Productos poliedrales en el problema de Kahn

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Realizability. How to play

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Are finite groups realizable in $HoTop_*$?

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$G \cong \mathcal{E}(X)$ for some X?

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Which finite groups are realizable by 1-connected finite type Sullivan algebras?

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Introduce graphs on the picture

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Theorem (Frucht'39, Realizability in $\mathcal{C} = \mathit{Graphs}$)

Every finite group G is realizable by a finite, connected and simple graph $\mathcal{G}$.

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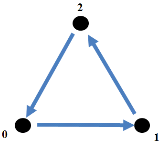
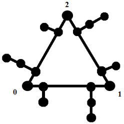
Example ($G = \mathbb{Z}_3$, Cayley graph $\to$ simple graph)





Our problem revisited

Problem 1

Let $\mathcal{G} = (V, E)$ be a finite, simple, connected graph (with more than one vertex). Does there exist a space X such that $\mathrm{Aut}(\mathcal{G}) \cong \mathcal{E}(X)$?

Solving Problem 1

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- $\mathcal{G} = (V, E)$, $|V| > 1$
- $f : \mathcal{G}_1 \hookrightarrow \mathcal{G}_2$ such that $[v, w]$ edge of $\mathcal{G}_1$ iff $[f(v), f(w)]$ edge of $\mathcal{G}_2$

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(based on an example of Arkowitz-Lupton)

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- generators in dimensions: $|x_1| = 8, |x_2| = 10, |y_1| = 33, |y_2| = 35,$ $|y_3| = 37, |z| = 119, |x_v| = 40, |z_v| = 119,$

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- differentials:

$$\begin{array}{llll} d(x_1) = & 0 & d(y_3) = & x_1 x_2^3 \\ d(x_2) = & 0 & d(x_v) = & 0 \\ d(y_1) = & x_1^3 x_2 & d(z) = & y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6 + x_1^{15} + x_2^{12} \\ d(y_2) = & x_1^2 x_2^2 & d(z_v) = & x_v^3 + \sum_{[v,w] \in E} x_v x_w x_2^4 \end{array}$$

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- Let $X_{\mathcal{G}}$ the rational space whose Sullivan minimal model is $A_{\mathcal{G}}$. The monoid of self-homotopy classes of $X_{\mathcal{G}}$ is

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Theorem

Every finite group G is realized by infinitely many (non homotopically equivalent) rational elliptic spaces X. That is, $G \cong \mathcal{E}(X)$.

Applications

Idea (Crowley-Löh, 2015)

Degree theorems "à la Gromov" are related to the existence of inflexible manifolds

Definition (Inflexible manifold)

An oriented closed connected manifold M is inflexible if

$$\{\deg f \mid f : M \to M \text{ continuous}\} \subset \{-1, 0, 1\}$$

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But $d \equiv 0 \pmod 4$, modifying our construction we get ...

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Theorem

For every finite group G, there exist infinitely many inflexible manifolds M_G such that

$$\mathcal{E}((M_G)_{\mathbb{Q}}) \cong G$$

What happens if G acts on a module M?

Realizability level 2. How to play

- Algebraic structure (G, M)

  G is a group, M is a finitely generated $\mathbb{Z}G$-module

- Homotopy invariant $\big(\mathcal{E}(-), \pi_k(-)\big)$

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- G acts on $\mathbb{Q}[V]$: for $g \in G$, $p \in \mathbb{Q}[V]$, $(gp)(v) = p(g^{-1}v)$.

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(Characterization of finite $G \leq GL(V)$, Hilbert, Noether)

Let V be a finitely generated and faithful $\mathbb{Q}G$-module. Then, there exists algebraic forms $p_1, \ldots, p_r \in \mathbb{Q}[V]^G$ such that, for $f \in GL(V)$

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we modify those algebraic forms

Solving Problem 2

Lemma

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For an arbitrary realizable family, and any $n > deg(q_{r+1}) \dots$

Solving Problem 2

$$\mathcal{M}_{(\mathcal{Q},n)} = \Big(\Lambda(x_1, x_2, y_1, y_2, y_3, z, v_j \mid j = 1, \dots, N), d\Big)$$

$$\begin{aligned}
\deg x_1 &= 8, & d(x_1) &= 0\\
\deg x_2 &= 10, & d(x_2) &= 0\\
\deg y_1 &= 33, & d(y_1) &= x_1^3 x_2\\
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\deg v_j &= 40, & d(v_j) &= 0\\
\deg z &= 80n + 39, & d(z) &= \sum_{i=1}^{r+1} q_i x_1^{10n+5-5\deg(q_i)} + q_0(x_1^{10n-5} + x_2^{8n-4})\\
& & &+ x_1^{10(n-1)}(y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_1^6)\\
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Let G be a finite group, and V a finitely generated faithful $\mathbb{Q}G$-module. Then, there exists a Postnikov piece X such that, for some $k \geq 2$,

$$\big(G, V\big) \cong \big(\mathcal{E}(X), \pi_k X\big)$$

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Example (realization of infinite groups)

Let $\mathcal{O}(m; k) < GL_{m+k}(\mathbb{R})$ preserving:

$$q_0 = x_1^2 + x_2^2 + \ldots + x_m^2 - x_{m+1}^2 - \ldots - x_{m+k}^2.$$

The family $\mathcal{Q} = \{q_0, (q_0)^{m+k+1}\} \subset \mathbb{Q}[x_1, \ldots, x_{m+k}]$ is realizable. Then,

- $O(\mathcal{Q})$ can be realized by infinitely many (rational) spaces.
- $O(\mathcal{Q}) \cong \mathcal{O}(m; k)(\mathbb{Q})$, which is an infinite group for $m \geq 2$.

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\deg x_1 &= 5k-2, & d(x_1) &= 0\\
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- ▷ Rational homotopy theory (finite type over $\mathbb{Q}$, not over $\mathbb{Z}$).

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Following our approach for $\mathbb{Q}$

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Idea

Introduce Toric Topology in the picture

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Corollary

$\mathcal{E}(\mathbb{H}P^\infty) = \{1\}$

Polyhedral product functor

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- For $\sigma \subseteq V$ face of K, the σ−power of X is:

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$CAT(K)$ is the small category: faces of K ($\emptyset$ initial object) and inclusions $i_{\sigma,\tau} : \sigma \subseteq \tau$.

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For $X = BS^1$, $(BS^1)^K \simeq DJ(K)$ where $H^*(DJ(K);\mathbb{Z}) \underbrace{\cong \mathbb{Z}[K]}_{\text{face ring of } K}$.

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Recall that: $\mathbb{Z}[K] = S_{\mathbb{Z}}(V)/\underbrace{(v_U : U \notin K)}_{\text{square free monomials}}$.

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- Then, identify $\mathcal{E}\big((BS^3)^K\big)/\mathcal{E}^*\big((BS^3)^K\big)$ to the image of

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- Finally, the entries of $M_f \in GL(n,\mathbb{Z})$ induced by $H^4(f;\mathbb{Z})$ are non negative integers (degrees of self-maps of BS^3). Then M_f and $M_{f^{-1}}$ are permutation matrices, and $Im\,\psi = Aut(K)$. $\square$

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Solving Conjecture

Theorem

Let K be a simplicial complex of dimension 1. Then

$$\mathcal{E}^*\left((BS^3)^K\right) \cong \{1\}$$

Proof (techniques of Dwyer-Mislin, Jackowski-McClure-Oliver, Nothbom-Ray)

Fix notation $X = BS^3$.

▷ Step1 We have:

$$\begin{array}{ccccccc} [X^K, X^K] & \overset{injection}{\rightsquigarrow} & [X^K, X^n] & \overset{\{\pi_j\}_1^n}{\rightsquigarrow} & [X^K, X] & \overset{injection}{\rightsquigarrow} & \prod_p [X^K, X_p^\wedge] \\ f & \rightsquigarrow & f & \rightsquigarrow & \{f_j\}_1^n & \rightsquigarrow & \{f_{j_p}^\wedge \mid p\}_1^n \end{array}$$

we also have, for a face σ of K:

$$[X^\sigma, X] \underset{(Iwase)}{\cong} \{\underbrace{(0, 0, \ldots, a_i, 0)}_{\dim\sigma+1} \mid a_i = 0 \text{ or } a_i \text{ odd square}\}$$

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▷ Step 2 We then have, for every $j = 1, \dots, n$, for every p prime:

$$\begin{array}{ccc} \mathcal{E}^*(X^K) & \rightsquigarrow & \left\{ [X^\sigma, X_p^\wedge] \mid \sigma \in CAT(K) \right\} \\ f & \rightsquigarrow & f_j^\sigma \simeq_p \begin{cases} \pi_j & \text{if } v_j \in \sigma \\ * & \text{if } v_j \notin \sigma \end{cases} \end{array}$$

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Is there $f \not\simeq Id_{X^K}$ inducing the same family?

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Is there $f \not\simeq Id_{X^K}$ inducing the same family?

▷ Step 3 The obstruction for the unicity lies in $\lim^i {\Pi_i}^p$ for

$$\begin{array}{rcl} {\Pi_i}^p : CAT^{op}(K) & \to & \mathcal{A}b \\ \sigma & \mapsto & \pi_i(map(X^\sigma, X_p^\wedge)_{f_j^\sigma}) \end{array}$$

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that can be computed as the cohomology of a cochain complex

$$N^n(\Pi_i{}^p) = \prod_{\sigma_0 \to \sigma_1 \to \cdots \to \sigma_n} \Pi_i{}^p(\sigma_n)$$

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$$N^n(\Pi_i{}^p) = \prod_{\sigma_0 \to \sigma_1 \to \cdots \to \sigma_n} \Pi_i{}^p(\sigma_n)$$

As dim K = 1, $N^{\geq 3}(\Pi_i{}^p) = 0$, $N^2(\Pi_2{}^p) = 0$, and $H^1(N^*(\Pi_1{}^p)) = 0$. □

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Corollary 1

Let K be a simplicial complex of dimension 1. Then

$$\mathcal{E}\left((BS^3)^K\right) \cong Aut(K)$$

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Corollary 2

Every finite group is realizable by infinitely many integral spaces.