Functor calculus as a new tool in algebra

Métodos Categóricos y Homotopicos en Álgebra, Geometría y Topología, Universidad de la Rioja, november 2016

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Contents

Framework

1. Varieties: categories whose

- objects are sets S endowed with a given set of operations $\mu\colon S^n \to S$, $\mu \in \mathcal{O}_n$, satisfying a given set of equational axioms

- morphisms are set maps preserving all given operations

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are called algebraic categories or varieties.

2. Semi-abelian categories by definition are pointed, finitely complete and cocomplete, protomodular and Barr-exact categories [Janelidze, Márki, Tholen 2002]. In particular, each morphism $f\colon X \to Y$ admits a natural regular epi–mono factorisation

$$X \twoheadrightarrow X/\mathrm{Ker}(f) \rightarrowtail Y\,.$$

Hence f is monic iff $\mathrm{Ker}(f) = 0$.

Properties of semi-abelian categories

Moreover, a semi-abelian category $\mathbb{A}$ has the following strong properties:

1. All regular epimorphisms are cokernels (of their kernel).

2. The full subcategory of abelian group objects in $\mathbb{A}$ is an abelian category, called the abelian core of A denoted by $\mathrm{Ab}(\mathbb{A})$.

3. For any morphism $f\colon X \to Y$ and subobject $S \leq X$ in $\mathcal{C}$,

$$f^{-1}f(S) = S \vee \mathrm{Ker}(f).$$

In fact, a variety is semi-abelian if and only if it satisfies this property.

Convention: throughout this presentation, $\mathcal{D}$ denotes a semi-abelian category.

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- the category of split extensions of a given object G in a semi-abelian category ....

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, called the Smith-Pedicchio commutator of R and S (subsumes the Freese-McKenzie commutator in semi-abelian varieties).

3. New, more general concept [H, L, VdL]: subobjects $X_1, \ldots, X_n$ of $X \Longrightarrow [X_1, \ldots, X_n]_{Hig} \leq X$, called the Higgins commutator of $X_1, \ldots, X_n$; in general NOT reducable to nested binary commutators!

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2. Recall that there is a bijective correspondence called normalization between equivalence relations on X and normal subobjects of X, defined by

$$\mathrm{Nor}\left(\begin{tikzcd} R \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & X \arrow[l, "s_0" description] \end{tikzcd} \right) = \left(d_1 : \mathrm{Ker}(d_0) \rightarrowtriangle X \right)$$

while $\mathrm{Nor}^{-1}(N \lhd X)$ is the kernel pair of $G \twoheadrightarrow G/N$.

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Now let $K, L \triangleleft X$ be the normalisations of two equivalence relations R and S on X. Then

$$\mathrm{Nor}\,([R, S]_{S\text{-}P}) = [K, L]_{Hig} \vee [K, L, X]_{Hig}.$$

Basic (algebraic) functor calculus

In the sequel, $F\colon \mathcal{C} \to \mathcal{D}$ denotes a functor between categories satisfying
- $\mathcal{C}$ is pointed and has finite sums ($=$ coproducts)
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In particular, $cr_1F(X) = \mathrm{Ker}\Big(F(0)\colon F(X) \to F(0)\Big)$ and

$$cr_2F(X,Y) = \mathrm{Ker}\Big(r_{12}\colon F(X+Y) \twoheadrightarrow F(X) \times F(Y)\Big).$$

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- Let *Groups* denote the category of groups. Then for groups $X_1, \ldots, X_n$ and elements $x_k \in X_k$, $k = 1, \ldots, n$, we have

$$[x_1, \ldots, x_n] \in Id_{Gr}(X_1| \ldots |X_n).$$

If $n = 2$ these elements generate $Id_{Gr}(X_1|X_2)$ (freely if one takes $x_1, x_2 \neq e$).

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- Let *Loops* denote the category of loops. Then for loops X_1, X_2, X_3 and elements $x_k \in X_k$ the associator

$$A(x_1, x_2, x_3) = (x_1(x_2x_3)) \backslash ((x_1x_2)x_3) \in Id_{Lp}(X_1|X_2|X_3).$$

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- Preservation of “pseudo-right-exactness” [Van der Linden]: If F preserves coequalizers of reflexive parallel pairs of morphisms (reflexive meaning that these morphisms admit a common section) then so does $cr_n F$ in all variables, for any n.

Preservation of coequalizers of reflexive parallel pairs

A functor $F\colon \mathcal{C} \to \mathcal{D}$ as before preserves coequalizers of reflexive parallel pairs iff for any right-exact sequence

$$A \xrightarrow{a} B \xrightarrow{b} C \longrightarrow 0$$

in $\mathcal{C}$ the sequence

$$F(A) \;+\; F(A|B) \xrightarrow{\left\langle \substack{F(a) \\ \delta} \right\rangle} F(B) \xrightarrow{F(b)} F(C) \longrightarrow 0$$

in $\mathcal{D}$ is exact, where

$$\delta\colon F(A|B) \xrightarrow{F(a|1_B)} F(B|B) \rightarrowtail F(B+B) \xrightarrow{F(\nabla^2)} F(B)\,.$$

Operadic structure of cross-effects

- Operadic structure: Let $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ be reduced functors where the category $\mathcal{E}$ is semi-abelian, too.

Denote "multi-objects", i.e. sequences of objects in $\mathcal{C}$, by $\underline{X}_j = X_{j,1}, \ldots, X_{j,k_j}$ and concatination of such by

$$\underline{X}_1 \cup \ldots \cup \underline{X}_n = X_{1,1}, \ldots, X_{1,k_1}, \ldots, X_{n,1}, \ldots, X_{n,k_n}.$$

Then there is a natural transformation

$$\begin{array}{c} cr_n G\Big(cr_{k_1} F(\underline{X}_1), \ldots, cr_{k_n} F(\underline{X}_n)\Big) \\ \Big\downarrow \\ cr_{k_1+\ldots+k_n}(G\circ F)\big(\underline{X}_1 \cup \ldots \cup \underline{X}_n\big) \end{array}$$

rendering a certain canonical diagram commutative.

Commutators via functor calculus

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor as before.

For subobjects $x_k : X_k \rightarrowtail X$, $k = 1, \ldots, n$, of an object X of $\mathcal{C}$ define the commutator of $X_1, \ldots, X_n$ relatively to F, denoted by $[X_1, \ldots, X_n]_F$ to be the image of the morphism

$$F(X_1|\ldots|X_n) \rightarrowtail F(X_1 + \ldots + X_n) \xrightarrow{F(x_1,\ldots,x_n)} F(X)$$

Note that $[X_1, \ldots, X_n]_F \leq F(X)$.

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Note that $[X_1, \ldots, X_n]_F \le F(X)$. We call

$$[X_1, \ldots, X_n]_{Hig} = [X_1, \ldots, X_n]_{Id_{\mathcal{D}}} \le X$$

the Higgins commutator of $X_1, \ldots, X_n$. Note that

$$[X_1]_F = \mathrm{Im}\Big(cr_1F(X_1) \rightarrowtail F(X_1) \xrightarrow{F(x_1)} F(X)\Big).$$

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- $[X_1, \ldots, X_n]_{Id_{Gr}}$ is generated as a subgroup by all nested commutators (with arbitrary bracketting) of elements $x_1 \in X_{k_1}, \ldots, x_m \in X_{k_m}$ such that $\{k_1, \ldots, k_m\} = \{1, \ldots, n\}$. [B. Loiseau]

Example: Higgins commutators in loops

2. If $\mathcal{D}$ is the category of loops then

- $[X_1, X_2]_{Id_{Lp}}$ is the normal subloop of $\langle X_1 \cup X_2 \rangle$ generated by the elements $[x_2, x_1]$, $A(x_1, y_1, y_2)$, $A(x_1, x_2, y_1)$, $A(x_1, x_2, y_2)$ $A(x_2, x_1, y_2)$, and $A_3(x_1, x_2, x_1, y_2)$ where $x_i, y_i \in X_i$ and

$$[a, b] = ba \backslash ab$$

$$A(a, b, c) = a(bc) \backslash (ab)c$$

$$A_3(a, b, c, d) = (A(a, b, c)A(a, b, d)) \backslash A(a, b, cd)\,.$$

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3. If $\mathcal{D}$ is a category of ω-loops then $[X_1, X_2]_{Id_{\mathcal{D}}}$ is the normal subobject of $X_1 \vee X_2$ generated by the elements

$$[(x_1, \ldots, x_n), (y_1, \ldots, y_n)]_\theta =$$
$$\theta(x_1 y_1, \ldots, x_n y_n) \big/ \big(\theta(x_1, \ldots, x_n)\theta(y_1, \ldots, y_n)\big)$$

where $x_1, \ldots, x_n \in X_1$, $y_1, \ldots, y_n \in X_2$ and θ runs through a family of strongly generating operations of $\mathcal{D}$.

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- 3a) If $\mathcal{D} = Groups$ then $[x, y]_{inv} = y^{-1}x^{-1}yx$ and $[(x_1, x_2), (y_1, y_2)]_{prod} = {}^{x_1}[y_1, x_2]$.

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where $x_1, \dots, x_n \in X_1$, $y_1, \dots, y_n \in X_2$ and θ runs through a family of strongly generating operations of $\mathcal{D}$.

- 3a) If $\mathcal{D} = Groups$ then $[x, y]_{inv} = y^{-1}x^{-1}yx$ and $[(x_1, x_2), (y_1, y_2)]_{prod} = {}^{x_1}[y_1, x_2]$.

- 3b) If $\mathcal{D} = Loops$ then

$$[(x_1, x_2), (y_1, y_2)]_{prod} = ((x_1 y_1)(x_2 y_2)) / ((x_1 x_2)(y_1 y_2)).$$

Example: Higgins commutators in ω-loops

3. If $\mathcal{D}$ is a category of ω-loops then $[X_1, X_2]_{Id_{\mathcal{D}}}$ is the normal subobject of $X_1 \vee X_2$ generated by the elements

$$[(x_1, \ldots, x_n), (y_1, \ldots, y_n)]_\theta =$$
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In particular,

$$[(e, x_2), (y_1, e)]_{prod} = (y_1 x_2)/(x_2 y_1) = [y_1, x_2]_r$$
$$[(x_1, e), (y_1, y_2)]_{prod} = ((x_1 y_1) y_2)/(x_1 (y_1 y_2)) = A_r(x_1, y_1, y_2).$$

Example: Higgins commutators in algebras over a linear operad

4. If $\mathcal{D}$ is the category of $\mathcal{P}$-algebras $\mathcal{P}\text{-}Alg$, then

$$[X_1, \ldots, X_n]_{\mathcal{P}\text{-}Alg} = \sum_{p_k \geq 1} \mu_p(X_1^{\otimes p_1} \otimes \ldots \otimes X_n^{\otimes p_n} \otimes \mathcal{P}(p)).$$

where $p = p_1 + \ldots + p_n$.

Commutators relatively to composite functors [T. Defourneau]

Let $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ be reduced functors where the category $\mathcal{E}$ is semi-abelian, too. Suppose that G preserves coequalizers of reflexive parallel pairs. Let Z be an object of C and $X, Y \leq Z$. Then

$$[X, Y]_{G \circ F} =$$

$$\big[[X,Y]_F\big]_G \vee \big[[X]_F, [Y]_F\big]_G$$

$$\vee \big[[X,Y]_F, [X]_F\big]_G \vee \big[[X,Y]_F, [Y]_F\big]_G$$

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The relations below indicated in red color are valid only if $\mathcal{C}$ is semi-abelian and F preserves regular epimorphisms:

- Removing internal brackets or repetitions of subobjects enlarges the commutator:

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- Preservation by morphisms: For $f : X \to Y$ in $\mathcal{C}$,

$$F(f)([X_1, \ldots, X_n]_F) = [f(X_1), \ldots, f(X_n)]_F.$$

Applications of commutators: the kernel of (the image under a functor of) a cokernel

Suppose that $\mathcal{C}$ is semi-abelian and that F preserves coequalizers of reflexive graphs, that is, of parallel morphisms admitting a commun section. Then for any subobject $u\colon U \rightarrowtail X$ in $\mathcal{C}$, there is a natural short exact sequence in $\mathcal{D}$

$$0 \to [U]_F \vee [U,X]_F \to F(X) \to F(\mathrm{Coker}(u)) \to 0$$

Taking $F = Id_{\mathcal{C}}$ we obtain the following description of the normal closure $\lhd U \rhd$ of U in X [H & Loiseau]:

$$\lhd U \rhd \ = \ U \vee [U,X]_{Id_{\mathcal{C}}}$$

Consequently, U is normal in X iff $[U,X]_{Id_{\mathcal{C}}} \subset U$ [cf. also Mantovani & Metere].

Application: $[U,X]$ is always normal since $[[U,X],X] \subset [U,X,X] \subset [U,X]$.

Lower central series

For an object X of $\mathcal{D}$ let

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$$[N_{k_1}, \ldots, N_{k_n}]_{Id_{\mathcal{D}}} \subset N_{k_1+\ldots+k_n}$$

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for $N_k = \gamma_k^F(X)$. In particular, taking $F = Id_{\mathcal{D}}$ we obtain the Higgins lower central series (H.l.c.s.) of X,

$$X = \gamma_1^{Id_{\mathcal{D}}}(X) \geq \gamma_2^{Id_{\mathcal{D}}}(X) \geq \ldots$$

Examples: Higgins lower central series in the fundamental varieties

1. If $\mathcal{D}$ is the category of groups, then the Higgins lower central series coincides with the classical l.c.s.

2. If $\mathcal{D}$ is the category of loops, then the categorically defined lower central series coincides with the commutator-associator filtration introduced by Mostovoy.

3. If $\mathcal{D}$ is the category of $\mathcal{P}$-algebras $\mathcal{P}\text{-}Alg$, then

$$\gamma_n^{Id_{\mathcal{P}-Alg}}(X) = \sum_{k \geq n} \mu_k(X^{\otimes k} \otimes \mathcal{P}(k)).$$

How to prove this?

Characterisation of the c.l.c.s.

Theorem. Let $\mathcal{X}(X)\colon X = X_1 \geq X_2 \geq \ldots$ be a natural filtration of all objects X in $\mathcal{D}$ by normal subobjects X_n of X. Then $\mathcal{X}(X)$ coincides with the Higgins l.c.s. of X for all X if and only if

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1. Factorisations $\overline{m_n}$ exist and are cokernels rendering the following diagrams commutative:

$$\begin{tikzcd}
M_n(X,\dots,X) \arrow[r, "m_n"] \arrow[d, two heads, "t_{\underline{1}}"] & X_n \arrow[d, two heads, "q_n"] \\
(T_{\underline{1}}M_n)(X,\dots,X) \arrow[r, dotted, two heads, "\overline{m_n}"] & X_n/X_{n+1}
\end{tikzcd}$$

2. The images of the maps

$$m_k \colon M_k(X, \ldots, X) \to X_k \hookrightarrow X_n,$$

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Proof of the identity $\gamma_n^{Id_{\mathcal{D}}}(X) = \gamma_n(X)$ in $\mathcal{D} = Groups$:

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Application:

Proof of the identity $\gamma_n^{Id_{\mathcal{D}}}(X) = \gamma_n(X)$ in $\mathcal{D} = Groups$: take

- $M_n(X_1, \ldots, X_n)$ to be the free group generated by the set $X_1 \times \ldots \times X_n$ modulo the normal subgroup generated by the tuples $(x_1, \ldots, x_n)$ where one of the x_k's is trivial

- m_n to send a basis element $(x_1, \ldots, x_n) \in X^n$ to $[x_1, \ldots, x_n]$.

Example: Lower central series of the group ring functor

Let $F\colon$ *Groups* $\to$ Ab be the functor sending a group G to its group ring $\mathbb{Z}[G]$ (but forgetting the multiplication!). Then

$$\gamma_n^F(G) = I^n(G)$$

where $I^n(G)$ is the n-th power of the augmentation ideal of $\mathbb{Z}[G]$.

Linearisation of algebraic structures

Reminder: Relations between groups and Lie algebras

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2. The associated graded of arbitrary groups: For any group G and elements $x, y \in G$ let $[x, y] = (xy)(yx)^{-1}$. An N-series of G is a filtration

$$\mathcal{N}\colon G = N_1 \supset N_2 \supset \dots$$

of G by subgroups N_n such that $[N_i, N_j] \subset N_{i+j}$.

Linearisation of algebraic structures

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of G by subgroups N_n such that $[N_i, N_j] \subset N_{i+j}$. Then $\mathrm{Gr}_n^{\mathcal{N}}(G) = N_n/N_{n+1}$ is an abelian group, and

$$\mathrm{Gr}^{\mathcal{N}}(G) = \sum_{k\geq 1} \mathrm{Gr}_n^{\mathcal{N}}(G)$$

is a graded Lie ring whose bracket is induced by the commutator of G.

Examples of N-series

1. The lower central series

$$\gamma\colon G = \gamma_1(G) \supset \gamma_2(G) \supset \dots$$

where $\gamma_n(G) = \langle [x_1, \dots, x_n] \mid x_1, \dots, x_n \in G \rangle$ with

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2. The dimension series: let $\mathbb{K}$ be a commutative ring. Then the subgroups

$$D_{n,\mathbb{K}}(G) = G \cap (1 + I^n_{\mathbb{K}}(G))$$

form an N-series where $I^n_{\mathbb{K}}(G)$ denotes the n-th power of the augmentation ideal of the group algebra $\mathbb{K}(G)$.

Relations between groups and Lie algebras - sequel

3. Mal'cev/Lazard equivalence: There is a canonical equivalence between the categories of radicable n-step nilpotent groups and n-step nilpotent Lie algebras over $\mathbb{Q}$, based on the Baker-Campbell-Hausdorff formula.

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4. Primitive operations on group algebras: Primitive elements of Hopf algebras (the bialgebra type of group algebras) form a Lie algebra under the usual ring commutator.

Relations between groups and Lie algebras - Summary

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2. The associated graded of arbitrary groups
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GOAL: develop semi-abelian Lie theory!

That is, given a semi-abelian variety (generalizing groups),

- exhibit a related linear structure $=$ type of algebras $\sim$ linear operad (generalizing Lie algebras)

- generalize the relations 1. to 4. above to this situation.

Examples: varieties of loops I

semi-abelian variety: Moufang loops

$$(a(bc))a = (ab)(ca)$$

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$$[[x,y],[x,z]] = [[[x,y],z],x] + [[[y,z],x],x] + [[[z,x],x],y]$$

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semi-abelian variety: Bruck loops:

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Linearization: Lie triple systems: left antisymmetric triple bracket $[-,-,-]$ satisfying the Jacobi identity and

$$[u,v,[x,y,z]] = [[u,v,x],y,z] + [x,[u,v,y],z] + [x,y,[u,v,z]]$$

Examples: varieties of loops II

semi-abelian variety: Bol loops (contain Moufang and Bruck loops)

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Examples: varieties of loops II

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Linearization: Bol algebras:

- antisymmetric binary bracket $[-,-]$
- Lie triple system $[-,-,-]$

satisfying

$$\begin{aligned} [x,y,[u,v]] &= [[x,y,u],v] + [u,[x,y,v]] \\ &\quad + [u,v,[x,y]] + [[x,y],[u,v]] \end{aligned}$$

Example: arbitrary loops

semi-abelian variety: Loops

Linearization: Sabinin algebras: multilinear operations

- $\langle x_1, x_2, \ldots, x_m; y, z \rangle$, $m \geq 0$
- $\Phi(x_1, x_2, \ldots, x_m; y_1, y_2, \ldots, y_n)$, $m \geq 1$, $n \geq 2$

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- $\langle x_1, \ldots, x_r, u, v, x_{r+1}, \ldots, x_m; y, z\rangle -$
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Due to Miheev and Sabinin; Shestakov and Umirbaev; Mostovoy, Pérez-Izquierdo

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Due to Miheev and Sabinin; Shestakov and Umirbaev; Mostovoy, Pérez-Izquierdo and Shestakov, who also developed a general theory treating arbitrary varieties of loops, and explicitly treated many more examples of those.

Linearisation of arbitrary semi-abelian varieties

- use algebraic functor calculus to construct a suitable notion of commutators and a suitable operad in abelian groups, satisfying relation 2. (basically done, see below)

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- try to generalize relation 1. (needs magic from differential geometry, FAR beyond the author's reach...)

Linearisation of a semi-abelian variety

Construction of an operad in abelian groups associated with a semi-abelian variety $\mathbb{A}$:

Let E be a free object of rank 1 in $\mathbb{A}$ and write $\bar{E} = E^{ab}$.
Let $F_n = \mathrm{MultiLin}(cr_n Id_{\mathbb{A}}) \colon \mathbb{A}^n \to \mathbb{A}$ be the multilinearisation of the n-th cross-effect of $Id_{\mathbb{A}}$, i.e.

$$F_n(X_1, \ldots, X_n) =$$

$$\mathrm{Coker}\Big(\coprod_{k=1}^{n} Id(X_1, \ldots, X_k, X_k, \ldots, X_n) \to Id(X_1, \ldots, X_n) \Big)$$

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which is an abelian object in $\mathbb{A}$. Then a (right) operad $\mathcal{P} = \mathrm{AbOp}(\mathbb{A})$ in Ab is defined by

$$\mathcal{P}(n) = \mathbb{A}\left(\bar{E}, F_n(\bar{E}, \dots, \bar{E})\right)$$

and composition operations

$$\gamma_{k_1,\dots,k_n;n}\colon \mathcal{P}(k_1) \otimes \dots \otimes \mathcal{P}(k_n) \otimes \mathcal{P}(n) \to \mathcal{P}(k_1 + \dots + k_n),$$

$$f_1 \otimes \dots \otimes f_n \otimes g \mapsto \mu_{k_1,\dots,k_n} \circ F_n(f_1, \dots, f_n) \circ g$$

where

$$\mu_{k_1,\dots,k_n}\colon F_n\left(F_{k_1}(\underline{\bar{E}}), \dots, F_{k_n}(\underline{\bar{E}})\right) \to F_{k_1+\dots+k_n}(\underline{\bar{E}})$$

is induced by the composition operations for the cross-effects of the composable functors $Id_{\mathbb{A}}, Id_{\mathbb{A}}$.

Examples

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3. If $\mathcal{Q}$ is an operad in k-modules and $\mathbb{A}$ is the category of $\mathcal{Q}$-algebras then $\mathrm{AbOp}(\mathbb{A}) = \mathcal{Q}$.

Lazard equivalence/BCH-formula

Let $\mathcal{D}$ be a 2-step nilpotent semi-abelian variety. Then there exists a (non unique) 2-step nilpotent group structure among the operations in $\mathcal{D}$, denoted by $+_M$.

Let E be a distinguished free object of rank 1 in $\mathcal{D}$ and let e denote its basis element.

Suppose that $1_{E^{ab}} + 1_{E^{ab}}$ is invertible in the ring $\mathcal{D}(E^{ab}, E^{ab})$. Then the 2-step nilpotent (right) operad $\mathrm{AbOp}(\mathcal{D})$ actually is an operad in the monoidal category of $Z[\frac{1}{2}]$-modules as

$$\mathrm{AbOp}(\mathcal{D})(1) = |E^{ab}|$$

$$\mathrm{AbOp}(\mathcal{D})(2) = |Id_{\mathcal{D}}(E^{ab}|E^{ab})|$$

Lazard equivalence/BCH-formula

Theorem: There exists a Lazard equivalence

$$L^*\colon \mathrm{Alg}\text{-}\mathrm{AbOp}(\mathcal{D}) \to \mathcal{D}$$

given by $|L^*(A)| = |A|$ and the following Baker-Campbell-Hausdorff formula: an n-ary operation θ of the variety $\mathcal{D}$ acts on $|L^*(A)|$ by

$$\theta(x_1,\ldots,x_n) = \sum_{p=1}^{n}\left(\lambda_1(x_p\otimes\overline{\theta_p(e)}) + \frac{1}{2}\lambda_2(x_p\otimes x_p\otimes H(\theta_p))\right)$$
$$+\ \frac{1}{2}\sum_{1\le p<q\le n}\lambda_2\Big(x_p\otimes x_q\otimes\gamma_{1,1;2}(\overline{\theta_p(e)}\otimes\overline{\theta_q(e)}\otimes[e_1,e_2])\Big)$$
$$+\ \sum_{1\le p<q\le n}\lambda_2(x_p\otimes x_q\otimes\Delta(\theta_{pq}))$$

with $\Delta(\theta_{pq}) = \theta_{pq}(e_1,e_2) - (\theta_p(e_1) +_M \theta_q(e_2))$.

Here

- θ_p is the unary operation of $\mathcal{D}$ given by $\theta_p(a) = \theta(0,\dots,0,a,0,\dots,0)$ where a is placed in the p-th variable. Similarly, θ_{pq} is the binary operation of $\mathcal{D}$ given by $\theta_{pq}(a,b) = \theta(0,\dots,0,a,0,\dots,0,b,0,\dots,0)$ where a, b are placed in the p-th and q-th variable, respectively;

- for any unary operation ϑ of $\mathcal{D}$, $H(\vartheta) = \vartheta_{E+E}(i_1e +_M i_2e) -_M (i_1\vartheta_E(e) +_M i_1\vartheta_E(e))$ where $i_p\colon E \to E+E$ is the injection of the p-th summand;

- for $k = 1,2$, $e_k = i_ke \in E+E$. Furthermore, $[a,b]_M = (a +_M b) -_M (b +_M a)$.

Internal actions

Let G be an object of a category $\mathcal{C}$ which has a 0-object and kernels. An internal action of G on some object A of $\mathcal{C}$ morally should be some additional data linking G and A which is equivalent with a split extension in $\mathcal{C}$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"'] & X \arrow[r, shift right, "p"'] & G \arrow[l, shift right, "s"'] \arrow[r] & 0 \end{tikzcd}$$

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THEOREM [Bourn-Janelidze]: If $\mathcal{C} = \mathbb{A}$ is semi-abelian then there exists a monad $\mathbb{T}_G$ on $\mathbb{A}$ whose algebras A are equivalent with split extensions as above. The underlying functor T_G of $\mathbb{T}_G$ is given by $T_G(A) = \mathrm{Ker}(r_G \colon A + G \to G)$.

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Example: In $\mathbb{A} = \mathit{Groups}$,
$T_G(A) = \langle {}^g a \in A + G \mid a \in A, g \in G \rangle$ and $g \cdot_\xi a = \xi({}^g a)$.

Action cores

Observing that there is a split short exact sequence

$$0 \longrightarrow Id(A|G) \xrightarrow{j_A} T_G(A) \underset{r_A}{\overset{i_A}{\leftrightarrows}} A \longrightarrow 0$$

we may restrict an action $\xi\colon T_G(A) \to A$ to a map $\psi_\xi = \xi \circ j_A\colon Id(A|G) \to A$ which is the non-unital part of ξ; we call it the action core of ξ [H-Loiseau].

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Definition: An (abstract, or strict) action core of G on A is a morphism $\psi\colon Id(A|G) \to A$ rendering the following three diagrams commutative.

Axioms for abstract action cores

$$\begin{tikzcd} Id(Id(A|G)|A) \arrow[r, "C^A_{A,G}"] \arrow[d, "Id(\psi|1_A)"'] & Id(A|G) \arrow[d, "\psi"] \\ Id(A|A) \arrow[r, "c^A_2"'] & A \end{tikzcd}$$

$$\begin{tikzcd} Id(Id(A|G)|G) \arrow[r, "C^G_{A,G}"] \arrow[d, "Id(\psi|1_G)"'] & Id(A|G) \arrow[d, "\psi"] \\ Id(A|G) \arrow[r, "\psi"'] & A \end{tikzcd}$$

$$\begin{tikzcd} Id(Id(A|G)|A|G) \arrow[rr, "C^{A,G}_{A,G}"] \arrow[d, "Id(\psi|1_A|1_G)"'] & & Id(A|G) \arrow[d, "\psi"] \\ Id(A|A|G) \arrow[r, "S^{A,G}_{2,1}"] & Id(A|G) \arrow[r, "\psi"] & A \end{tikzcd}$$

Equivalence of actions with action cores

THEOREM [H-Loiseau]: Assigning the action core ψ_ξ with an action ξ establishes an equivalence between actions and action cores, and thus of action cores with split extensions in $\mathcal{C}$

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2. of G on a normal subobject $N \overset{n}{\triangleright\!\!\longrightarrow} G$ is given by the unique map $c_{N,G} \colon Id(N|G) \to N$ rendering the following square commutative.

$$\begin{tikzcd} Id(N|G) \arrow[r, dotted, "c_{N,G}"] \arrow[d, "Id(n|1_G)"'] & N \arrow[d, "n"] \\ Id(G|G) \arrow[r, "c_2^G"] & G \end{tikzcd}$$

Action cores and semi-direct products

There is a natural construction of a semidirect product of G with A along a given action core $\psi\colon Id(A|G) \to A$, which is a natural split short exact sequence

$$0 \longrightarrow A \xrightarrow{i_\psi} A \rtimes_\psi G \underset{p_\psi}{\overset{s_\psi}{\leftrightarrows}} G \longrightarrow 0$$

in $\mathcal{C}$ such that

$$s_\psi^* \, c_{A \rtimes_\psi G, A} = \psi.$$

Representations

Recall that a representation of a

- group G is a group homomorphism $G \to \mathrm{Aut}(A)$ for some abelian group A
- associative algebra A is an associative algebra homomorphism $\mathfrak{g} \to \mathrm{End}(V)$ for some vector space V
- Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{End}(V)$ for some vector space V
- loop Q is ???

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Definition [Beck]: Let $\mathcal{C}$ be a category with pullbacks. Then a representation of an object G of $\mathcal{C}$ (or Beck module over G) is an abelian group object in the category $\mathcal{C}/G$.

Recall that

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Hence an abelian group object in $\mathcal{C}/G$ is a triple

$$(p\colon X \to G, \quad \eta\colon 1_G \to p, \quad \mu\colon p \times p \to p)$$

satisfying the usual axioms. In particular, $\eta\colon G \to X$ is a splitting of p, so that the extension

$$0 \longrightarrow A \longrightarrow X \underset{p}{\overset{\eta}{\leftrightarrows}} G \longrightarrow 0$$ with $A = \mathrm{Ker}(p)$ is equivalent with an action core $\psi\colon Id(A|G) \to A$ of G on A if $\mathcal{C} = \mathbb{A}$ is semi-abelian.

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Moreover, in $\mathbb{A}$ the multiplication μ is unique if it exists, in which case ψ is said to be a G-module structure on A.

Characterisation of module structures

Examples:

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What is the general obstruction of an action on an abelian object to be a module?

THEOREM [H-VdL]: An action core $\psi\colon Id(A|G) \to A$ of an object G of $\mathbb{A}$ on an abelian object A of $\mathbb{A}$ is a G-module structure iff the composite map

$$Id(A|A|G) \xrightarrow{S_{2,1}} Id(A|G) \xrightarrow{\psi} A$$

is trivial.

In $\mathbb{A} = \mathit{Groups}$, $Id(A|A|G)$ is the normal subgroup of $A + A + G$ generated by the three subgroups

$$[i_1 A, [i_2 A, i_3 G]], \quad [i_2 A, [i_3 G, i_1 A]], \quad [i_3 G, [i_1 A, i_2 A]]$$

which all map to commutators in the abelian group A, hence to the trivial element, under the map $\psi \circ S_{2,1}$ followed by the injection $i_\psi \colon A \rightarrowtail A \rtimes_\psi G$.

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In $\mathbb{A} = Loops$, $Id(A|A|G)$ contains the associators $A(i_1a, i_1a', i_3g)$, which may map to non-trivial associators in $A \rtimes_\psi G$ under the map $\psi \circ S_{2,1}$ followed by the injection $i_\psi \colon A \rightarrowtail A \rtimes_\psi G$.

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For example this happens when $A \rtimes_\psi G$ is the quaternion loop $H = \{\pm 1, \pm i, \pm j, \pm k\}$ [H-VdL]: Here $A = \{\pm 1, \pm j\}$ is the Klein 4-group and $G = \{1, i\}$ is the cyclic group of order 2, while the associator $A(j, j, i) = -1 \neq 1$.

Representations of a given object G of $\mathbb{A}$

Examples:

In $\mathbb{A} = \mathit{Groups}$, G-modules are equivalent with modules over the ring $\mathbb{Z}[G]$.

In $\mathbb{A} = \mathit{Lie}_k$, $\mathfrak{g}$-modules are equivalent with modules over the enveloping algebra $\mathrm{U}(\mathfrak{g})$.

In $\mathbb{A} = \mathit{Loops}$, Q-modules are equivalent with modules over the group ring of a certain group $\mathrm{U}(Q, \mathit{Loops})_e$.

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Note that modules over a ring R are algebras over the linear monad $\mathbb{T}_R$ on the abelian category Ab with underlying functor $T_R = R \otimes -$.

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THEOREM [H]: Representations of an object G of a semi-abelian category $\mathbb{A}$ are equivalent with algebras over the monad $\mathrm{Lin}(\mathbb{T}_G)$ on $\mathrm{Ab}(\mathbb{A})$ which is the linearization of the monad $\mathbb{T}_G$ on $\mathbb{A}$ (whose algebras are G-actions).

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In particular, the underlying functor of $\mathrm{Lin}(\mathbb{T}_G)$ is given by restricting to $\mathrm{Ab}(\mathbb{A})$ the linearisation $\mathrm{Lin}(T_G)$ of the underlying functor T_G of $\mathbb{T}_G$.

Representations of a given object G of $\mathbb{A}$

THEOREM [H]: Representations of an object G of a semi-abelian category $\mathbb{A}$ are equivalent with algebras over the monad $\mathrm{Lin}(\mathbb{T}_G)$ on $\mathrm{Ab}(\mathbb{A})$ which is the linearization of the monad $\mathbb{T}_G$ on $\mathbb{A}$ (whose algebras are G-actions).

In particular, the underlying functor of $\mathrm{Lin}(\mathbb{T}_G)$ is given by restricting to $\mathrm{Ab}(\mathbb{A})$ the linearisation $\mathrm{Lin}(T_G)$ of the underlying functor T_G of $\mathbb{T}_G$.

Moreover, $\mathrm{Lin}(T_G)$ is right-exact, and even preserves all colimits if $\mathbb{A}$ is a variety.

Here the linearisation $\mathrm{Lin}(F)\colon \mathcal{C} \to \mathbb{A}$ of a functor $F\colon \mathcal{C} \to \mathbb{A}$ is given by

$$\mathrm{Lin}(F)(X) = \mathrm{Coker}\Big(F(X|X) \rightarrowtail F(X+X) \xrightarrow{F(\nabla)} F(X)\Big)\,.$$

Representations of a given object G of $\mathbb{A}$

Now suppose that $\mathbb{A}$ is a variety, and let E be a free object of rank 1 in $\mathbb{A}$. Then

$$I_{\mathbb{A}}(G) = \mathbb{A}\big(E^{ab}, \mathrm{Lin}(Id_{\mathbb{A}}(-|G)(E^{ab})\big)$$

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is a bimodule over the ring $R = \mathrm{End}_{\mathbb{A}}(E^{ab})$ equipped with a non-unital, R-bilinear ring structure $I_{\mathbb{A}}(G) \otimes_R I_{\mathbb{A}}(G) \to I_{\mathbb{A}}(G)$. Thus we have a ring

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THEOREM [Gray,H]: Representations of an object G of a semi-abelian variety are equivalent with modules over a ring [Gray]. This ring can be chosen to be $\mathrm{U}(G)$ [H].