1. Exterior spaces

Definition 1 Let \((X, \tau)\) be a topological space, where \(X\) is the subjacent set and \(\tau\) its topology. An exnterior topology on \((X, \tau)\) is a non-empty collection \(\epsilon\) of open subsets which is closed under finite intersections and such that \(\emptyset \notin \epsilon\), \(X, U \in \epsilon\) and \(C \subseteq \epsilon\) if and only if \(C \cap U \in \epsilon\). If an open subset is a member of \(\epsilon\) is said to be an exterior open subset.

As exterior space \((X, \epsilon)\) consists of a space \((X, \tau)\) together with an exnterior topology \(\epsilon\).

4 map \(f : (X, \tau) \to (X', \tau')\) is said to be an exterior map if it is continuous and \(f^{-1}(C) \in \epsilon\), for all \(C \in \epsilon\).

The image \(f(X)\) of an exterior space \((X, \epsilon)\) is defined by \(f(X) = f \cdot \bigcup \epsilon = \bigcup f^{-1}(C)\). When the exterior topology is defined by \(\epsilon\) and the category of topological spaces is defined by \(\mathbf{Top}\).

We can consider the functor \((\cdot) \circ \mathbf{Top} \to \mathbf{E}\) given by the following construction:

Let \((X, \tau), (Y, \tau')\) be exterior spaces, \((Y, \tau)\) a topological space and for \(g \in Y\) denote by \((\tau_g)\) the family of open neighborhoods of \(Y\) at \(g\). We consider on \((X, \tau)\) the product topology \(\tau_{\tau'}\) and the exnterior topology \(\tau_{\tau'} = \tau_{\tau} \cup \tau_g\) given by these \(E \in \tau_g\) such that for each \(g \in Y\) there exists \(U \in (8_g)\) such that \(T \cap X \subseteq \text{E}\). This exterior space will be denoted by \((X, \tau')\) in order to avoid a possible confusion with the product topologyn.

3. Exterior discrete semiflows

Given a discrete semiflow \(\psi : \mathbb{N} \times X \to X\), we will consider the limit space \(\lim \psi(X)\) and the end point \(\lim \psi(X)\).

Definition 2 Given an exterior space \(X\) with exnterior topology \(\tau\), the basins of end points are determined by \(\lim \psi(X)\).

2. Discrete semiflows

Next we recall some basic notions about discrete semiflows.

Definition 3 A discrete semiflow \((X, \tau)\) is a continuous map \(\tau : \mathbb{N} \times X \to X\) such that

\(1)\quad \tau(0, x) = x, \forall x \in X\)

\(2)\quad \tau(n + 1, \tau(n, x)) = \tau(n, \tau(n, x)) \quad \forall n \in \mathbb{N}, \forall x \in X\)

We will denote by \(\mathbf{EDS}(\mathbb{N})\) the category of discrete semiflows.

Given a discrete semiflow \(\psi : \mathbb{N} \times X \to X\), we will denote by \(\psi^n(X)\) the composition of \(\psi\) with itself \(n\) times.

4. Basins of end points induced by a rational function on \(C \cup \{\infty\}\)

Consider a function \(h : C \to C\) of the form \(h(z) = \frac{P(z)}{Q(z)}\), where \(z \in C\setminus\{0\}\), \(P(z) = (z - z_1) \cdots (z - z_n)\) and \(Q(z) = (z - z_{n+1}) \cdots (z - z_k)\), and \((z_1, \cdots, z_n) \neq 0\).

Then the set of poles of \(h\) is \(\{z_1, \cdots, z_n\}\) and the set of poles of \(h^{-1}\) is \(\{z_{n+1}, \cdots, z_k\}\).

The map \(h \cdot \mathbf{E}\) induces a map \(f = h^{-1}\) on \(C \cup \{\infty\}\) where \(h^{-1}(0) = \infty\) and \(h^{-1}(\infty) = 0\).

The map \(f\) is given by \(C \cup \{\infty\}\) the structure a discrete semiflow by the formula \(n \mapsto f^n(\infty)\). When we consider the canonical exnterior topology \(\tau(C \cup \{\infty\}\) associated to this semiflow, we obtain an exterior discrete semiflow and the corresponding map \(\omega(p) = \lim_{n \to \infty} f^n(p)\).

In the following example, we consider the rational map \(h(z) = P(z)/Q(z)\), where \(P(z) = 1 + 4z^2\) and \(Q(z) = 5z^2\). In this case, the induced map \(f = h^{-1}\) has six fixed points, \(\infty, -0.909017 - 0.587786i, -0.909017 + 0.587786i, 0.390917 - 0.951967i, 0.390917 + 0.951967i\), \(0\). We consider the space decomposed into seven pieces. We have associated to each piece a different color.

In the piece \(G\) one has points whose basin correspond to an end point which is not associated to a fixed point (for example end points associated to 2-cycles) or points such that the induced sequence has not converged, after a prefixed limit number of iterations, to any fixed point yet.

The other colors correspond to points that are in the basin of an end point associated to a fixed point. We also describe if a point is a repeller or an attractor.

In the first picture, we can see the intersection of the basins with the unit disk. In the second, we also can see, up to inversion, the intersection of the basin with the complement of the unit disk in \(C \cup \{\infty\}\).

It is interesting to note that in this example the basin of end point associated to the fixed point \((\neq f)\) of \(f\) is the same that the attraction basin of the Newton-Raphson numerical method when it is applied to find the roots of the equation \(z^3 - 1 = 0\).