii), the given category equivalence permits to study exterior homotopy categories of $(n-1)$-connected rayed cofibrant exterior spaces, and in case (iv), rayed cofibrant exterior spaces with finitely many non trivial exterior homotopy groups $\pi_q^{\mathbb{N}}$, $n \leq q \leq m$, using the corresponding global towers of pointed spaces, see [**?**] .

Finally, we remark that spaces such as open separable triangulated manifolds or locally finite, finite dimensional, separable CW-complexes with the cocompact externology and a suitable base ray are cofibrant rayed exterior spaces. Therefore the set of proper rayed homotopy classes between spaces of this type can be analyzed using these localized categories.

2. The category of proper and exterior spaces

A continuous map $f : X \to Y$ is said to be proper if for every closed compact subset K of Y, $f^{-1}(K)$ is a compact subset of X. The category of topological spaces and the subcategory of spaces and proper maps will be denoted by **Top** and **P**, respectively. This last category and its corresponding proper homotopy category are very useful for the study of non compact spaces. Nevertheless, one has the problem that **P** does not have enough limits and colimits and then we can not develop the usual homotopy constructions like loops, homotopy limits and colimits, et cetera.

An answer to this problem is given by the notion of exterior space. The new category of exterior spaces and maps is complete and cocomplete and contains as a full subcategory the category of spaces and proper maps, see [**?**, **?**]. For more properties and applications of exterior homotopy categories we refer the reader to [**?**, **?**] and for a survey of proper homotopy to [**?**].

Definition 1. *Let (X, τ) be a topological space. An externology on (X, τ) is a non empty collection ε of open subsets which is closed under finite intersections and such that if $E \in \varepsilon$, $U \in \tau$ and $E \subset U$ then $U \in \varepsilon$. If an open subset is a member of ε is said to be an e-open subset. An exterior space $(X, \varepsilon \subset \tau)$ consists of a space (X, τ) together with an externology ε. A map $f : (X, \varepsilon \subset \tau) \to (X', \varepsilon' \subset \tau')$ is said to be exterior if it is continuous and $f^{-1}(E) \in \varepsilon$, for all $E \in \varepsilon'$.*

The category of exterior spaces and maps will be denoted by $\mathbf{E}$.

Given a space (X, τ), we can always consider the trivial exterior space taking $\varepsilon = \{X\}$ and the total exterior space if one takes $\varepsilon = \tau$. The one point topological space is denoted by $*$, the corresponding trivial exterior space is also denoted by $*$ and the total exterior space by $\star$.

An important role is played by the family ε_{cc}^{X} of the complements of closed-compact subsets of a topological space X, that will be called the cocompact externology. We denote by $\mathbb{N}$ and $\mathbb{R}_+$ the exterior spaces of non negative integers and non negative real numbers having the usual topology and the cocompact externology.

The category of spaces and proper maps can be considered as a full subcategory of the category of exterior spaces via the full embedding $(\cdot)_{cc} : \mathbf{P} \hookrightarrow \mathbf{E}$.

The functor $(\cdot)_{cc}$ carries a space X to the exterior space X_{cc} which is provided with the topology of X and the externology ε_{cc}^{X}. A map $f\colon X \to Y$ is carried to the exterior map $f_{cc}\colon X_{cc} \to Y_{cc}$ given by $f_{cc} = f$. It is easy to check that a continuous map $f : X \to Y$ is proper if and only if $f = f_{cc}\colon X_{cc} \to Y_{cc}$ is exterior.

Let $\mathbf{E}^{\mathbb{N}}$ be the category of exterior spaces under $\mathbb{N}$ and $\mathbf{E}^{\mathbb{R}_+}$ the category of spaces under $\mathbb{R}_+$. If (X, λ) is an object in the category $\mathbf{E}^{\mathbb{R}_+}$, where $\lambda\colon \mathbb{R}_+ \to X$ is a base ray, the natural restriction $\lambda|_{\mathbb{N}}\colon \mathbb{N} \to X$ gives a sequence base in X. Then we have a canonical forgetful functor $\mathbf{E}^{\mathbb{R}_+} \to \mathbf{E}^{\mathbb{N}}$.

Definition 2. *An exterior space is said to be first countable at infinity, if there is a countable base of the externology; that is, there is a decreasing sequence*

$$X = X_0 \supset X_1 \supset X_2 \supset \cdots$$

such that for every $j \geq 0$ the interior of X_j is e-open and for every e-open E there is $i \geq 0$ such that $X_i \subset E$.

The full subcategories determined by exterior spaces which are first countable at infinity will be denoted by $\mathbf{E}_{fc}$, $\mathbf{E}_{fc}^{\mathbb{N}}$, $\mathbf{E}_{fc}^{\mathbb{R}_+}$.

Let X be an exterior space and $L \subset X$, we say that L is e-compact if $L \setminus E$ is a compact subset, for all E e-open subset of X.

Let X, Z be exterior spaces, then we define $Z^X = Hom_{\mathbf{E}}(X, Z)$ with the topology generated by the subsets of the form:

$$(K, U) = \{\alpha \in Z^X : \alpha(K) \subset U\}$$

$$(L, E) = \{\alpha \in Z^X : \alpha(L) \subset E\}$$

where $K \subset X$ is a compact subset, $U \subset Z$ is an open subset, $L \subset X$ is an e-compact subset and $E \subset Z$ an e-open subset. This construction gives a functor $\mathbf{E}^{op} \times \mathbf{E} \to \mathbf{Top}$.

Let X be an exterior space, Y a topological space, we consider on $X \times Y$ the product topology $\tau_{X\times Y}$ and the externology $\varepsilon_{X\bar{\times}Y}$ given by those $E \in \tau_{X\times Y}$ such that for each $y \in Y$ there exists $U_y \in \tau_Y$, $y \in U_y$ and $E_y \in \varepsilon_X$ such that $E_y \times U_y \subset E$. This exterior space will be denoted by $X\bar{\times}Y$ in order to avoid a possible confusion with the product externology. This construction gives a functor $\mathbf{E} \times \mathbf{Top} \to \mathbf{E}$. When Y is a compact space we can prove that E is an e-open subset if and only if it is an open subset and there exists $G \in \varepsilon_X$ such that $G \times Y \subset E$. Furthermore, if Y is a compact space and $\varepsilon_X = \varepsilon_{cc}^X$ then $\varepsilon_{X\bar{\times}Y}$ coincides with the externology of complements of closed-compact subsets of $X \times Y$.

Let Y be a topological space and Z an exterior space, then we consider on $Z^Y = Hom_{\mathbf{Top}}(Y, Z)$ the compact-open topology and the externology given by the open subsets E of Z^Y such that E contains a subset of the form (K, G), where K is a compact subset of Y and G is an e-open subset of Z. This construction gives a functor $\mathbf{Top}^{op} \times \mathbf{E} \to \mathbf{E}$. It is not very difficult to see that, if Y is a compact space, $E \in \tau_{Z^Y}$ is an e-open subset if and only if it contains a subset of the form (Y, G).

Proposition 1. *Let X, Z be exterior spaces and Y a topological space, then*

(i) If X is a Hausdorff, locally compact space and $\varepsilon_X = \varepsilon_{cc}^X$, there is a natural bijection

$$Hom_{\mathbf{E}}(X\bar{\times}Y, Z) \cong Hom_{\mathbf{Top}}(Y, Z^X).$$

(ii) If Y is a locally compact space, there is a natural bijection

$$Hom_{\mathbf{E}}(X\bar{\times}Y, Z) \cong Hom_{\mathbf{E}}(X, Z^Y).$$

Let S^q denote the q-dimensional pointed sphere. For an exterior space X, using the exterior space $\mathbb{N}\bar{\times}S^q$ and the topological space $X^{\mathbb{N}}$, one has a canonical isomorphism

$$Hom_{\mathbf{E}}(\mathbb{N}\bar{\times}S^q, X) \cong Hom_{\mathbf{Top}}(S^q, X^{\mathbb{N}}).$$

Definition 3. *Let $(X, *)$ be in $\mathbf{Top}^*$ and for an integer $q \geq 0$, denote its q-th homotopy group by $\pi_q(X, *)$. A continuous map $f \colon (X, *) \to (X', *')$ is said to be a weak equivalence if $\pi_q(f)$ is an isomorphism for every $q \geq 0$. Similarly, given a set S of non negative integers, f is said to be a weak S-equivalence if $\pi_q(f)$ is an isomorphism for every $q \in S$.*

Let Σ denote the class of weak equivalences and Σ^S the class of weak S-equivalences. The category of fractions $\mathbf{Top}^*]\Sigma^S]^{-1}$ ($\mathbf{Top}^*[\Sigma]^{-1}$) will be called the category of S-types (types) of pointed spaces.

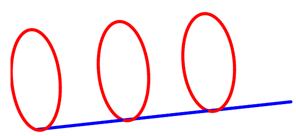
Definition 4. *Let (X, λ) be in $\mathbf{E}^{\mathbb{R}_+}$ and take an integer $q \geq 0$. The q-th $\mathbb{N}$-exterior homotopy group of (X, λ) is given by*

$$\pi_q^{\mathbb{N}}(X, \lambda) = \pi_q(X^{\mathbb{N}}, \lambda|_{\mathbb{N}})$$

where π_q is the q-th homotopy group functor.

An exterior map $f\colon (X, \lambda) \to (X', \lambda')$ is said to be an exterior weak $\mathbb{N}$-equivalence if $\pi_q^{\mathbb{N}}(f)$ is an isomorphism for every $q \geq 0$. Similarly, given a set S of non negative integers, f is said to be an exterior weak $\mathbb{N}$-S-equivalence if $\pi_q^{\mathbb{N}}(f)$ is an isomorphism for every $q \in S$. Let $\Sigma_{\mathbb{N}}$ denote the class of exterior weak $\mathbb{N}$-equivalences and $\Sigma_{\mathbb{N}}^S$ the class of exterior weak $\mathbb{N}$-S-equivalences.

The elements of $\pi_q^{\mathbb{N}}(X, \lambda)$ are represented by a sequence of spheres, as the given below for $q = 1$, converging to an infinity point of X.



Definition 5. *The category of fractions*

$$\mathbf{E}^{\mathbb{R}_+}[\Sigma_{\mathbb{N}}^S]^{-1}$$

is said to be the category of exterior $\mathbb{N}$-S-types, and the full subcategory determined by $\mathbf{P}$ will be denoted by

$$\mathbf{P}^{\mathbb{R}_+}[\Sigma_{\mathbb{N}}^S]^{-1}$$

and it is said to be the category of proper $\mathbb{N}$-S-types.

Remark 1. *Some subsets S determine interesting categories of fractions, for example:*

(a) *If S is the set of non negative integer, the category of fractions $\mathbf{E}^{\mathbb{R}_+}[\Sigma_{\mathbb{N}}^S]^{-1}$ is equivalent to the exterior homotopy category of rayed cofibrant exterior spaces (any $\mathbb{N}$-complex is a cofibrant exterior space). This is a consequence of the existence of a suitable closed model structure in $\mathbf{E}$, see* [**?**, **?**].

(b) *If $S = \{0\}$, $\mathbf{E}^{\mathbb{R}_+}[\Sigma_{\mathbb{N}}^S]^{-1}$ is the category of exterior 0-types which can be determined by the set of path-components ($\pi_0(X)$) and the space of 'end points' ($\lim_{E \in \varepsilon_X} \pi_0(E)$) of an exterior space X.*

(c) *If* $S=\{1\}$*,* $\mathbf{E}^{\mathbb{R}_+}[\Sigma_{\mathbb{N}}^{S}]^{-1}$ *is equivalent to the exterior homotopy category of rayed cofibrant exterior spaces such that* $\pi_q^{\mathbb{N}}(X,\lambda)$ *is trivial for* $q\neq 1$*. An exterior space* X *satisfying these properties is said to be an exterior* $\mathbb{N}$*-Eilenberg-Mac Lane space* $X=K_{\mathbb{N}}(G,1)$*, where* $\pi_1^{\mathbb{N}}X\cong G$*. For example* $\mathbb{R}^2$ *with the cocompact externology satisfies this property, and* $\mathbb{R}^2=K_{\mathbb{N}}(\Pi_0^{\infty}\mathbb{Z}/\oplus_0^{\infty}\mathbb{Z},1)$.

(d) *For* $S=\{1,2\}$*,* $\mathbf{E}^{\mathbb{R}_+}[\Sigma_{\mathbb{N}}^{S}]^{-1}$ *is said to be the category of exterior* $\mathbb{N}$*-*$\{1,2\}$*-types. This category is equivalent to the exterior homotopy category of rayed cofibrant exterior spaces* (X,λ) *such that* X *is* 0*-connected, has one 'end point' and* $\pi_q^{\mathbb{N}}(X,\lambda)$ *is trivial for* $q\geq 3$.

3. Exterior $\mathbb{N}$-S-types and S-types of global towers of pointed spaces

Given a category $\mathcal{C}$ we can consider the following induced categories:

$$\mathrm{pro}\,\mathcal{C},\ \mathrm{pro}^{+}\mathcal{C},\ \mathrm{tow}\,\mathcal{C},\ \mathrm{tow}^{+}\mathcal{C}.$$

$\mathrm{pro}\,\mathcal{C}$ is the category whose objects are pro-objects X in $\mathcal{C}$; that is, functors of the form $X\colon J\to\mathcal{C}$, where J is a left-filtering small category.

$\mathrm{pro}^{+}\mathcal{C}$ is the category whose objects are global pro-objects Y in $\mathcal{C}$; that is, functors $Y\colon K\to\mathcal{C}$, where K is a left-filtering small category with final object, and pro-morphisms are compatible with the final object. An alternative description of this category, denoted as $(\mathrm{pro}\,\mathcal{C},\mathcal{C})$, is given by Edwards-Hastings in [**?**] as the full subcategory of $\mathrm{Maps}(\mathrm{pro}\,\mathcal{C})$, the category of maps in $\mathrm{pro}\,\mathcal{C}$, determined by maps of the form $Y\to Y_0$, where Y_0 is a constant pro-object. For any information and properties related to these categories $\mathrm{pro}\,\mathcal{C}$ and $\mathrm{pro}^{+}\mathcal{C}=(\mathrm{pro}\,\mathcal{C},\mathcal{C})$, we refer the reader to [**?**, **?**].

$\mathrm{tow}\,\mathcal{C}$ is the category whose objects are towers X in $\mathcal{C}$, which are functors $X\colon\mathbf{N}\to\mathcal{C}$, where $\mathbf{N}$ is the canonical category induced by the set of ordered non negative integer numbers.

$\mathrm{tow}^{+}\mathcal{C}$ is the category whose objects are global towers Y in $\mathcal{C}$, which are functors $Y\colon\mathbf{N}^{+}\to\mathcal{C}$, where $\mathbf{N}^{+}$ is the category $\mathbf{N}$ together with a chosen object, 0, and with the condition that morphisms of global towers have to be compatible with this chosen object.

Given $X=\{X_i\}_{i\in\mathbf{N}^{+}}$, $Y=\{Y_i\}_{i\in\mathbf{N}^{+}}$ in $\mathrm{tow}^{+}\mathcal{C}$, a global morphism $f\colon X\to Y$ in $\mathrm{tow}^{+}\mathcal{C}$ can be represented by a sequence of morphisms $\{f_i\colon X_{\varphi(i)}\to Y_i\}_{i\in\mathbf{N}^{+}}$, where $\varphi\colon\mathbf{N}^{+}\to\mathbf{N}^{+}$ satisfies that $\varphi(0)=0$, $\varphi(i)\geq i$ and if $i<j$, then $\varphi(i)<\varphi(j)$. Two sequences $\{f_i\colon X_{\varphi(i)}\to Y_i\}_{i\in\mathbf{N}^{+}}$, $\{g_i\colon X_{\psi(i)}\to Y_i\}_{i\in\mathbf{N}^{+}}$, represent the same global morphism if there exists $\theta\colon\mathbf{N}^{+}\to\mathbf{N}^{+}$ (satisfying the same properties as above) such that $\theta \geq \varphi$, $\theta \geq \phi$ and $f_i X^{\theta(i)}_{\varphi(i)} = g_i X^{\theta(i)}_{\psi(i)}$, where $X^j_i \colon X_j \to X_i$, $j \geq i$, denotes the bonding morphism of the global pro-object $X = \{X_i\}$.

If the map φ is the identity $\varphi(i) = i, i \in \mathbf{N}^+$, it is said that the f is a level global morphism. For a more detailed description of this type of category see again [**?**].

Let **Gr** be the category of groups. For $\mathbf{Top}^*$ and **Gr**, we have the corresponding global categories:

$$\mathrm{pro}^+\mathbf{Top}^*, \quad \mathrm{pro}^+\mathbf{Gr}, \quad \mathrm{tow}^+\mathbf{Top}^*, \quad \mathrm{tow}^+\mathbf{Gr}.$$

The homotopy group π_q induces the functors:

$$\mathrm{pro}^+\mathbf{Top}^* \xrightarrow{\mathrm{pro}^+\pi_q} \mathrm{pro}^+\mathbf{Gr}$$

$$\mathrm{tow}^+\mathbf{Top}^* \xrightarrow{\mathrm{tow}^+\pi_q} \mathrm{tow}^+\mathbf{Gr}$$

Given a family of groups $\{G_\alpha | \alpha \in A\}$, denote by $\sqcup_{\alpha\in A} G_\alpha$ the coproduct (free product) in the category of groups. Associated with the infinite cyclic group $\mathbb{Z}$, we can consider the global tower of groups:

$$\mathcal{Z} = \{\cdots \to \sqcup_2^\infty \mathbb{Z} \to \sqcup_1^\infty \mathbb{Z} \to \sqcup_0^\infty \mathbb{Z}\}$$

Then one can consider the following global version of Brown's functor, see [**?**, **?**, **?**]:

$$\mathcal{P} \colon \mathrm{tow}^+\mathbf{Gr} \to \mathbf{Gr}, \quad \mathcal{P}(\mathcal{G}) = \mathbf{Hom}_{\mathrm{tow}^+\mathbf{Gr}}(\mathcal{Z}, \mathcal{G})$$

where $\mathcal{G}$ is an object in $\mathrm{tow}^+\mathbf{Gr}$.

The usual version of Brown's functor is given by

$$\mathcal{P}^\infty \colon \mathrm{tow}\mathbf{Gr} \to \mathbf{Gr}, \quad \mathcal{P}^\infty(\mathcal{G}) = \mathbf{Hom}_{\mathrm{tow}\mathbf{Gr}}(\mathcal{Z}, \mathcal{G})$$

where now $\mathcal{Z}, \mathcal{G}$ are considered as objects in $\mathrm{tow}\mathbf{Gr}$.

Given a global tower of groups $\{G_i\}$, the group $\mathcal{P}(\{G_i\})$ can be described as follows: Consider sequences $\alpha \colon \mathbf{N}^+ \to \mathbf{N}^+$ such that if $i \leq j$, then $\alpha(i) \leq \alpha(j)$, $\alpha(0) = 0$ and $\alpha(i) \to \infty$. Take sequences of elements $\{g_{\alpha(i)}\}_0^\infty$ such that $g_{\alpha(i)} \in G_{\alpha(i)}$. Two sequences $\{g_{\alpha(i)}\}_0^\infty$, $\{h_{\beta(i)}\}_0^\infty$ are related if there exists a sequence γ such that $\gamma \leq \alpha$, $\gamma \leq \beta$ and $G^{\alpha(i)}_{\gamma(i)}(g_{\alpha(i)}) = G^{\beta(i)}_{\gamma(i)}(h_{\beta(i)})$, for every $i \geq 0$ (G^j_i denotes the bonding morphism). The group $\mathcal{P}(\{G_i\})$ is given by the set of equivalence classes of the form $[\{g_{\alpha(i)}\}_0^\infty]$.

The group $\mathcal{P}^\infty(\{G_i\})$ is the quotient of $\mathcal{P}(\{G_i\})$ obtained as follows: two elements of $\mathcal{P}(\{G_i\})$ are related if they can be represented by sequences $\{g_{\alpha(i)}\}_0^\infty$, $\{h_{\alpha(i)}\}_0^\infty$ such that there is $i_0 \geq 0$ with $g_{\alpha(i)} = h_{\alpha(i)}$ for every $i \geq i_0$. The class represented by $\{g_{\alpha(i)}\}_0^\infty$ will be denoted by

$[\{g_{\alpha(i)}\}_0^\infty]^\infty \in \mathcal{P}^\infty(\{G_i\})$. We remark that the description given of the functors $\mathcal{P}, \mathcal{P}^\infty$ is also valid for global towers of pointed sets. Denote by **Set*** the category of pointed sets.

We refer the reader to [**?**, Section 3, Corollary 1], that contains for towers of groups (and pointed sets) a proof of the fact that Brown's functor preserves and reflects isomorphisms:

Lemma 1. (Grossman) *A morphism* $f\colon \mathcal{G} \to \mathcal{H}$ *in* tow**Gr** (tow**Set***) *is an isomorphism if and only if* $\mathcal{P}^\infty(f)\colon \mathcal{P}^\infty(\mathcal{G}) \to \mathcal{P}^\infty(\mathcal{H})$ *is an isomorphism.*

Now we can use Grossman's result to prove a global version:

Lemma 2. *A global morphism* $f\colon \mathcal{G} \to \mathcal{H}$ *in* tow$^+$**Gr** (tow$^+$**Set***) *is an isomorphism if and only if* $\mathcal{P}(f)\colon \mathcal{P}(\mathcal{G}) \to \mathcal{P}(\mathcal{H})$ *is an isomorphism.*

Proof. Suppose that $f\colon \mathcal{G} \to \mathcal{H}$ is a global morphism, where $\mathcal{G} = \{G_i\}$ and $\mathcal{H} = \{H_i\}$. Then one has the following commutative diagram:

$$\begin{tikzcd} \mathcal{P}(\mathcal{G}) \arrow[r] \arrow[d, "\mathcal{P}(f)"] & \mathcal{P}^\infty(\mathcal{G}) \arrow[d, "\mathcal{P}^\infty(f)"] \\ \mathcal{P}(\mathcal{H}) \arrow[r] & \mathcal{P}^\infty(\mathcal{H}) \end{tikzcd}$$

It is easy to check that if $\mathcal{P}(f)$ is surjective, then f_0 is surjective: If $h \in H_0$ is not in the image of $f_0\colon G_0 \to H_0$, then any element of the form $[\{h_{\alpha(i)}\}_0^\infty]$ with $h_{\alpha(0)} = h$ is not in the image of $\mathcal{P}(f)$. We can also see that if $\mathcal{P}(f)$ is injective, then f_0 is injective: If we have two elements $g, g' \in G_0$ such that $f_0(g) = f_0(g')$, one can consider the sequence $\{g_{\alpha(i)}\}_0^\infty$ such that $g_{\alpha(0)} = g$, $g_{\alpha(i)} = 1$, $i > 0$ and similarly $\{g'_{\alpha(i)}\}_0^\infty$. Then, we have that $\mathcal{P}(f)[\{g_{\alpha(i)}\}_0^\infty] = \mathcal{P}(f)[\{g'_{\alpha(i)}\}_0^\infty]$. This implies that $g = g_{\alpha(0)} = g'_{\alpha(0)} = g'$.

Since $\mathcal{P}(\mathcal{H}) \to \mathcal{P}^\infty(\mathcal{H})$ is surjective and taking into account that the diagram is commutative, we have that if $\mathcal{P}(f)$ is surjective, then $\mathcal{P}^\infty(f)$ is also surjective. Assume that $\mathcal{P}(f)$ is injective and suppose that we have two elements $x, x' \in \mathcal{P}^\infty(\mathcal{G})$, $x = [\{g_{\alpha(i)}\}_0^\infty]^\infty$, $x' = [\{g'_{\alpha(i)}\}_0^\infty]^\infty$ such that $\mathcal{P}^\infty(f)(x) = \mathcal{P}^\infty(f)(x')$. This implies that we can modify a finite number of elements of the sequences $\{g_{\alpha(i)}\}_0^\infty$, $\{g'_{\alpha(i)}\}_0^\infty$ to obtain new sequences $\{\bar{g}_{\alpha(i)}\}_0^\infty$, $\{\bar{g}'_{\alpha(i)}\}_0^\infty$ such that $[\{\bar{g}_\alpha(i)\}_0^\infty]^\infty = x$, $[\{\bar{g}'_\alpha(i)\}_0^\infty]^\infty = x'$, $\bar{g}_{\alpha(i)} = 1 = \bar{g}'_{\alpha(i)}$, for $0 \leq i \leq i_0$ and $\bar{g}_{\alpha(i)} = g_{\alpha(i)}$, $\bar{g}'_{\alpha(i)} = g'_{\alpha(i)}$ for $i \geq i_0$ and the elements $\bar{x} = [\{\bar{g}_\alpha(i)\}_0^\infty], \bar{x}' = [\{\bar{g}'_\alpha(i)\}_0^\infty] \in \mathcal{P}(\mathcal{G})$ verify $\mathcal{P}(f)(\bar{x}) = \mathcal{P}(f)(\bar{x}')$. Since $\mathcal{P}(f)$ is injective, it follows that $\bar{x} = \bar{x}'$. This implies $x = x'$.

Therefore if $\mathcal{P}(f)$ is an isomorphism, it follows that f_0 and $\mathcal{P}^\infty(f)$ are isomorphisms. By Lemma **??**, one has that $f\colon \mathcal{G} \to \mathcal{H}$ in tow**Gr** (tow**Set***) is an isomorphism (and f_0 is an isomorphism). This implies that $f\colon \mathcal{G} \to \mathcal{H}$ in tow$^+$**Gr** (tow$^+$**Set***) is also an isomorphism.

□

Definition 6. *Suppose that* $\{f_i\colon X_{\varphi(i)} \to Y_i\}_{i\in\mathbf{N}^+}$ *represents a global morphism* f *in* tow$^+$**Top*** *and* S *is a set of non negative integers. It is said that* f *is a weak* S*-equivalence if* $\mathrm{tow}^+\pi_q(f)$ *is an isomorphism for every* $q \in S$*. We denote by* Σ^S *the corresponding class of* S*-weak equivalences and by* tow$^+$**Top**$^*[\Sigma^S]^{-1}$ *the localization induced by* Σ^S*. This category will be called the category of* S*-types of global towers of pointed spaces.*

Next we want to compare exterior $\mathbb{N}$-S-types with S-types of global towers.

Given a exterior space $(X,\lambda) \in \mathbf{E}^{\mathbb{R}+}$ we can factorize the map $\lambda\colon \mathbb{R}_+ \to X$ as $\lambda = q'\lambda'$, where λ' is an exterior cofibration and q' is an exterior homotopy equivalence. A map $i\colon A \to X$ is said to be an exterior cofibration if i is a closed map (in the category of topological spaces) and i has the exterior homotopy extension property, see [**?**].

Take the exterior space $X' = ((\mathbb{R}_+ \bar{\times} I) \cup X)/(t,1) \sim \lambda(t), t \in \mathbb{R}_+$, and consider the exterior maps $\lambda'\colon \mathbb{R}_+ \to X'$, $\lambda'(t) = [(t,0)]$ and $q'\colon X' \to X$ is given by $q'[(t,s)] = \lambda(t)$, $q'[x] = x$, $t \in \mathbb{R}_+$, $s \in I$, $x \in X$.

$$\begin{tikzcd} & X' \arrow[d, "q'"] \\ \mathbb{R}_+ \arrow[ur, "\lambda'"] \arrow[r, "\lambda"'] & X \end{tikzcd}$$

If $\mathbf{E}^{\mathbb{R}+}_w$ denotes the full subcategory of $\mathbf{E}^{\mathbb{R}+}$ determined by cofibrant exterior spaces, that is, objects (X,ν) where ν is an exterior cofibration (well rayed exterior spaces), the above factorization induces a functor $(\cdot)'\colon \mathbf{E}^{\mathbb{R}+} \to \mathbf{E}^{\mathbb{R}+}_w$ and together with the inclusion functor $\mathbf{E}^{\mathbb{R}+}_w \to \mathbf{E}^{\mathbb{R}+}$ one has an induced equivalence of categories

$$\mathbf{E}^{\mathbb{R}+}[\Sigma^S_{\mathbb{N}}]^{-1} \simeq \mathbf{E}^{\mathbb{R}+}_w[\Sigma^S_{\mathbb{N}}]^{-1}. \tag{1}$$

This follows from the fact that q' is an exterior homotopy equivalence, which is also a weak $\mathbb{N}$-S-equivalence.

It is interesting to note that if $X \in \mathbf{E}^{\mathbb{R}+}$ is first countable at infinity, one can find a sequence $\{X'_n\}$ which is a countable base of the externology of X' and $c_n \in \mathbb{R}_+$ such that $\lim c_n = \infty$ and $[0,c_n] \cong \lambda'(\mathbb{R}_+) \cap X'_n \to X'_n$ is a cofibration of topological spaces, see [**?**].

Denote by $\mathbf{E}^{\mathbb{R}+}_{fcw}$ the full subcategory of first countable well rayed exterior spaces; that is, the base ray $\lambda\colon \mathbb{R}_+ \to X$ is an exterior cofibration and there exists a countable externology base $\{X_n\}$ of X and $c_n \in \mathbb{R}_+$, $\lim c_n = \infty$, such that $X_0 = X$ and $[0, c_n] \cong \lambda(\mathbb{R}_+) \cap X_n \to X_n$ is a cofibration of topological spaces. We can see that the equivalence given in (**??**) induces a new category equivalence

$$\mathbf{E}^{\mathbb{R}+}_{fc}[\Sigma^S_{\mathbb{N}}]^{-1} \simeq \mathbf{E}^{\mathbb{R}+}_{fcw}[\Sigma^S_{\mathbb{N}}]^{-1}. \tag{2}$$

Taking the one-point space $\star$ with the trivial externology, one can consider the following pushout in $\mathbf{E}$

$$\begin{tikzcd} \mathbb{R}_+ \arrow[r] \arrow[d, "\lambda"'] & \star \arrow[d] \\ X \arrow[r] & \bar{X} \end{tikzcd}$$

This gives a new functor $(\bar{\cdot})\colon \mathbf{E}^{\mathbb{R}+} \to \mathbf{E}^{\star}$. Note that if (X, λ) is a well rayed exterior space, then $(\bar{X}, \star)$ is a well pointed exterior space; that is, $\star \to X$ is an exterior cofibration. Moreover, if X is an object in $\mathbf{E}^{\mathbb{R}+}_{fcw}$, then $(\bar{X}, \star)$ is in $\mathbf{E}^{\star}_{fcw}$. Then, we obtain the corresponding restriction $(\bar{\cdot})\colon \mathbf{E}^{\mathbb{R}+}_{fcw} \to \mathbf{E}^{\star}_{fcw}$, where $\mathbf{E}^{\star}_{fcw}$ is the full subcategory determined by first countable well pointed exterior spaces. Taking into account that if X is in $\mathbf{E}^{\mathbb{R}+}_{fcw}$, the canonical map $X \to \bar{X}$ is a weak $\mathbb{N}$-S-equivalence, one has that the restriction functor $(\bar{\cdot})\colon \mathbf{E}^{\mathbb{R}+}_{fcw} \to \mathbf{E}^{\star}_{fcw}$ induces an equivalence of categories

$$\mathbf{E}^{\mathbb{R}+}_{fcw}[\Sigma^S_{\mathbb{N}}]^{-1} \simeq \mathbf{E}^{\star}_{fcw}[\Sigma^S_{\mathbb{N}}]^{-1}. \tag{3}$$

Observe that a pointed exterior space $\star \to X$ can be considered as a rayed space taking the canonical map $\mathbb{R}_+ \to \star \to X$. Then we also have exterior homotopy groups $\pi^{\mathbb{N}}_q(X)$ for any object X in $\mathbf{E}^{\star}$ and the corresponding notion of weak $\mathbb{N}$-S-equivalence.

Given an exterior space $X \in \mathbf{E}^{\star}$, the externology ε_X can be seen as a left-filtering category with a final object and we can consider the functor

$$\varepsilon(X)\colon \varepsilon_X \to \mathbf{Top}^*, \varepsilon(X)(E) = E$$

where E is a pointed space.

This induces a full embedding

$$\varepsilon\colon \mathbf{E}^{\star} \to \mathrm{pro}^{+}\mathbf{Top}^*.$$

Now for each space X in $\mathbf{E}^{\star}_{fc}$, we can choose a fixed countable base $\{X_n\}$ of the externology ε_X (we are using the Axiom of Choice for the collection $\{\mathcal{B}(X)|X \in \mathbf{E}^{\star}_{fc}\}$, where $\mathcal{B}(X)$ is the set of countable bases of the externology ε_X). This gives a functor

$$\varepsilon\colon \mathbf{E}^{\star}_{fc} \to \mathrm{tow}^{+}\mathbf{Top}^{*}\ , \varepsilon(X) = \{X_n\}.$$

Lemma 3. *The following diagram*

$$\begin{tikzcd} \mathbf{E}^{\star}_{fc} \arrow[r, "\varepsilon"] \arrow[d, "\pi_q^{\mathbb{N}}"'] & \mathrm{tow}^{+}\mathbf{Top}^{*} \arrow[d, "\mathrm{tow}^{+}\pi_q"] \\ \mathbf{Gr} & \mathrm{tow}^{+}\mathbf{Gr} \arrow[l, "\mathcal{P}"] \end{tikzcd}$$

is commutative up to isomorphism (for $q = 0$, one has to consider $\mathbf{Set}^{*}$ *instead of* $\mathbf{Gr}$*).*

Proof. Given an object X in $\mathbf{E}^{\star}_{fc}$ and $\{X_i\}$ a countable base of its externology, we can define a canonical map $\theta\colon \pi_q^{\mathbb{N}}(X) \to \mathcal{P}(\{\pi_q X_i\})$ as follows: An element of $\pi_q^{\mathbb{N}}(X)$ is represented by a sequence of maps $f_{\alpha(i)}\colon S^q \to X$ such that $f_{\alpha(i)}(S^q) \subset X_{\alpha(i)}$. Notice that two sequences $f_{\alpha(i)}\colon S^q \to X$, $g_{\beta(i)}\colon S^q \to X$ represent the same element if there is $\gamma\colon \mathbf{N}^{+} \to \mathbf{N}^{+}$ such that $\gamma \leq \alpha$, $\gamma \leq \beta$ and $f_{\alpha(i)}$ is pointed homotopic to $g_{\beta(i)}$ in $X_{\gamma(i)}$. Then, the canonical isomorphism is given by $\theta([\{f_{\alpha(i)}\}] = [\{[f_{\alpha(i)}]\}]$, where $[f_{\alpha(i)}] \in \pi_q(X_{\alpha(i)})$.

□

Proposition 2. *The functor $\varepsilon\colon \mathbf{E}^{\star}_{fc} \to \mathrm{tow}^{+}\mathbf{Top}^{*}$ satisfies that if $f\colon X \to Y$ is a morphism in $\mathbf{E}^{\star}_{fc}$, then $\pi_q^{\mathbb{N}}(f)$ is an isomorphism if and only if* $\mathrm{tow}^{+}\pi_q$ *(εf) is an isomorphism.*

Proof. Given a morphism $f\colon X \to Y$ in $\mathbf{E}_{fc}$, by Lemma **??**, we have that $\pi_q^{\mathbb{N}}(f) \cong \mathcal{P}\mathrm{tow}^{+}\mathrm{\pi_q}(\tilde{\varepsilon}(\mathrm{f}))$. Now the result follows from Lemma **??** that proves that $\mathcal{P}$ preserves and reflects isomorphims. □

We also consider the following Telescopic construction (a similar construction is given by Edwards-Hastings in [**?**], but that version gave a contractible topological space):

Given a global tower of pointed topological spaces

$$\{X_i\} = \{\ \cdots \longrightarrow X_2 \xrightarrow{X_1^2} X_1 \xrightarrow{X_0^1} X_0\ \},$$

the telescope of $\{X_i\}$ is constructed as the following quotient space

$$\mathrm{Tel}\{X_i\} = \left(X_0 \times \{0\} \cup \coprod_{1}^{\infty} X_i \times [i-1, i] \right) / \sim$$

where $(X_i^{i+1}(x), i) \sim (x, i)$, $x \in X_{i+1}$, $i \geq 0$. $\mathrm{Tel}\{X_i\}$ has a unique externology such that the family

$$\{E_n = \left(X_n \times \{n\} \cup \coprod_{n+1}^{\infty} X_i \times [i-1, i] \right) / \sim | n \geq 0\}$$

is a countable exterior neighbourhood base. We remark that $\mathrm{Tel}\{X_i\}$ has a canonical base ray $\lambda \colon \mathbb{R}_+ \to \mathrm{Tel}\{X_i\}$, $\lambda(t) = [(*, t)]$ and we can consider the associated exterior space $\overline{(\mathrm{Tel}\{X_i\})'}$.

Given a morphism f of global towers of pointed spaces represented by $\{f_i \colon X_{\varphi(i)} \to Y_i\}$, we get an induced exterior map $\mathrm{Tel}\{f_i\} \colon \mathrm{Tel}\{X_i\} \longrightarrow \mathrm{Tel}\{Y_i\}$.

If $\{f_i' \colon X_{\varphi'(i)} \to Y_i\}$ represents the same morphism f in $\mathrm{tow}^+\mathbf{Top}^*$, then one can check that $\overline{(\mathrm{Tel}(\{f_i\})'}, \overline{(\mathrm{Tel}(\{f_i'\})'}$ are exterior homotopic in $\mathbf{E}^\star_{fc}$.

Denote by $\tilde{\varepsilon} = \varepsilon(\bar{\cdot})(\cdot)'$ the composite

$$\mathbf{E}^{\mathbb{R}_+}_{fc} \xrightarrow{(\cdot)'} \mathbf{E}^{\mathbb{R}_+}_{fc} \xrightarrow{(\bar{\cdot})} \mathbf{E}^\star_{fcw} \xrightarrow{\varepsilon} \mathrm{tow}^+\mathbf{Top}^*$$

Using the construction Tel and the functor $\tilde{\varepsilon}$, we can obtain our main result:

Theorem 1. *Suppose that S is a set of non negative integer numbers. Then $\tilde{\varepsilon} \colon \mathbf{E}^{\mathbb{R}_+}_{fc} \to \mathrm{tow}^+\mathbf{Top}^*$ induces an equivalence of categories*

$$\mathbf{E}^{\mathbb{R}_+}_{fc}[\Sigma^S_{\mathbb{N}}]^{-1} \simeq \mathrm{tow}^+\mathbf{Top}^*[\Sigma^S]^{-1}.$$

That is, the category of exterior $\mathbb{N}$-S-types is equivalent to the category of S-types of global towers of pointed spaces.

Proof. We note that by Proposition **??**, the functor $\varepsilon \colon \mathbf{E}^\star_{fcw} \to \mathrm{tow}^+\mathbf{Top}^*$ preserves weak $\mathbb{N}$-S-equivalences. Then, there is an induced functor

$$\varepsilon \colon \mathbf{E}^\star_{fcw}[\Sigma^S_{\mathbb{N}}]^{-1} \to \mathrm{tow}^+\mathbf{Top}^*[\Sigma^{\mathrm{S}}]^{-1}.$$

Given a global tower of pointed topological spaces $X = \{X_i\} \in \mathrm{tow}^+\mathbf{Top}^*$, one has the telescope $\mathrm{Tel}(\{X_i\})$ with its corresponding countable neigbourhood base at infinity $\{E_n\}$ and canonical base ray λ. Then, we have maps

$$\mathrm{Tel}(\{X_i\}) \xrightarrow{q'} (\mathrm{Tel}\{X_i\})' \xrightarrow{\overline{q'}} \overline{(\mathrm{Tel}(\{X_i\})'}.$$

For each $n \geq 0$, $E^1_n = (\mathbb{R}_+ \times [0, \frac{1}{n+1}] \cup [n, \infty) \times I \cup E_n)/(t, 1) \sim \lambda(t), t \in [n, \infty)$ gives a countable neighbourhood base at infinity for $(\mathrm{Tel}\{X_i\})'$ and $E^2_n = E^1_n/\mathbb{R}_+$ gives a countable neighbourhood base at infinity for $\overline{(\mathrm{Tel}(\{X_i\})'}$. It is easy to check that the natural inclusion

$E_n \subset E_n^1$ and the quotient map $E_n^1 \to E_n^2$ are homotopy equivalences. We also have the following global level maps $\{E_n\} \to \{E_n^1\} \to \{E_n^2\}$.

Note that the map of towers, $\{p_n : E_n \to X_n\}$ given by $p_n[(x,t)] = X_n^i x$, $x \in X_i$, $i \geq n$, where X_n^i denotes the corresponding boundary composition, is a homotopy equivalence.

Therefore one has the following global level maps

$$X = \{X_n\} \leftarrow \{E_n\} \to \{E_n^1\} \to \{E_n^2\} = \varepsilon\left(\overline{(\mathrm{Tel}(X))}'\right)$$

which induce isomorphisms in the localized category $\mathrm{tow}^+\mathbf{Top}^*[\Sigma^{\mathrm{S}}]^{-1}$.

Given a global morphism g in $\mathrm{tow}^+\mathbf{Top}^*$, we have that

$$\varepsilon(\overline{(\mathrm{Tel}(g))}') \cong g$$

in $\mathrm{tow}^+\mathbf{Top}^*[\Sigma^S]^{-1}$. Then, $\mathrm{tow}^+\pi_q(g)$ is an isomorphism if and only if $\mathrm{tow}^+\pi_q(\varepsilon(\overline{(\mathrm{Tel}(g))}')$ is an isomorphism and this is equivalent, by Proposition **??**, to the fact that $\pi_q^{\mathbb{N}}(\overline{(\mathrm{Tel}(g))}')$ is an isomorphism.

This implies that the telescopic construction induces a well defined functor

$$\overline{(\mathrm{Tel})}' \colon \mathrm{tow}^+\mathbf{Top}^*[\Sigma^S]^{-1} \to \mathbf{E}^{\star}_{fcw}[\Sigma^S_{\mathbb{N}}]^{-1}$$

where $\{X_i\}$ is carried to $\overline{(\mathrm{Tel}(\{X_i\}))}'$.

Now take an object X in $\mathbf{E}^{\star}_{fcw}$ with $\varepsilon(X) = \{X_i\}$. Since each map $E_n^2 \to X_n$ is a homotopy equivalence, we have that the canonical map $\overline{(\mathrm{Tel}(\varepsilon(X))}' \to X$ is an exterior weak $\mathbb{N}$-equivalence. Then, $\overline{(\mathrm{Tel})}'(\varepsilon(X)) \to X$ induces an isomorphism in the category $\mathbf{E}^{\star}_{fcw}[\Sigma^S_{\mathbb{N}}]^{-1}$.

Therefore we have seen that the composites $\overline{(\mathrm{Tel})}'\,\varepsilon$, $\varepsilon\,\overline{(\mathrm{Tel})}'$ are isomorphic to identity functors on the localized categories. Then we obtain the category equivalence $\mathbf{E}^{\star}_{fcw}[\Sigma^S_{\mathbb{N}}]^{-1} \simeq \mathrm{tow}^+\mathbf{Top}^*[\Sigma^S]^{-1}$. Taking into account the equivalences given in (**??**) and (**??**), we have the desired category equivalence. □

Remark 2. *A direct description of the functor from first countable rayed exterior spaces to global towers of pointed spaces (without passing through the auxiliary notions of well-rayed and well-pointed exterior spaces) can be given as follows: Given a rayed exterior space, $\lambda \colon \mathbb{R}_+ \to X$, if X is first countable at infinity we can choose a sequence of 'neighbourhoods at infinity' $\{X_n\}$ such that $X_0 = X$ and*

$\lambda^{-1}(X_n) = [0, c_n]$, $\lim c_n = \infty$, $c_{n+1} \geq c_n$. *Consider the pushout*

$$\begin{tikzcd} [0, c_n] \arrow[r] \arrow[d, "\lambda|_{[0,c_n]}"] & \mathbb{R}_+ \arrow[d] \\ X_n \arrow[r] & \hat{X}_n \end{tikzcd}$$

where $0 \in \mathbb{R}_+$ *determines a base point in* $\hat{X}_n$. *It is easy to check that the functor* $\tilde{\varepsilon}\colon \mathbf{E}^{\mathbb{R}_+}_{fc}[\Sigma^S_{\mathbb{N}}]^{-1} \to \mathrm{tow}^+\mathbf{Top}^*[\Sigma^S]^{-1}$ *is isomorphic to the functor* $\hat{\varepsilon}\colon \mathbf{E}^{\mathbb{R}_+}_{fc}[\Sigma^S_{\mathbb{N}}]^{-1} \to \mathrm{tow}^+\mathbf{Top}^*[\Sigma^S]^{-1}$, $\hat{\varepsilon}(X) = \{\hat{X}_n\}$. *In the proof of theorem above, we have preferred to use well-rayed exterior spaces because this is a canonical procedure that changes an object by a cofibrant substitute for exterior spaces.*

Remark 3. *As a consequence of the theorem above, one has*

(a) *If S is the set of non negative integers, the exterior homotopy category of rayed cofibrant first countable exterior spaces, see* [**?**, **?**], *is equivalent to the category of fractions* $\mathrm{tow}^+\mathbf{Top}^*[\Sigma^S]^{-1}$.

(b) *If* $S = \{0\}$, $\mathbf{E}^{\mathbb{R}_+}[\Sigma^{\{0\}}_{\mathbb{N}}]^{-1}$ *is the category of exterior* 0*-types. On the other hand, it is well known that* $\pi_0\colon \mathbf{Top}^* \to \mathbf{Set}^*$ *induces an equivalence between* 0*-types of pointed spaces and pointed sets. Therefore* $\mathrm{tow}^+\mathbf{Top}^*[\Sigma^{\{0\}}]^{-1}$ *is equivalent to* $\mathrm{tow}^+\mathbf{Set}^*$.

(c) *If* $S = \{1\}$, *the functor* $\pi_1\colon \mathbf{Top}^* \to \mathbf{Gr}$ *induces an equivalence between* 1*-types of pointed spaces and groups. Then, in this case, one has that the exterior homotopy category of exterior* $\mathbb{N}$*-Eilenberg-Mac Lane spaces which are first countable at infinity is equivalent to the category of global towers of groups.*

(d) *For* $S = \{1, 2\}$, *it is well known that the fundamental categorical group functor* $\rho_2\colon \mathbf{Top}^* \to \mathbf{CG}$ *and the classifying functor* $B\colon \mathbf{CG} \to \mathbf{Top}^*$ *induce an equivalence between the category of 2-types of pointed 0-connected spaces and categorical groups up to weak equivalence, see* [**?**, **?**]. *Therefore, in this case, the functor* $(\mathrm{tow}^+\rho_2)\,\tilde{\varepsilon}$ *and the construction* $\mathrm{Tel}\,(\mathrm{tow}^+\mathrm{B})$ *induce an equivalence of categories*

$$\mathbf{E}^{\mathbb{R}_+}_{fc}[\Sigma^{\{1,2\}}_{\mathbb{N}}]^{-1} \simeq \mathrm{tow}^+\mathbf{CG}[\Sigma]^{-1},$$

where Σ *is the class of maps in* $\mathrm{tow}^+\mathbf{CG}$ *given by the saturation of the class of the level weak equivalences of categorical groups.*

Remark 4. *For exterior spaces with a base ray* (X, λ) *we can also consider the q-th* $\mathbb{R}_+$*-exterior homotopy group given by*

$$\pi^{\mathbb{R}_+}_q(X, \lambda) = \pi_q(X^{\mathbb{R}_+}, \lambda).$$

This permits to introduce new categories of fractions taking $\Sigma^S_{\mathbb{R}_+}$ *the class of exterior weak* $\mathbb{R}_+$*-*S*-equivalences. There are interesting relations between the categories* $\mathbf{E}^{\mathbb{R}_+}_{fc}[\Sigma^S_{\mathbb{N}}]^{-1}$ *and* $\mathbf{E}^{\mathbb{R}_+}_{fc}[\Sigma^S_{\mathbb{R}_+}]^{-1}$ *that will be analyzed in a further paper. For example, for* $S = \{0, \cdots, n+1\}$ *and* $S' = \{0, \cdots, n\}$ *one has a canonical functor* $\mathbf{E}^{\mathbb{R}_+}_{fc}[\Sigma^S_{\mathbb{N}}]^{-1} \to \mathbf{E}^{\mathbb{R}_+}_{fc}[\Sigma^{S'}_{\mathbb{R}_+}]^{-1}$*. The formulation of our main result in terms of rayed exterior spaces and tower of pointed spaces permits to compare in a natural way* $\mathbb{N}$*-*S*-types,* $\mathbb{R}_+$*-*S*-types,* S*-types of global towers of pointed spaces, global towers of* S*-types of pointed spaces and standard* S*-types of pointed spaces.*

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