Fusion systems, groups, partial groups and simplicial sets

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Métodos Categóricos y Homotópicos en Álgebra, Geometría y Topología

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Outline

Introduction

In a paper published in 2003, together with Ran Levi and Bob Oliver, we introduced the concept of p-local finite group.

For a fixed prime p, this is a triple $(S, \mathcal{F}, \mathcal{L})$, where

- S is a finite p-group
- $\mathcal{F}$ is a saturated fusion system over S, and
- $\mathcal{L}$ is an associated centric linking system.

This last is a category extending the fusion system and we define the classifying space of $(S, \mathcal{F}, \mathcal{L})$ as the p-completed nerve $|\mathcal{L}|^\wedge_p$

The aim of this talk is to describe the homotopy fixed point set

$$\left(|\mathcal{L}|^\wedge_p\right)^{h\pi}$$

by the action of a finite p-group π on $(S, \mathcal{F}, \mathcal{L})$.

We describe new models for the classifying space due to Andy Chermak (2013) and point to a precise one, $\mathbb{L}_?$, that carries an action of π, and for which we can prove

$$\left[(\mathbb{L}_?)^\wedge_p\right]^{h\pi} \simeq \coprod_{\sigma \in H^1(\pi, \mathbb{L}_?)} (\mathbb{L}_?^\sigma)^\wedge_p$$

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Fusion systems

Definition

Let G be a finite group, p a prime number, and $S \in \mathrm{Syl}_p(G)$.
The fusion system of G consists of

- *Objects: $P \leq S$, the subgroups of S, and*
- *Morphisms:*

$$\begin{aligned}\mathrm{Hom}_{\mathcal{F}_S(G)}(P,Q) &= \{\varphi\colon P \to Q \mid \exists g \in G, \varphi(x) = gxg^{-1}\} \\ &\cong N_G(P,Q)/C_G(P)\end{aligned}$$

that compose as group homomorphisms.

Fusion systems

Definition (Puig)

A fusion system $\mathcal{F}$ over a finite p-group S consists of a set $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ for every pair P,Q of subgroups of S such that

$$\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$$

and form a category where every morphism decomposes as an isomorphism followed by an inclusion.

It is saturated if it satisfies some extra axioms. ▸ Axioms

- If G is a finite group and $S \in \mathrm{Syl}_p(G)$, then $\mathcal{F}_S(G)$ is a saturated fusion system.
- We will say that a saturated fusion system $\mathcal{F}$ is *exotic* if it is not of this form.

- Brauer (1964) defines abstract fusion between elements in p-groups.

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Fusion systems

There are exotic saturated fusion systems at each prime p, but only one family is know at the prime 2:

Solomon, 1974. There exists a well defined 2-local structure over the Sylow 2-subgroup of $\mathrm{Spin}_7(q)$ (q odd prime power) which contains a unique conjugacy class of involutions. But there is no finite group with such structure.

Benson, 1994. There is a space $B\mathrm{Sol}(q)$, related to Dwyer-Wilkerson exotic 2-local finite loop space $DI(4)$, which supports the 2-local structure defined by Solomon: A classifying space for a non-existing group !

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Fusion systems show up while studying the homotopy type of p-completed classifying spaces of finite groups.

- Martino-Priddy conjecture (1996):

$$BG_p^\wedge \simeq BH_p^\wedge \quad \Longleftrightarrow \quad \mathcal{F}_p(G) \simeq \mathcal{F}_p(H)$$

Two sorts of objects are introduced in the arguments of the above results

- Linking systems (B-Levi-Oliver)
- Partial groups and localities (Chermak)

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Fusion systems

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Fix a saturated fusion system $\mathcal{F}$ over a finite p-group S. Let Δ be the family of $\mathcal{F}$-centric subgroups of S.

An associated centric linking system is a category defined with objects Δ and extending $\mathcal{F}$ in the sense that

$$\mathrm{Mor}_{\mathcal{F}}(P,Q) = \mathrm{Mor}_{\mathcal{L}}(P,Q)/Z(P)\,.$$

Some ▸ Axioms *should be satisfied.*

- $|\mathcal{L}|_p^\wedge$, is called the classifying space of $(S,\mathcal{F},\mathcal{L})$.

Fusion systems

- If G is a finite group and $S \in \mathrm{Syl}_p(G)$, then we construct a centric linking system $\mathcal{L}_S^c(G)$ associated to $\mathcal{F}_S(G)$ as the category with
  - Objects: $P \leq S$ such that $C_G(P) = Z(P) \times C'_G(P)$ with $(p, |C'_G(P)|) = 1$
  - Morphisms: $\mathrm{Mor}_{\mathcal{L}_S^c(G)}(P, G) = N_G(P, G)/C'_G(P)$
- Then, $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ is a p-local finite group with classifying space

$$|\mathcal{L}_S^c(G)|_p^\wedge \simeq (BG)_p^\wedge$$

- The Martino-Priddy conjecture is first reduced to showing that a fusion system $\mathcal{F}_S(G)$ admits a unique associated centric linking system.
- Oliver (2004, 2006): M-P conjecture is true: $\mathcal{L}_S(G)$ is the only possible centric linking system associated to $\mathcal{F}_S(G)$. (The proof depends on the classification of finite simple groups.)
- Chermak (2013): any (abstract) fusion system admits a unique associated centric linking system. Still depending on CFSG,
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Partial groups

Definition

A partial group is a simplicial set $\mathbb{M}$ satisfying

(P1) $\mathbb{M}_0$ *consists of a unique vertex*

(P2) *The spine operator* $e^n : \mathbb{M}_n \longrightarrow (\mathbb{M}_1)^n$ *is injective for all* $n \geq 1$.

(P3) *There is an inversion* $(-)^{-1} : \mathbb{M} \longrightarrow \mathbb{M}$ *such that for each* $\boldsymbol{u} \in \mathbb{M}_n$, $n \geq 1$:

- (I1) *There is a simplex* $[\nu(\boldsymbol{u})|\boldsymbol{u}] \in \mathbb{M}_{2n}$; *and*
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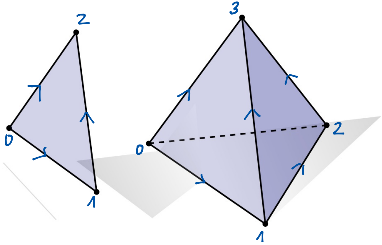

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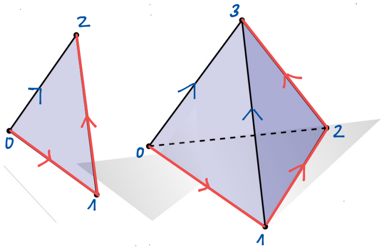

the sequence of edges joining the ordered vertices from 0 *through* n.

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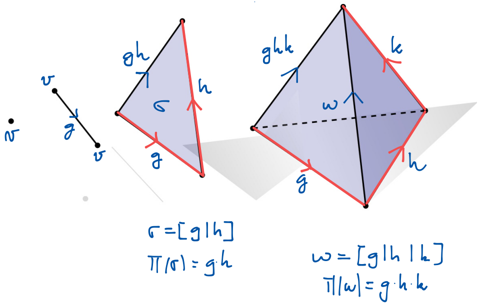
- *We write* $\boldsymbol{\omega} = [g_1|g_2|\dots|g_n]$. *if* $\boldsymbol{\omega} \in \mathbb{M}_n$ *and* $\boldsymbol{e}^n(\boldsymbol{\omega}) = (g_1, g_2, \dots, g_n)$*:*



We write $1 = s_0(v)$ *for* $v \in \mathbb{M}_0$, *and* $\Pi[x_1|x_2|\dots|x_n] = x_1 \cdot x_2 \cdot \dots \cdot x_n$.

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- A homomorphism of partial groups is a simplicial map
- A extension of partial groups is a fibre bundle $\mathbb{M} \xrightarrow{\imath} \mathbb{E} \xrightarrow{\tau} \mathbb{H}$
- It turns out that if $\mathbb{H}$ and $\mathbb{M}$ are partial groups, then so is $\mathbb{E}$

Partial groups

We now concentrate in extensions

$$\mathbb{M} \xrightarrow{\imath} \mathbb{E} \xrightarrow{\tau} BG$$

where G is a finite group.

- There is a fibration $B^2Z(\mathbb{M}) \to B\,\mathrm{aut}(\mathbb{M}) \to B\,\mathrm{Out}(\mathbb{M})$.
- The classification of extensions works exactly as in the case of finite groups
- The extension is regular split extension if it admits a regular section. A regular section defines an action of G on $\mathbb{M}$, and $\mathbb{E}$ can be described as a semidirect product.
- For a regular split extension, $H^1(G, \mathbb{M})$ classifies equivalence classes of sections.

Localities

Definition

Let $\mathbb{L}$ be a partial group with finite $\mathbb{L}_1$, $S \le \mathbb{L}$ a p-subgroup and Δ a family of subgroups of S. Then, $(\mathbb{L}, S)$ is a locality via Δ if the following holds

(O1) $[u_1|\ldots|u_n] \in \mathbb{L}_n$ *if and only if there is a string of composable conjugation maps between objects of* Δ*:*

$$X_0 \xleftarrow{u_1} X_1 \xleftarrow{u_2} \ldots \xleftarrow{u_n} X_n\,, \qquad X_i \in \Delta\,.$$

(O2) Δ *is closed by overgroups. Also, if $X \in \Delta$ and $u \in \mathbb{L}_1$ are such that ${}^uX \le S$, then ${}^uX \in \Delta$.*

(L1) S *is maximal in the poset (ordered by inclusion) of p-subgroups of $\mathbb{L}$.*

- Example: Let G be a finite group and fix $S \in \mathrm{Syl}_p(G)$, set $\mathcal{F} = \mathcal{F}_S(G)$, and let Γ be a non-empty $\mathcal{F}$-invariant collection of subgroups of S, closed under taking overgroups. Then

$$\mathbb{L}_\Gamma(G) = \Big\{ [g_1|g_2|\ldots|g_n] \in BG \;\Big|\; \exists P_0, P_1, \ldots, P_n \in \Gamma\,, \\ P_0 \xleftarrow{g_1} P_1 \xleftarrow{g_2} \ldots \xleftarrow{g_n} P_n \Big\} \subseteq BG$$

is a partial group, and $(\mathbb{L}_\Gamma(G), S)$ a locality via Γ.

Localities

- The locality $\mathbb{L}(\mathcal{L})$ of a p-local finite group $(S, \mathcal{F}, \mathcal{L})$.

  This is a construction due to Chermak.

  Let $\equiv$ be the equivalence relation defined on $\mathrm{Iso}(\mathcal{L})$ such that $f \equiv g$ if one is a restriction of the other. Define

$$\mathbb{L}(\mathcal{L})_n = \{\, [\bar{f}_1 | \bar{f}_2 | \dots | \bar{f}_n] \mid \exists P_0, P_1, \dots, P_n \in \mathrm{Ob}(\mathcal{L})\,,\\ P_0 \xleftarrow{f_1} P_1 \xleftarrow{f_2} \dots \xleftarrow{f_n} P_n \,\}$$

  Then, $(\mathbb{L}(\mathcal{L}), S)$ is a locality via $\Delta = \mathrm{Ob}(\mathcal{L})$

Localities

Theorem (B-González)

The natural projection $|\mathcal{L}| \longrightarrow \mathbb{L}$ is a weak equivalence of simplicial sets.

Furthermore, it is $|\mathcal{A}\mathrm{ut}_{\mathrm{typ}}(\mathbb{L})|$-equivariant and the action on $\mathbb{L}$ induces an isomorphism of simplicial groups

$$|\mathcal{A}\mathrm{ut}_{\mathrm{typ}}(\mathcal{L})| \cong \mathrm{aut}(\mathbb{L}, S)\,.$$

(This last is the simplicial subgroup of $\mathrm{aut}(\mathbb{L})$ that leaves S stable.)

It follows $B\,\mathrm{aut}(|\mathcal{L}|^\wedge_p) \simeq B|\mathcal{A}\mathrm{ut}_{\mathrm{typ}}(\mathcal{L})| \simeq B\,\mathrm{aut}(\mathbb{L}, S)$. $\mathcal{A}\mathrm{ut}_{\mathrm{typ}}$

• Localities provide models for classifying spaces of p-local finite groups. Some questions arise:

1. The homotopy type of $(\mathbb{L}, S)$ depends on the set of objects Δ. We need to adjust Δ so that we get to the right homotopy type.
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Localities

1. Concerning the homotopy type.

- If $(\mathbb{L}, S)$ is a locality via Δ and $H \in \Delta$, then

$$N_{\mathbb{L}}(H) = \{u \in \mathbb{L}_1 \,|\, {}^{u}H = H\}.$$

  and

$$C_{\mathbb{L}}(H) = \{u \in \mathbb{L}_1 \,|\, {}^{u}h = h,\ \forall h \in H\}.$$

  are subgroups of $\mathbb{L}$.

- This allows the association of fusion and linking systems, as we did for groups.
- A locality $(\mathbb{L}, S)$ via Δ is centric if it satisfies
  1. $C_{\mathbb{L}}(P)$ is a p-group for all $P \in \Delta$, and
  2. Δ contains all centric subgroups, that is, all $P \leq S$ such that $C_{\mathbb{L}}(P) = Z(P)$.
- Centric localities have *the right homotopy type.*

3. Centralizers and mapping spaces

- For a locality $(\mathbb{L}, S)$ and an arbitrary subgroup $T \leq S$, define the centralizer of T in $\mathbb{L}$ as the partial subgroup $\mathbb{C}_{\mathbb{L}}(T) \leq \mathbb{L}$ with n-simplices

$$\mathbb{C}_{\mathbb{L}}(T)_n = \left\{ [u_1|u_2|\ldots|u_n] \in \mathbb{L} \;\middle|\; {}^{u_i}h = h,\ \forall h \in H\,,\ i = 1, \ldots, n \right\}.$$

Proposition

Let $(\mathbb{L}, S)$ be a centric locality via Δ. If $T \leq S$ is fully centralized, then

- $(\mathbb{C}_{\mathbb{L}}(T), C_S(T))$ *is a centric locality via* $\Delta_T = \{C_P(T) \,|\, T \leq P \in \Delta\}$.

And the adjoint of the product map provides a homotopy equivalence

- $\mathbb{C}_{\mathbb{L}}(T)^{\wedge}_p \xrightarrow{\simeq} \mathrm{Map}(BT, \mathbb{L}^{\wedge}_p)_{\mathsf{incl}}$

Localities

2. Extensions

- Let $(\mathbb{L}, S)$ be a locality via Δ and let G be a discrete group. An extension

  $$\mathbb{L} \longrightarrow \mathbb{E} \longrightarrow BG$$

  is called *isotypical* if the structural group is $\mathrm{aut}(\mathbb{L}; S)$.

- If this is the case, then $\mathbb{E}$ is also a locality. More precisely:

  (a) There is an associated group extension

  $$\begin{array}{ccccc} BN_{\mathbb{L}}(S) & \longrightarrow & BN_{\mathbb{E}}(S) & \longrightarrow & BG \\ \downarrow & & \downarrow & & \Vert \\ \mathbb{L} & \longrightarrow & \mathbb{E} & \longrightarrow & BG \end{array}$$

  (b) Fix $\widetilde{S} \in \mathrm{Syl}_p(N_{\mathbb{E}}(S))$. Then $(\mathbb{E}, \widetilde{S})$ is a locality via $\widetilde{\Delta} = \{P \leq \widetilde{S} \mid P \cap S \in \Delta\}$.

  (c) If $\mathbb{L}$ is a centric locality, then $\mathcal{F}_{\widetilde{S}}(\mathbb{E})$ is saturated and $\widetilde{\Delta}$ contains all of the $\mathcal{F}_{\widetilde{S}}(\mathbb{E})$-centric $\mathcal{F}_{\widetilde{S}}(\mathbb{E})$-radical subgroups of $\widetilde{S}$.

Localities

Definition

Let $(\mathbb{L}, S)$ be a locality via Δ and $\mathbb{L} \xrightarrow{\imath} \mathbb{E} \xrightarrow{\tau} BG$ an isotypical extension, where G is a finite group.

A centric equivariant replacement for $(\mathbb{L}, \Delta, S)$ with respect to the extension τ is a partial group $\mathbb{L}_{\mathrm{eq}}$ together with a map of extensions

$$\begin{array}{ccccc} \mathbb{L} & \longrightarrow & \mathbb{E} & \xrightarrow{\tau} & BG \\ {\scriptstyle \jmath}\downarrow & & \downarrow & & \| \\ \mathbb{L}_{\mathrm{eq}} & \longrightarrow & \mathbb{E}_{\mathrm{eq}} & \xrightarrow[\tau_{\mathrm{eq}}]{} & BG \end{array}$$

where $\jmath \colon \mathbb{L} \to \mathbb{L}_{\mathrm{eq}}$ is a trivial cofibration and $\mathbb{E}_{\mathrm{eq}}$ is a centric locality.

Theorem (B-González)

Let $(\mathbb{L}, \Delta, S)$ be a centric locality.
If π is a finite p-group and $\mathbb{L} \longrightarrow \mathbb{E} \xrightarrow{\tau} B\pi$ is an isotypical extension, then, there exists a centric equivariant replacement of $(\mathbb{L}, \Delta, S)$ with respect to the extension τ.

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Homotopy fixed points

Definition

Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and π be a finite p-group. By an action of π on $(S, \mathcal{F}, \mathcal{L})$ we understand an action of π on a classifying space $X \simeq |\mathcal{L}|_p^\wedge$.

- Borel construction gives a fibration $X \longrightarrow X \times_\pi E\pi \longrightarrow B\pi$
- This fibration is the fibrewise p-completion of a fibre bundle $|\mathcal{L}| \longrightarrow Y \longrightarrow B\pi$ with structure group $|\mathcal{A}\mathrm{ut}_{\mathrm{typ}}(\mathcal{L})|$ (B-Levi-Oliver)
- Let $\mathbb{L} = \mathbb{L}(\mathcal{L})$ be the locality of $(S, \mathcal{F}, \mathcal{L})$, then the weak equivalence $|\mathcal{L}| \longrightarrow \mathbb{L}$ extends to a diagram of fibre bundles

$$\begin{array}{ccccc} \mathbb{L} & \xleftarrow{\simeq_w} & |\mathcal{L}| & \xrightarrow{\kappa_p} & X \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{E} & \xleftarrow{\simeq_w} & Y & \xrightarrow{\kappa_p} & X_{h\pi} \\ \downarrow & & \downarrow & & \downarrow \\ B\pi & = & B\pi & = & B\pi \end{array}$$

Homotopy fixed points

- The extension $\mathbb{L} \longrightarrow \mathbb{E} \longrightarrow B\pi$ admits a centric equivariant replacement:

$$\begin{array}{ccccccc}
\mathbb{L}_{\mathrm{eq}} & \xleftarrow{\simeq_w} & \mathbb{L} & \xleftarrow{\simeq_w} & |\mathcal{L}| & \xrightarrow{\kappa_p} & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{E}_{\mathrm{eq}} & \xleftarrow{\simeq_w} & \mathbb{E} & \xleftarrow{\simeq_w} & Y & \xrightarrow{\kappa_p} & X_{h\pi} \\
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\end{array}$$

- It follows that $X^{h\pi} \simeq \left[(\mathbb{L}_{\mathrm{eq}})^\wedge_p\right]^{h\pi}$.
- There are bijections

$$H^1(\pi;\mathbb{L}) \cong H^1(\pi;\mathbb{L}_{\mathrm{eq}}) \cong \pi_0((\mathbb{L}^\wedge_p)^{h\pi}).$$

- Since $\mathbb{E}_{\mathrm{eq}}$ is a centric locality, the adjoint of the evaluation provides a mod p homotopy equivalence

$$\mathbb{C}_{\mathbb{E}_{\mathrm{eq}}}(\sigma(\pi)) \xrightarrow{\simeq_p} \mathrm{Map}\big(B\pi, (\mathbb{E}_{\mathrm{eq}})^\wedge_p\big)_\sigma$$

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Homotopy fixed points

- The fixed points set $(\mathbb{L}_{\mathrm{eq}}^{\sigma}, S^{\sigma})$ is a centric locality and the above equivalence extends to

$$
\begin{array}{ccc}
\mathbb{L}_{\mathrm{eq}}^{\sigma} & \xrightarrow{\simeq_p} & [(\mathbb{L}_{\mathrm{eq}})_p^{\wedge}]_{\tilde{\sigma}}^{h\pi} \\
\downarrow & & \downarrow \\
\mathbb{C}_{\mathbb{E}_{\mathrm{eq}}}(\sigma(\pi)) & \xrightarrow{\simeq_p} & \mathrm{Map}\big(B\pi, (\mathbb{E}_{\mathrm{eq}})_p^{\wedge}\big)_{\sigma} \\
\downarrow & & \downarrow \\
BZ(\pi) & \xrightarrow{\simeq} & \mathrm{Map}(B\pi, B\pi)_{\mathsf{Id}}
\end{array}
$$

Theorem (B-González)

Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and π a finite p-group. Assume that π acts on a classifying space $X \simeq |\mathcal{L}|_p^{\wedge}$. Then,

(a) *π acts on the locality $\mathbb{L}(\mathcal{L})$, and*

(b) *if $\mathbb{L}_{\mathrm{eq}}$ is a centric equivariant replacement for $\mathbb{L}(\mathcal{L})$, then*

$$X^{h\pi} \simeq \coprod_{\sigma \in H^1(\pi; \mathbb{L})} [\mathbb{L}_{\mathrm{eq}}^{\sigma}]_p^{\wedge}$$

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\downarrow & & \downarrow \\
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End

Thank you for your attention

Return

Saturation Axioms for fusion systems

Let $\mathcal{F}$ be a fusion system over a p-group S.

1. A subgroup $P \leq S$ is *fully centralized in* $\mathcal{F}$ if $|C_S(P)| \geq |C_S(P')|$ for all $P' \leq S$ which is $\mathcal{F}$-conjugate to P.
2. A subgroup $P \leq S$ is *fully normalized in* $\mathcal{F}$ if $|N_S(P)| \geq |N_S(P')|$ for all $P' \leq S$ which is $\mathcal{F}$-conjugate to P.

Definition

A fusion system $\mathcal{F}$ over a p-group S is a saturated if the following two conditions hold:

(I) *For all $P \leq S$ which is fully normalized in $\mathcal{F}$, P is fully centralized in $\mathcal{F}$ and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p \mathrm{Aut}_{\mathcal{F}}(P)$.*

(II) *If $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ are such that φP is fully centralized, and if we set*

$$N_\varphi = \{g \in N_S(P) \,|\, \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi P)\},$$

then there is $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\overline{\varphi}|_P = \varphi$.

Return

Centric linking systems

Given a fusion system $\mathcal{F}$ over a finite p-group S we say that

- P is $\mathcal{F}$-conjugate to P' if there is an isomorphism $\varphi: P \longrightarrow P'$ in $\mathcal{F}$.
- $P \leq S$ is $\mathcal{F}$-centric if all P' $\mathcal{F}$-conjugate to P satisfies $C_S(P') = Z(P')$.

Let $\mathcal{F}$ be a fusion system over the p-group S. A *centric linking system* associated to $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of S, together with a functor

$$\pi: \mathcal{L} \longrightarrow \mathcal{F},$$

and "distinguished" monomorphisms $P \xrightarrow{\delta_P} \mathrm{Aut}_{\mathcal{L}}(P)$ for each $\mathcal{F}$-centric subgroup $P \leq S$, which satisfy the following conditions.

Return

Centric linking systems

(A) π is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $P, Q \in \mathcal{L}$, $Z(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P,Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \leq \mathrm{Aut}_{\mathcal{L}}(P)$), and π induces a bijection

$$\mathrm{Mor}_{\mathcal{L}}(P,Q)/Z(P) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{F}}(P,Q).$$

(B) For each $\mathcal{F}$-centric subgroup $P \leq S$ and each $g \in P$, π sends $\delta_P(g) \in \mathrm{Aut}_{\mathcal{L}}(P)$ to $c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$.

(C) For each $f \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ and each $g \in P$, the following square commutes in $\mathcal{L}$:

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ {\scriptstyle \delta_P(g)}\downarrow & & \downarrow{\scriptstyle \delta_Q(\pi(f)(g))} \\ P & \xrightarrow{f} & Q \end{array}$$

Return

Fusion systems

Definition

Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. A self-equivalence of $\mathcal{L}$ is called isotypical if it maps subgroups of S to isomorphic subgroups and inclusions to inclusions.

- $\mathrm{Aut}_{\mathrm{typ}}(\mathcal{L})$ is the group of isotypical self-equivalences of $\mathcal{L}$.
- $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ is the group of isotypical self equivalences of $\mathcal{L}$ modulo natural equivalence.
- $\mathcal{A}\mathrm{ut}_{\mathrm{typ}}(\mathcal{L})$ is the strict monoidal category with objects $\mathrm{Aut}_{\mathrm{typ}}(\mathcal{L})$ and morphisms the natural equivalences

Theorem

The nerve $|\mathcal{A}\mathrm{ut}_{\mathrm{typ}}(\mathcal{L})|$ is a simplicial group that acts naturally on the nerve $|\mathcal{L}|$, and

$$\pi_i(B|\mathcal{A}\mathrm{ut}_{\mathrm{typ}}(\mathcal{L})|) = \begin{cases} \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) & i = 1, \\ Z(\mathcal{L}) & i = 2 \\ 0 & i \geq 3 \end{cases}$$

Furthermore, $B|\mathcal{A}\mathrm{ut}_{\mathrm{typ}}(\mathcal{L})| \simeq B\,\mathrm{aut}(|\mathcal{L}|^{\wedge}_p)$.