Fourier-Dunkl system of the second kind and Euler-Dunkl polynomials*,†

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Abstract

We prove a partial fraction decomposition of a quotient of two functions $E_\alpha(itx)$ and $\mathcal{I}_\alpha(it)$ which are defined in terms of the Bessel functions J_α and $J_{\alpha+1}$ of the first kind. This expansion leads naturally to the introduction of an orthonormal system with respect to the measure $\frac{|x|^{2\alpha+1}\,dx}{2^{\alpha+1}\Gamma(\alpha+1)}$ in $[-1,1]$, which we call the Fourier-Dunkl system of the second kind. Euler-Dunkl polynomials $\mathcal{E}_{n,\alpha}(x)$ of degree n are defined by considering $E_\alpha(tx)/\mathcal{I}_\alpha(t)$ as a generating function. It is shown that the sum $\sum_{m=1}^\infty 1/j_{m,\alpha}^{2k}$, where $j_{m,\alpha}$ are the positive zeros of J_α, is equal (up to an explicit factor) to $\mathcal{E}_{2k-1,\alpha}(1)$. For $\alpha = 1/2$ this leads to classical results of Euler since the function $E_{1/2}(x)$ is the exponential function and $\mathcal{E}_{n,1/2}(x)$ are (essentially) the Euler polynomials. In the second part of the paper a sampling theorem of Whittaker-Shannon-Kotel'nikov type is established which is strongly related to the above-mentioned partial decomposition and which holds for all functions in the Payley-Wiener space defined by the Dunkl transform in $[-1,1]$.

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1 Introduction and results

For $\alpha \in \mathbb{C} \setminus \{-1, -2, \dots\}$, consider the entire functions

$$\mathcal{I}_\alpha(z) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(iz)}{(iz)^\alpha} = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{n!\, \Gamma(n+\alpha+1)}, \tag{1.1}$$

where J_α is the Bessel function of order α, and

$$E_\alpha(z) = \mathcal{I}_\alpha(z) + \frac{z}{2(\alpha+1)}\, \mathcal{I}_{\alpha+1}(z). \tag{1.2}$$

A simple computation gives

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\gamma_{n,\alpha}}$$

with

$$\gamma_{n,\alpha} = \begin{cases} 2^{2k} k!\, (\alpha+1)_k, & \text{if } n = 2k, \\ 2^{2k+1} k!\, (\alpha+1)_{k+1}, & \text{if } n = 2k+1, \end{cases}$$

where $(a)_n$ denotes the Pochhammer symbol

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

The entire function E_α is invariant under the Dunkl operator

$$\Lambda_\alpha f(x) = \frac{d}{dx} f(x) + \frac{2\alpha+1}{2} \left(\frac{f(x) - f(-x)}{x} \right)$$

(see [7, 10]).

For any complex value of α (except for the negative integers), we can order the zeros $j_{m,\alpha}$, $m \in \mathbb{Z} \setminus \{0\}$, of the Bessel function $J_\alpha(x)/x^\alpha$ so that $j_{m,\alpha} = -j_{-m,\alpha}$ and $0 < \operatorname{Re} j_{m,\alpha} \le \operatorname{Re} j_{m+1,\alpha}$, $m \ge 1$ ([11, § 15.41, p. 497]). The case $\alpha > -1$ is particularly relevant because then $j_{m,\alpha}$, $m \ge 1$, are positive numbers ([11, § 15.27, p. 483]).

It is easy to see that $\{j_{m,\alpha+1}\}_m$ are the zeros of $\operatorname{Im} E_\alpha(ix)$. We associate to the function E_α the so-called Fourier-Dunkl system ([5, 4, 2])

$$e_{\alpha,m}(x) = \frac{2^{\alpha/2} \Gamma(\alpha+1)^{1/2}}{|\mathcal{I}_\alpha(i j_{m,\alpha+1})|} E_\alpha(i j_{m,\alpha+1} x), \quad m \in \mathbb{Z} \setminus \{0\}, \quad x \in [-1,1], \tag{1.3}$$

$$e_{\alpha,0}(x) = 2^{(\alpha+1)/2} \Gamma(\alpha+2)^{1/2}. \tag{1.4}$$

The sequence of functions $\{e_{\alpha,m}\}_{m\in\mathbb{Z}}$ was introduced in [5]. It was proved there (Theorem 1) that, for any $\alpha > -1$, it is a complete orthonormal system in $L^2([-1,1], d\mu_\alpha)$, where

$$d\mu_\alpha(x) = \frac{|x|^{2\alpha+1}}{2^{\alpha+1}\Gamma(\alpha+1)}\,dx.$$

Note that for $\alpha = -1/2$, we have $E_{-1/2}(x) = e^x$, $\Lambda_{-1/2} = d/dx$ and

$$e_{-1/2,m}(x) = \left(\frac{\pi}{2}\right)^{1/4} e^{im\pi x}.$$

Then $(E_\alpha, \Lambda_\alpha, \{e_{\alpha,m}\}_{m\in\mathbb{Z}})$ can be regarded a generalization of $(e^x, d/dx, \{e^{m\pi ix}\}_{m\in\mathbb{Z}})$.

Now, instead of $\{j_{m,\alpha+1}\}$, which are the zeros of $\operatorname{Im} E_\alpha(ix)$, we can start from $\{j_{m,\alpha}\}$, which are the zeros of $\operatorname{Re} E_\alpha(ix)$, and define a different system, namely

$$f_{\alpha,m}(x) = \frac{2^{\alpha/2+1}(\alpha+1)\Gamma(\alpha+1)^{1/2}}{|j_{m,\alpha}\mathcal{I}_{\alpha+1}(ij_{m,\alpha})|} E_\alpha(ij_{m,\alpha}x), \quad m \in \mathbb{Z}\setminus\{0\}, \quad x \in [-1,1]. \tag{1.5}$$

The system $\{f_{\alpha,m}\}_m$ will be called the *Fourier-Dunkl system of the second kind*. This can also be regarded as a generalization of the trigonometric system, for in the case $\alpha = -1/2$ we obtain

$$f_{-1/2,m}(x) = \left(\frac{\pi}{2}\right)^{1/4} e^{im\pi x} e^{-i\pi x/2},$$

which, except for the common factor $e^{-i\pi x/2}$, is the trigonometric system.

The first result of this paper is the following partial fraction decomposition:

Theorem 1.1. *Let $\alpha \in \mathbb{C}\setminus\{-1,-2,-3,\dots\}$. Then*

$$\frac{E_\alpha(itx)}{\mathcal{I}_\alpha(it)} = \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{2(\alpha+1)E_\alpha(ij_{m,\alpha}x)}{j_{m,\alpha}\mathcal{I}_{\alpha+1}(ij_{m,\alpha})(j_{m,\alpha}-t)} \tag{1.6}$$

for $t \in \mathbb{C}\setminus\{j_{m,\alpha} : m \in \mathbb{Z}\setminus\{0\}\}$ and $x \in (-1,1)\setminus\{0\}$. Moreover, the right-hand side converges uniformly for t in bounded subsets of $\mathbb{C}\setminus\{j_{m,\alpha} : m \in \mathbb{Z}\setminus\{0\}\}$ and x in compact subsets of $(-1,1)\setminus\{0\}$. If, in addition, $\operatorname{Re}\alpha < 1/2$, then (1.6) holds also for $x = 0$, uniformly for t in compact subsets of $\mathbb{C}\setminus\{j_{m,\alpha} : m \in \mathbb{Z}\setminus\{0\}\}$.

The content of the paper is as follows. Theorem 1.1 and some other extensions will be proved in Section 2 using residues as the main tool.

In Section 3, we prove that the Fourier-Dunkl system of the second kind $\{f_{\alpha,m}\}_m$ is a complete orthonormal system in $L^2([-1,1], d\mu_\alpha)$ for $\alpha > -1$. In this notation, (1.6) reads as

$$\frac{E_\alpha(itx)}{\mathcal{I}_\alpha(it)} = \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^m \operatorname{sgn}(m) f_{\alpha,m}(x)}{2^{\alpha/2}\Gamma(\alpha+1)^{1/2}(t-j_{m,\alpha})}, \tag{1.7}$$

where $t \in \mathbb{C} \setminus \{j_{m,\alpha} : m \in \mathbb{Z} \setminus \{0\}\}$. Thus, besides the convergence of this expansion in $L^2([-1,1], d\mu_\alpha)$, Theorem 1.1 provides the uniform convergence on compact subsets of $(-1,1) \setminus \{0\}$ (and in $x = 0$ for $\operatorname{Re}\alpha < 1/2$).

We introduce in Section 3 the Euler-Dunkl polynomials $\{\mathcal{E}_{n,\alpha}\}_{n=0}^{\infty}$, $\alpha \in \mathbb{C} \setminus \{-1,-2,\dots\}$, by considering the generating function

$$\frac{E_\alpha(xt)}{\mathcal{I}_\alpha(t)} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\alpha}(x) \frac{t^n}{\gamma_{n,\alpha}}. \tag{1.8}$$

For the special case $\alpha = 1/2$ we obtain the classical results of Euler, namely

$$\frac{\mathcal{E}_{n,-1/2}(2x-1)}{2^n} = E_n(x),$$

where $\{E_n(x)\}_n$ are the Euler polynomials defined by the generating function

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}$$

(for the properties of the Euler polynomials, see [6], for instance). We prove for the Euler-Dunkl polynomials the following Dunkl counterpart of the Fourier expansion for the Euler polynomials: for $\alpha > -1$,

$$\mathcal{E}_n(x) = \frac{-(-i)^n \gamma_n}{2^{\alpha/2}\Gamma(\alpha+1)^{1/2}} \sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{(-1)^m \operatorname{sgn}(m)}{j_{m,\alpha}^{n+1}} f_{\alpha,m}(x)$$

in $L^2([-1,1], d\mu_\alpha)$. We actually prove that the convergence is uniform on compact subsets of $(-1,1) \setminus \{0\}$ for $n = 0$ or $[-1,1] \setminus \{0\}$ for $n \geq 1$ (the convergence can be extended to $x = 0$ assuming that $\alpha < n + 1/2$).

In Section 4 we show that the expansion (1.7) provides a nice Whittaker-Shannon-Kotel'nikov sampling theorem related to the Dunkl transform. In a similar way as the Fourier transform (which is the particular case $\alpha = -1/2$), the Dunkl transform of order $\alpha \geq -1/2$ is given by

$$\mathcal{F}_\alpha f(y) = \int_{\mathbb{R}} f(t) E_\alpha(-iyt)\, d\mu_\alpha(t), \quad y \in \mathbb{R}, \tag{1.9}$$

for $f \in L^1(\mathbb{R}, d\mu_\alpha)$.

The Dunkl transform can be extended to $L^2(\mathbb{R}, d\mu_\alpha)$, $\alpha > -1$, although the integral expression (1.9) is no longer valid in general (see [9] and [10] for details); however, and as usual, we will use it in a informal way. Moreover, $\mathcal{F}_\alpha$ is an isometric isomorphism on $L^2(\mathbb{R}, d\mu_\alpha)$ and

$$\mathcal{F}_\alpha^{-1} f(y) = \mathcal{F}_\alpha f(-y). \tag{1.10}$$

Associated to the Dunkl transform, the Paley-Wiener type space PW_α is formed by all functions $f \in L^2(\mathbb{R}, d\mu_\alpha)$ such that

$$f(t) = \int_{-1}^{1} u(x)E_\alpha(itx)\, d\mu_\alpha(x), \quad u \in L^2([-1,1], d\mu_\alpha), \tag{1.11}$$

endowed with the norm of $L^2(\mathbb{R}, d\mu_\alpha)$.

The sampling theorem is the following:

Theorem 1.2. *If $f \in PW_\alpha$, $\alpha > -1$, then*

$$\frac{f(t)}{\mathcal{I}_\alpha(it)} = -2(\alpha+1) \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{f(j_{m,\alpha})}{j_{m,\alpha}\mathcal{I}_{\alpha+1}(ij_{m,\alpha})(t - j_{m,\alpha})}, \tag{1.12}$$

which converges uniformly on compact subsets of $\mathbb{C} \setminus \{j_{m,\alpha} : m \in \mathbb{Z}\setminus\{0\}\}$.

The case $\alpha = 1/2$ is usually known as the Whittaker-Shannon-Kotel'nikov Theorem:

$$f(t) = \sum_{k=-\infty}^{\infty} f(\pi k)\frac{\sin(t - k\pi)}{t - k\pi},$$

where $f(t) = \int_{-1}^{1} u(x)e^{ixt}\, dx$, $u \in L^2([-1,1], dx)$.

According to Theorem 12 of [1], for $\alpha \geq -1/2$, a function $f \in PW_\alpha$ if and only if it is an entire function satisfying

$$|f(z)| \leq ce^{|\operatorname{Im} z|}, \quad z \in \mathbb{C}, \tag{1.13}$$

and $f(x) \in L^2(\mathbb{R}, d\mu_\alpha)$. We prove that if $\alpha \in \mathbb{C} \setminus \{-1,-2,\dots\}$ with $\operatorname{Re}\alpha < -1/2$, sampling Theorem 1.2 holds for any entire function f satisfying (1.13) (see Theorem 4.1).

In [5] we have established a sampling theorem for the system $\{e_{\alpha,m}\}_{m\in\mathbb{Z}}$ using different methods. We shall show here that this result is now a simple consequence of Theorem 1.2 (see Theorem 4.3).

From now on, unless necessary, we omit α and write simply j_m instead of $j_{m,\alpha}$.

2 Proof of Theorem 1.1 and consequences

Proof of Theorem 1.1. Take a large circle $D = \{z \in \mathbb{C} : |z| = A\}$ of radius $A > |t|$ with the only condition, at the moment, that none of the points j_m, $m \in \mathbb{Z}\setminus\{0\}$, must lie on D, and consider

$$\frac{1}{2\pi i}\int_D \frac{E_\alpha(iwx)}{(w-t)\mathcal{I}_\alpha(iw)}\, dw.$$

The poles of $\frac{E_\alpha(iwx)}{(w-t)\mathcal{I}_\alpha(iw)}$ inside D, all of them simple, are t and those j_m with $|j_m| < A$. The residue at t is, obviously,

$$\frac{E_\alpha(itx)}{\mathcal{I}_\alpha(it)};$$

and the residue at each j_m is

$$\lim_{w\to j_m} \frac{(w-j_m)E_\alpha(iwx)}{(w-t)\mathcal{I}_\alpha(iw)} = \frac{E_\alpha(ij_m x)}{(j_m-t)i\mathcal{I}'_\alpha(ij_m)}.$$

From the identity

$$\mathcal{I}'_\alpha(z) = \frac{z}{2(\alpha+1)}\mathcal{I}_{\alpha+1}(z), \tag{2.1}$$

which follows immediately from (1.1), we get

$$\frac{E_\alpha(ij_m x)}{(j_m-t)i\mathcal{I}'_\alpha(ij_m)} = -\frac{2(\alpha+1)E_\alpha(ij_m x)}{j_m\mathcal{I}_{\alpha+1}(ij_m)(j_m-t)}. \tag{2.2}$$

Thus, the calculus of residues gives

$$\frac{1}{2\pi i}\int_D \frac{E_\alpha(iwx)}{(w-t)\mathcal{I}_\alpha(iw)}\,dw = \frac{E_\alpha(itx)}{\mathcal{I}_\alpha(it)} - \sum_{|j_m|<A} \frac{2(\alpha+1)E_\alpha(ij_m x)}{j_m\mathcal{I}_{\alpha+1}(ij_m)(j_m-t)}. \tag{2.3}$$

Using arguments similar to those of [11, § 15.41, p. 498], let us see that the value of A can be chosen arbitrarily large and such that there exists some constant $c > 0$ independent of A (but depending on α) satisfying

$$c\,\frac{e^{|\operatorname{Im} w|}}{|w|^{1/2}} \le |J_\alpha(w)| \tag{2.4}$$

for $w \in D$. This follows from the equality

$$2J_\alpha(w) = H_\alpha^{(1)}(w) + H_\alpha^{(2)}(w), \tag{2.5}$$

where the Bessel functions of the third kind satisfy the estimates

$$H_\alpha^{(1)}(w) = \left(\frac{2}{\pi w}\right)^{1/2} e^{i(w-\frac{1}{2}\alpha\pi-\frac{1}{4}\pi)}\{1+\eta_{1,\alpha}(w)\}, \tag{2.6}$$

$$H_\alpha^{(2)}(w) = \left(\frac{2}{\pi w}\right)^{1/2} e^{-i(w-\frac{1}{2}\alpha\pi-\frac{1}{4}\pi)}\{1+\eta_{2,\alpha}(w)\}, \tag{2.7}$$

$\eta_{1,\alpha}(w)$ and $\eta_{2,\alpha}(w)$ being $\mathcal{O}(1/w)$ for large $|w|$ [11, § 15.4, p. 496]. Therefore,

$$\frac{1}{2}\left(\frac{2}{\pi|w|}\right)^{1/2} e^{-\operatorname{Im} w+\frac{\pi \operatorname{Im}\alpha}{2}} \le \left|H_\alpha^{(1)}(w)\right| \le 2\left(\frac{2}{\pi|w|}\right)^{1/2} e^{-\operatorname{Im} w+\frac{\pi \operatorname{Im}\alpha}{2}},$$

$$\frac{1}{2}\left(\frac{2}{\pi|w|}\right)^{1/2} e^{\operatorname{Im} w-\frac{\pi \operatorname{Im}\alpha}{2}} \le \left|H_\alpha^{(2)}(w)\right| \le 2\left(\frac{2}{\pi|w|}\right)^{1/2} e^{\operatorname{Im} w-\frac{\pi \operatorname{Im}\alpha}{2}},$$

for $|w|$ large enough. This, together with (2.5), gives

$$\begin{aligned}2|J_\alpha(w)| &\geq \frac{1}{2}\left(\frac{2}{\pi|w|}\right)^{1/2} e^{|\operatorname{Im} w-\frac{\pi \operatorname{Im}\alpha}{2}|} - 2\left(\frac{2}{\pi|w|}\right)^{1/2} e^{-|\operatorname{Im} w-\frac{\pi \operatorname{Im}\alpha}{2}|}\\ &= \frac{1}{2}\left(\frac{2}{\pi|w|}\right)^{1/2} e^{|\operatorname{Im} w-\frac{\pi \operatorname{Im}\alpha}{2}|}\left(1-4e^{-2|\operatorname{Im} w-\frac{\pi \operatorname{Im}\alpha}{2}|}\right)\end{aligned}$$

for $|w|$ large enough, which proves (2.4) if, say, $|\operatorname{Im} w - \frac{\pi \operatorname{Im}\alpha}{2}| \geq 1$. On the two arcs of D with $|\operatorname{Im} w - \frac{\pi \operatorname{Im}\alpha}{2}| \leq 1$, according to (2.5), (2.6) and (2.7), the problem reduces essentially to get a lower bound for $|\cos(w-\frac{1}{2}\alpha\pi-\frac{1}{4}\pi)|$, which can be done by simply choosing A so that to avoid the zeros of the cosine function. This proves (2.4).

Furthermore, (2.5), (2.6) and (2.7) give also

$$|J_\alpha(z)| \leq C\,\frac{e^{|\operatorname{Im} z|}}{|z|^{1/2}} \tag{2.8}$$

for $|z|$ large enough, with a constant $C>0$ depending only on α. Therefore, for any compact set $K \subset (-1,1)\setminus\{0\}$ the radius A can be chosen with the additional property that there exists $C>0$ such that, for any $w \in D$ and any $x \in K$,

$$|J_\alpha(wx)| \leq C\,\frac{e^{|\operatorname{Im}(wx)|}}{|wx|^{1/2}}, \tag{2.9}$$

$$|J_{\alpha+1}(wx)| \leq C\,\frac{e^{|\operatorname{Im}(wx)|}}{|wx|^{1/2}}. \tag{2.10}$$

Using (2.4), (2.9) and (2.10), we get, for $x \in K$ and $w \in D$,

$$\begin{aligned}\left|\frac{E_\alpha(iwx)}{\mathcal{I}_\alpha(iw)}\right| &= \left|\frac{\mathcal{I}_\alpha(iwx)}{\mathcal{I}_\alpha(iw)} + \frac{iwx\mathcal{I}_{\alpha+1}(iwx)}{2(\alpha+1)\mathcal{I}_\alpha(iw)}\right|\\ &= \left|\frac{J_\alpha(wx)+iJ_{\alpha+1}(wx)}{J_\alpha(w)}\cdot\frac{w^\alpha}{(wx)^\alpha}\right| \leq \tilde{c}\,\frac{e^{(|x|-1)|\operatorname{Im} w|}}{|x|^{\operatorname{Re}\alpha+1/2}},\end{aligned}$$

for some constant $\tilde{c}$ depending only on α and K.

To finish the proof of (1.6), it is enough to prove that the left-hand side of (2.3) goes to 0 as A goes to infinity, uniformly on $x \in K$ and t in a bounded subset of $\mathbb{C}\setminus\{j_{\pm1}, j_{\pm2}, j_{\pm3}, \dots\}$. The obvious parametrization of the circle D and the above bound give

$$\left|\frac{1}{2\pi i}\int_D \frac{E_\alpha(iwx)}{(w-t)\mathcal{I}_\alpha(iw)}\,dw\right| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{A}{A-|t|}\cdot\tilde{c}\,\frac{e^{(|x|-1)A|\sin s|}}{|x|^{\operatorname{Re}\alpha+1/2}}\,ds,$$

which is easy to see that goes to 0 uniformly on x and t, as A goes to infinity (it is at this point where the values $x=\pm1$ must be excluded).

Only the final statement about $x = 0$ remains to be proved. We can follow a similar procedure for $x = 0$, using (2.4) and the bound $\sin s \geq \frac{2}{\pi}s$ on $[0, \frac{\pi}{2}]$, to get

$$\left|\frac{1}{2\pi i}\int_D \frac{E_\alpha(iwx)}{(w-t)\mathcal{I}_\alpha(iw)}\,dw\right| = \left|\frac{1}{2\pi i}\int_D \frac{dw}{(w-t)\mathcal{I}_\alpha(iw)}\right| \leq \frac{\tilde{c}}{2\pi}\frac{A^{\operatorname{Re}\alpha+3/2}}{A-|t|}\int_{-\pi}^{\pi} e^{-A|\sin s|}\,ds$$
$$\leq \frac{\tilde{c}}{2\pi}\frac{A^{\operatorname{Re}\alpha+3/2}}{A-|t|}4\int_0^{\pi/2} e^{-\frac{2A}{\pi}s}\,ds = \tilde{c}\frac{A^{\operatorname{Re}\alpha+1/2}}{A-|t|}(1-e^{-A}),$$

which goes to 0 uniformly on t when A goes to infinity, if $\operatorname{Re}\alpha < 1/2$. □

The previous result can be extended to the t-derivatives of $E_\alpha(itx)/\mathcal{I}_\alpha(it)$:

Corollary 2.1. *Let $\alpha \in \mathbb{C} \setminus \{-1,-2,-3,\dots\}$ and let n be a positive integer. Then,*

$$\frac{d^n}{dt^n}\left(\frac{E_\alpha(itx)}{\mathcal{I}_\alpha(it)}\right) = \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{2(\alpha+1)n!E_\alpha(ij_m x)}{j_m\mathcal{I}_{\alpha+1}(ij_m)(j_m-t)^{n+1}}$$

for $t \in \mathbb{C}\setminus\{j_{\pm1}, j_{\pm2}, j_{\pm3},\dots\}$ and $x \in [-1,1]\setminus\{0\}$, where the right-hand side converges uniformly for t in bounded subsets of $\mathbb{C}\setminus\{j_{\pm1}, j_{\pm2}, j_{\pm3},\dots\}$ and for x in compact subsets of $[-1,1]\setminus\{0\}$.

If, in addition, $\operatorname{Re}\alpha < n + 1/2$, then the equality holds also for $x = 0$, uniformly for t in bounded subsets of $\mathbb{C}\setminus\{j_{\pm1}, j_{\pm2}, j_{\pm3},\dots\}$.

Proof. The proof of Theorem 1.1 is still valid, with only the following changes: the starting point is now the integral

$$\frac{1}{2\pi i}\int_D \frac{E_\alpha(iwx)}{(w-t)^{n+1}\mathcal{I}_\alpha(iw)}\,dw.$$

The calculus of residues gives

$$\frac{1}{2\pi i}\int_D \frac{E_\alpha(iwx)}{(w-t)^{n+1}\mathcal{I}_\alpha(iw)}\,dw = \frac{1}{n!}\frac{d^n}{dt^n}\left(\frac{E_\alpha(itx)}{\mathcal{I}_\alpha(it)}\right) - \sum_{|j_m|<A}\frac{2(\alpha+1)E_\alpha(ij_m x)}{j_m\mathcal{I}_{\alpha+1}(ij_m)(j_m-t)^{n+1}}.$$

Now,

$$\left|\frac{1}{2\pi i}\int_D \frac{E_\alpha(iwx)}{(w-t)^{n+1}\mathcal{I}_\alpha(iw)}\,dw\right| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{A}{(A-|t|)^{n+1}}\cdot\tilde{c}\frac{e^{(|x|-1)A|\sin s|}}{|x|^{\operatorname{Re}\alpha+1/2}}\,ds$$
$$\leq C\frac{A}{(A-|t|)^{n+1}|x|^{\operatorname{Re}\alpha+1/2}},$$

for some C independent of t and $x \in [-1,1]$. Now there is no need to exclude the values $x = \pm1$, so we can allow the compact K to be contained in $[-1,1]\setminus\{0\}$, and still conclude that the integral goes to 0 as A goes to infinity, uniformly on $x \in K$ and t in a bounded subset of $\mathbb{C} \setminus \{j_{\pm1}, j_{\pm2}, j_{\pm3}, \dots\}$.

For $x = 0$, we get

$$\left| \frac{1}{2\pi i} \int_D \frac{E_\alpha(iwx)}{(w-t)^{n+1}\mathcal{I}_\alpha(iw)}\, dw \right| \le \tilde{c} \frac{A^{\operatorname{Re}\alpha+1/2}}{(A-|t|)^{n+1}}(1-e^{-A}),$$

which goes to 0 uniformly on t when A goes to infinity, if $\operatorname{Re}\alpha < n + 1/2$. □

The pointwise expansion (1.6) fails at $x = \pm1$: the series on the right-hand side of (1.6) converges for $x = \pm1$, but its sum is not equal to the left-hand side. Indeed, for $x = \pm1$, it becomes clear from (1.2), (2.1), and (2.2) that the right-hand side of (1.6) reduces to

$$\sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{\pm i}{j_m - t},$$

which, by the partial fraction decomposition of $J_{\alpha+1}(t)/J_\alpha(t)$ [11, § 15.41, p. 498], equals

$$\pm i \frac{J_{\alpha+1}(t)}{J_\alpha(t)} = \pm \frac{it\mathcal{I}_{\alpha+1}(it)}{2(\alpha+1)\mathcal{I}_\alpha(it)} = \pm \frac{(E_\alpha(it) - E_\alpha(-it))/2}{\mathcal{I}_\alpha(it)},$$

showing that the convergence in $x = \pm1$ is to a combination of the side limits $E_\alpha(it)$ and $E_\alpha(-it)$ of $E_\alpha(ixt)$.

Let us mention that for $\operatorname{Re}\alpha \ge \frac{1}{2}$ the expansion (1.6) does not converge at $x = 0$. Indeed, for $x = 0$, we get

$$\sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{2(\alpha+1)E_\alpha(0)}{j_m\mathcal{I}_{\alpha+1}(ij_m)(j_m - t)} = \sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{2(\alpha+1)}{j_m\mathcal{I}_{\alpha+1}(ij_m)(j_m - t)}.$$

Then, (2.8) gives

$$\left| \frac{1}{j_m\mathcal{I}_{\alpha+1}(ij_m)(j_m - t)} \right| \ge C_\alpha |j_m|^{\operatorname{Re}\alpha - 1/2},$$

which does not go to 0 if $\operatorname{Re}\alpha \ge \frac{1}{2}$.

Using the previous approach and the definitions (1.3) and (1.4), one can prove the Fourier expansion in Theorem 5.5 of [2] also for any complex number α (except for the negative integers) with pointwise convergence. The proof is similar to Theorem 1.1 and Corollary 2.1, so we omit it.

Theorem 2.2. *Let $\alpha \in \mathbb{C} \setminus \{-1, -2, -3, \dots\}$. Then,*

$$\frac{E_\alpha(itx)}{\mathcal{I}_{\alpha+1}(it)} = 1 + \frac{1}{2(\alpha+1)} \sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{tE_\alpha(ij_{m,\alpha+1}x)}{\mathcal{I}_\alpha(ij_{m,\alpha+1})(t - j_{m,\alpha+1})}, \tag{2.11}$$

if $t \in \mathbb{C} \setminus \{j_{\pm 1,\alpha+1}, j_{\pm 2,\alpha+1}, \dots\}$ *and* $x \in (-1,1) \setminus \{0\}$*, where the right-hand side converges uniformly for* t *in bounded subsets of* $\mathbb{C} \setminus \{j_{\pm 1,\alpha+1}, j_{\pm 2,\alpha+1}, \dots\}$ *and* x *in compact subsets of* $(-1,1) \setminus \{0\}$.

Moreover, for each positive integer n*,*

$$\frac{d^n}{dt^n}\left(\frac{E_\alpha(itx)}{\mathcal{I}_{\alpha+1}(it)}\right) = \frac{(-1)^n n!}{2(\alpha+1)} \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{j_{m,\alpha+1} E_\alpha(ij_{m,\alpha+1}x)}{\mathcal{I}_\alpha(ij_{m,\alpha+1})(t-j_{m,\alpha+1})^{n+1}}, \tag{2.12}$$

if $t \in \mathbb{C} \setminus \{j_{\pm 1,\alpha+1}, j_{\pm 2,\alpha+1}, \dots\}$ *and* $x \in [-1,1] \setminus \{0\}$*, where the right-hand side converges uniformly for* t *in bounded subsets of* $\mathbb{C} \setminus \{j_{\pm 1,\alpha+1}, j_{\pm 2,\alpha+1}, \dots\}$ *and* x *in compact subsets of* $[-1,1] \setminus \{0\}$.

If, in addition, $\operatorname{Re}\alpha < 1/2$ *or* $\operatorname{Re}\alpha < n + 1/2$, $n \geq 1$*, then the expansions* (2.11) *or* (2.12)*, respectively, also hold for* $x = 0$*, uniformly for* t *in bounded subsets of* $\mathbb{C} \setminus \{j_{\pm 1,\alpha+1}, j_{\pm 2,\alpha+1}, \dots\}$.

3 The Fourier-Dunkl system of the second kind and Euler-Dunkl polynomials

The orthogonality and completeness of the Fourier-Dunkl system of the second kind can be proved as that of its relative Fourier-Dunkl system (see Theorem 1 in [5]), just by using the completeness of the corresponding Bessel and Dini systems (see [11, Chapter XVIII]).

Since for $\alpha > -1$ the zeros j_m of the Bessel function J_α are real and simple and separate the zeros of $J_{\alpha+1}$, we have

$$|j_m \mathcal{I}_{\alpha+1}(ij_m)| = (-1)^{m+1} \operatorname{sgn}(m) j_m \mathcal{I}_{\alpha+1}(ij_m). \tag{3.1}$$

This and (1.5) give the expansion (1.7) as an easy consequence of Theorem 1.1.

For a complex number α (except for the negative integers), we define the Euler-Dunkl polynomials $\{\mathcal{E}_{n,\alpha}\}_{n=0}^\infty$ by means of the generating function (1.8). As a consequence, each $\mathcal{E}_{n,\alpha}$ is a polynomial of degree n. As usual, unless necessary we will denote them simply by $\mathcal{E}_n$. From the definition (1.8), it follows easily that the polynomials $\{\mathcal{E}_n\}_{n=0}^\infty$ are an Appell-Dunkl sequence, so they satisfy

$$\Lambda_\alpha(\mathcal{E}_n) = \theta_{n,\alpha}\mathcal{E}_{n-1}$$

with

$$\theta_{n,\alpha} := \frac{\gamma_{n,\alpha}}{\gamma_{n-1,\alpha}} = n + (\alpha + 1/2)(1 - (-1)^n).$$

Moreover,

1. $\mathcal{E}_{2n}$ is an even polynomial, $n \geq 0$, which vanishes at 1 (and hence at -1) for $n \geq 1$;
2. $\mathcal{E}_{2n+1}$ is an odd polynomial, $n \geq 0$.

The first Euler-Dunkl polynomials are

$$\mathcal{E}_0(x) = 1, \qquad \mathcal{E}_1(x) = x,$$
$$\mathcal{E}_2(x) = x^2 - 1, \qquad \mathcal{E}_3(x) = x^3 - \frac{\alpha+2}{\alpha+1}\,x,$$
$$\mathcal{E}_4(x) = x^4 - 2\,\frac{\alpha+2}{\alpha+1}\,x^2 + \frac{\alpha+3}{\alpha+1}, \qquad \mathcal{E}_5(x) = x^5 - 2\,\frac{\alpha+3}{\alpha+1}\,x^3 + \frac{\alpha+3}{(\alpha+1)^2}\,x.$$

By comparing the Euler-Dunkl polynomials with the Apostol-Euler-Dunkl polynomials $\{\mathfrak{E}_{n,\alpha,u}\}_n$, $u \in \mathbb{C} \setminus \{j_{m,\alpha+1} : m \in \mathbb{Z}\}$, introduced in [2], one can check that the sequence $\{\mathcal{E}_{n,\alpha}\}_n$ differs from $\{\mathfrak{E}_{n,\alpha,u}\}_n$ for all values of u. On the other hand, more properties of both the Bernoulli-Dunkl polynomials and the Euler-Dunkl polynomials (and some of their generalizations) can be found in [3].

For the Euler-Dunkl polynomials we have this expansion:

Theorem 3.1. *For $\alpha \in \mathbb{C} \setminus \{-1, -2, \dots\}$*

$$\mathcal{E}_n(x) = 2(\alpha+1)(-i)^n \gamma_n \sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{E_\alpha(ij_m x)}{j_m^{n+2} \mathcal{I}_{\alpha+1}(ij_m)} \tag{3.2}$$

with uniform convergence on compact subsets of $(-1,1)\setminus\{0\}$ for $n = 0$, and $[-1,1]\setminus\{0\}$ for $n \geq 1$. The convergence can be extended to $x = 0$ assuming that $\operatorname{Re}\alpha < n + 1/2$.

Proof. From the definition (1.8), we get that

$$\mathcal{E}_n(x) = \frac{\gamma_n}{i^n n!} \frac{d^n}{dt^n} \left(\frac{E_\alpha(itx)}{\mathcal{I}_\alpha(it)} \right) \bigg|_{t=0}, \qquad n = 0, 1, 2, \dots.$$

The theorem is then a consequence of Theorem 1.1 and Corollary 2.1. □

The Fourier-Dunkl expansion (1.7) for the Euler-Dunkl polynomials is then an easy consequence of the previous theorem, (3.1), and (1.5).

The Euler-Dunkl polynomials can be used to sum the reciprocal of the zeros of the Bessel functions. Indeed, by setting $n = 2k-1$ and inserting $x = 1$ in (3.2), and using (1.2), we get for any $\alpha \in \mathbb{C} \setminus \{-1, -2, \dots\}$ and $k \geq 1$,

$$\sum_{m=1}^{\infty} \frac{1}{j_m^{2k}} = \frac{(-1)^{k+1}}{2^{2k}(k-1)!\,(\alpha+1)_k}\, \mathcal{E}_{2k-1}(1). \tag{3.3}$$

Identities of this type are well known in the literature and we refer to a discussion and further references in [2]. By inserting $x = 0$ and assuming in addition $\operatorname{Re}\alpha < 2k - \frac{3}{2}$, we get

$$\sum_{m=1}^{\infty} \frac{1}{\mathcal{I}_{\alpha+1}(ij_m) j_m^{2k}} = \frac{(-1)^{k+1}}{2^{2k}(k-1)!\,(\alpha+1)(\alpha+1)_{k-1}} \mathcal{E}_{2k-2}(0). \tag{3.4}$$

With the notation of [2, Sect. 4], we have

$$\begin{aligned} \sigma_k(\alpha) &= \sum_{m=1}^{\infty} \frac{1}{j_{m,\alpha+1}^{2k}}, \\ \varrho_k(\alpha) &= \sum_{m=1}^{\infty} \frac{1}{\mathcal{I}_{\alpha}(ij_{m,\alpha+1}) j_{m,\alpha+1}^{2k}}. \end{aligned} \tag{3.5}$$

Hence, the identity (3.3) provides an explicit expression for $\sigma_k(\alpha - 1)$ in terms of the Euler-Dunkl polynomials (compare with [2, formula (4.7)]).

Although the series in (3.4) and (3.5) seem to be different kinds of "alternate" series, we show next that they are actually the same. Indeed, since $z(J_{\alpha-1}(z) + J_{\alpha+1}(z)) = 2\alpha J_\alpha(z)$, we have

$$x^2 \mathcal{I}_{\alpha+1}(ix) = 4\alpha(\alpha+1)(\mathcal{I}_\alpha(ix) - \mathcal{I}_{\alpha-1}(ix)),$$

and then

$$j_{m,\alpha}^2 \mathcal{I}_{\alpha+1}(ij_{m,\alpha}) = -4\alpha(\alpha+1)\mathcal{I}_{\alpha-1}(ij_{m,\alpha}). \tag{3.6}$$

Consequently,

$$\sum_{m=1}^{\infty} \frac{1}{\mathcal{I}_{\alpha+1}(ij_{m,\alpha}) j_{m,\alpha}^{2k}} = \frac{-1}{4\alpha(\alpha+1)} \sum_{m=1}^{\infty} \frac{1}{\mathcal{I}_{\alpha-1}(ij_{m,\alpha}) j_{m,\alpha}^{2k-2}} = \frac{-\varrho_{k-1}(\alpha-1)}{4\alpha(\alpha+1)}.$$

The identity (3.4) provides then an explicit expression for $\varrho_{k-1}(\alpha - 1)$ in terms of the Euler-Dunkl polynomials (compare with [2, formula (4.8)]). Using the pointwise convergence at $x = 0$ of the expansion (3.2), we can improve our results in [2]: the convergence of (3.5) was proved in [2] for $\operatorname{Re}\alpha < 2k - 3/2$, while in the present paper we have proved it for $\operatorname{Re}\alpha < 2k + 1/2$.

The Bernoulli-Dunkl polynomials $\{\mathfrak{B}_{n,\alpha}(x)\}_{n=0}^{\infty}$ are defined by the generating function

$$\frac{E_\alpha(xt)}{\mathcal{I}_{\alpha+1}(t)} = \sum_{n=0}^{\infty} \frac{\mathfrak{B}_{n,\alpha}(x)}{\gamma_{n,\alpha}} t^n$$

(see equation (2.6) of [2]). Hence, using Theorem 2.2 and proceeding as in the proof of Theorem 3.1, we can prove the expansion

$$\mathfrak{B}_{n,\alpha}(x) = \frac{-(-i)^n \gamma_{n,\alpha}}{2(\alpha+1)} \sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{E_\alpha(ij_{m,\alpha+1}x)}{\mathcal{I}_\alpha(ij_{m,\alpha+1}) j_{m,\alpha+1}^n},$$

for $\alpha \in \mathbb{C} \setminus \{-1, -2, \dots\}$, where the convergence is uniform on compact subsets of $(-1, 1) \setminus \{0\}$ for $n = 1$, or $[-1, 1] \setminus \{0\}$ for $n \geq 2$. The convergence can be extended to $x = 0$ assuming that $\operatorname{Re} \alpha < n + 1/2$. Notice that for $\alpha > -1$, and using (1.3), we recover the expansion (5.8) of [2] but now with pointwise convergence.

4 A Whittaker-Shannon-Kotel'nikov sampling theorem

The expansion (1.7) provides a very simple proof of Whittaker-Shannon-Kotel'nikov sampling Theorem 1.2.

Proof of Theorem 1.2. For $f \in PW_\alpha$, let us consider the corresponding function $u \in L^2([-1, 1], d\mu_\alpha)$ such that

$$f(t) = \int_{-1}^{1} u(x) E_\alpha(itx)\, d\mu_\alpha(x). \tag{4.1}$$

We associate to the function u the continuous operator T_u on $L^2([-1, 1], d\mu_\alpha)$ defined by

$$T_u(g) = \int_{-1}^{1} u(x) g(x)\, d\mu_\alpha(x).$$

For each $t \in \mathbb{C} \setminus \{j_m : m \in \mathbb{Z} \setminus \{0\}\}$, consider the function

$$\phi_t(x) = \frac{E_\alpha(itx)}{\mathcal{I}_\alpha(it)} \in L^2([-1, 1], d\mu_\alpha).$$

The identity (4.1), the fact that the Fourier-Dunkl expansion (1.7) converges in $L^2([-1, 1], d\mu_\alpha)$, and formulas (1.5) and (3.1), show that

$$\begin{aligned}
\frac{f(t)}{\mathcal{I}_\alpha(it)} = T_u(\phi_t(x)) &= \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{(-1)^m \operatorname{sgn}(m)}{2^{\alpha/2} \Gamma(\alpha + 1)^{1/2} (t - j_m)} T_u(f_{\alpha,m}(x)) \\
&= \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{(-1)^m \operatorname{sgn}(m) 2(\alpha + 1)}{|j_m \mathcal{I}_{\alpha+1}(i j_m)| (t - j_m)} \int_{-1}^{1} u(x) E_\alpha(i j_m x)\, d\mu_\alpha(x) \\
&= -2(\alpha + 1) \sum_{m \in \mathbb{Z} \setminus \{0\}} \frac{f(j_m)}{j_m \mathcal{I}_{\alpha+1}(i j_m)(t - j_m)},
\end{aligned}$$

which proves the pointwise convergence of (1.12) for $t \in \mathbb{C} \setminus \{j_m : m \in \mathbb{Z} \setminus \{0\}\}$.

The uniform convergence on compact sets can be proved as follows. Using (1.5) and the fact that

$$\left(\int_{-1}^{1} |f_{\alpha,m}(x)|^2\, d\mu_\alpha(x) \right)^{1/2} = 1,$$

we get

$$\begin{aligned}|f(j_m)| &= \left|\int_{-1}^{1} u(x)E_\alpha(ij_m x)\,d\mu_\alpha(x)\right| \\ &\le \|u\|_2 \left(\int_{-1}^{1} |E_\alpha(ij_m x)|^2\,d\mu_\alpha(x)\right)^{1/2} = c\,|j_m\mathcal{I}_{\alpha+1}(ij_m)|,\end{aligned}$$

for certain positive constant c which does not depend on m. This shows that the series

$$\sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{f(j_m)}{j_m\mathcal{I}_{\alpha+1}(ij_m)(t-j_m)^2}$$

converges uniformly on compact subsets of $t\in\mathbb{C}\setminus\{j_m : m\in\mathbb{Z}\setminus\{0\}\}$. Since the terms in this series are the derivatives of the terms in the series (1.12), we can deduce that the series (1.12) also converges uniformly on compact subsets of $t\in\mathbb{C}\setminus\{j_m : m\in\mathbb{Z}\setminus\{0\}\}$. □

According to Theorem 12 of [1], for $\alpha\ge -1/2$, a function $f\in PW_\alpha$ has a double feature: on the one hand f is an entire function satisfying

$$|f(z)|\le ce^{|\operatorname{Im} z|},\quad z\in\mathbb{C}, \tag{4.2}$$

for certain constant $c>0$ (the *complex* feature); on the other hand for $x\in\mathbb{R}$, $f(x)\in L^2(\mathbb{R},d\mu_\alpha)$ (the *harmonic* feature). For $\alpha\ge -1/2$, both complex and harmonic features seem to be necessary for f to satisfy sampling Theorem 1.2. Indeed, the function $f(t)=E_\alpha(it)$ (which satisfies (4.2)) shows that the *complex* feature is not enough to guarantee the sampling theorem: for $f(t)=E_\alpha(it)$, the partial fraction decomposition of $\frac{J_{\alpha+1}(t)}{J_\alpha(t)}$ (see [11, § 15.41, (1), p. 498]) gives, for the right-hand side of (1.12),

$$\begin{aligned}-2(\alpha+1)\sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{E_\alpha(ij_m)}{j_m\mathcal{I}_{\alpha+1}(ij_m)(t-j_m)} &= \sum_{m\in\mathbb{Z}\setminus\{0\}}\frac{-i}{t-j_m} \\ &= i\,\frac{J_{\alpha+1}(t)}{J_\alpha(t)} = i\,\frac{t\mathcal{I}_{\alpha+1}(it)}{2(\alpha+1)\mathcal{I}_\alpha(it)} \ne \frac{E_\alpha(it)}{\mathcal{I}_\alpha(it)}.\end{aligned}$$

Conversely, the following result shows (taking $N=0$) that for $\alpha<-1/2$, the *complex* feature of a function f alone implies the sampling theorem (and hence, for $\alpha<-1/2$ the *harmonic* feature does not seem to play any role there).

Theorem 4.1. *Let N be an integer and let f be an entire function satisfying*

$$|f(z)|\le C(1+|z|)^N e^{|\operatorname{Im} z|},\quad z\in\mathbb{C}, \tag{4.3}$$

for certain positive constant $C > 0$. If $\alpha \in \mathbb{C}\setminus\{-1,-2,\dots\}$ satisfies $\operatorname{Re}\alpha + N < -1/2$ then

$$\frac{f(t)}{\mathcal{I}_\alpha(it)} = -2(\alpha+1) \sum_{m\in\mathbb{Z}\setminus\{0\}} \frac{f(j_m)}{j_m \mathcal{I}_{\alpha+1}(ij_m)(t-j_m)}, \tag{4.4}$$

which converges uniformly on bounded subsets of $\mathbb{C}\setminus\{j_m : m \in \mathbb{Z}\setminus\{0\}\}$.

Proof. Take a large circle $D = \{z \in \mathbb{C} : |z| = A\}$ of radius $A > |t|$ with the only condition, at the moment, that none of the points j_m, $m \in \mathbb{Z}\setminus\{0\}$, must lie in D. The calculus of residues gives

$$\frac{1}{2\pi i}\int_D \frac{f(w)}{(w-t)\mathcal{I}_\alpha(iw)}\,dw = \frac{f(t)}{\mathcal{I}_\alpha(it)} + \sum_{|j_m|<A} \frac{2(\alpha+1)f(j_m)}{j_m\mathcal{I}_{\alpha+1}(ij_m)(t-j_m)}. \tag{4.5}$$

The estimate

$$|J_\alpha(w)| \geq c\,\frac{e^{|\operatorname{Im} w|}}{|w|^{1/2}} \tag{4.6}$$

for $w \in D$ and certain constant $c > 0$ independent of A (but depending on α) was proved in Theorem 1.1 (see (2.4)). Using (4.6) and (4.3), we get, for $w \in D$,

$$\left|\frac{f(w)}{\mathcal{I}_\alpha(iw)}\right| \leq \tilde{c}\,|w|^{\operatorname{Re}\alpha + N + 1/2}. \tag{4.7}$$

For $\operatorname{Re}\alpha + N + 1/2 < 0$, it is then easy to prove that the left-hand side of (4.5) goes uniformly to 0 as A goes to infinity. □

We illustrate Theorems 1.2 and 4.1 with some examples. In what follows, when we use $f(x,t) \in PW_{\alpha,t}$ or $\mathcal{F}_{\alpha,t}(f(x,t))(y)$, we are indicating that x plays the role of a parameter and t is the variable in the integrals which define PW_α or $\mathcal{F}_\alpha$.

Proposition 4.2. *For real numbers α, β, x and a nonnegative integer n with $\alpha > -1$, $\beta > \alpha + 2n$ and $0 < |x| \leq 1$, we have $t^{2n}E_\beta(ixt) \in PW_{\alpha,t}$. For $\alpha > -1$ and $0 < |x| \leq 1$, we also have $(E_\alpha(ixt) - \mathcal{I}_\alpha(it))/t \in PW_{\alpha,t}$.*

Proof. Using (1.10), the first part of the proposition will follow if we prove that $\mathcal{F}_{\alpha,t}(t^{2n}E_\beta(ixt))(y) = 0$ for $|y| > 1$, that is

$$\int_{\mathbb{R}} t^{2n}E_\beta(ixt)E_\alpha(iyt)\,d\mu_\alpha(t) = 0.$$

But, due to

$$E_\nu(iz) = 2^\nu\Gamma(\nu+1)\left(\frac{J_\nu(z)}{z^\nu} + \frac{J_{\nu+1}(z)}{z^{\nu+1}}\,zi\right)$$

and using the parity of the functions involved in the integrals, this follows easily from the fact that

$$\int_0^\infty t^{-(\beta-\alpha-1-2n)} J_\beta(xt) J_\alpha(yt)\, dt = \int_0^\infty t^{-(\beta-\alpha-1-2n)} J_{\beta+1}(xt) J_{\alpha+1}(yt)\, dt = 0$$

for $\alpha > -1-n$, $\beta > \alpha+2n$, and $0 < x < y$, which are particular cases of the identity [8, Ch. 8.11, (9), p. 48].

For the second part of the proposition, take into account that

$$\frac{E_\alpha(ixt) - \mathcal{I}_\alpha(it)}{t} = \frac{\mathcal{I}_\alpha(ixt) - \mathcal{I}_\alpha(it)}{t} + \frac{ix}{2(\alpha+1)}\, \mathcal{I}_{\alpha+1}(ixt).$$

As in the first part, for $n = 0$ and $\beta = \alpha + 1$, we have $\mathcal{I}_{\alpha+1}(ixt) \in PW_{\alpha,t}$, so it is enough to prove that

$$\frac{\mathcal{I}_\alpha(ixt) - \mathcal{I}_\alpha(it)}{t} \in PW_{\alpha,t}.$$

Proceeding as before this will follow if we prove that

$$\mathcal{F}_{\alpha,t}\Big((\mathcal{I}_\alpha(ixt) - \mathcal{I}_\alpha(it))/t\Big)(y) = 0$$

for $|y| > 1$, that is

$$\int_{\mathbb{R}} \frac{\mathcal{I}_\alpha(ixt) - \mathcal{I}_\alpha(it)}{t} E_\alpha(iyt)\, d\mu_\alpha(t) = 0.$$

But this is an easy consequence of

$$\int_0^\infty \frac{J_\alpha(xt)}{x^\alpha} J_{\alpha+1}(yt)\, dt = \frac{1}{y^{\alpha+1}}$$

for $\alpha > -1$ and $0 < x < y$, which is the identity [8, Ch. 8.11, (3), p. 47]. □

In Figure 1, we illustrate sampling Theorem 1.2 with the particular example of the function $f(t) = t^{2n} E_\beta(it) \in PW_\alpha$ for $\beta = 7.1$, $n = 1$ and $\alpha = 2.7$. More precisely, the left and right pictures correspond to the real and imaginary parts of that function (solid lines) and their approximations (dotted lines) according to Theorem 1.2, respectively. We have used m up to 6 in (1.12) and to simplify the picture, we have multiplied the identity (1.12) by the function $\mathcal{I}_\alpha(it)$. The red points are the real and imaginary parts of the evaluation of $f(t)$ at the roots $\pm j_{m,\alpha}$, $m = 1, \dots, 6$.

In Figure 2, we illustrate sampling Theorem 4.1 with the particular example of the function $f(t) = te^{it/2}$ and $\alpha = -9/4$. More precisely, the left and right pictures correspond to the real and imaginary parts of that function (solid lines) and their approximations (dotted lines) according to Theorem 4.1, respectively. We have used m up to 6 in (1.12) and to simplify the picture, we have multiplied the identity (4.4)

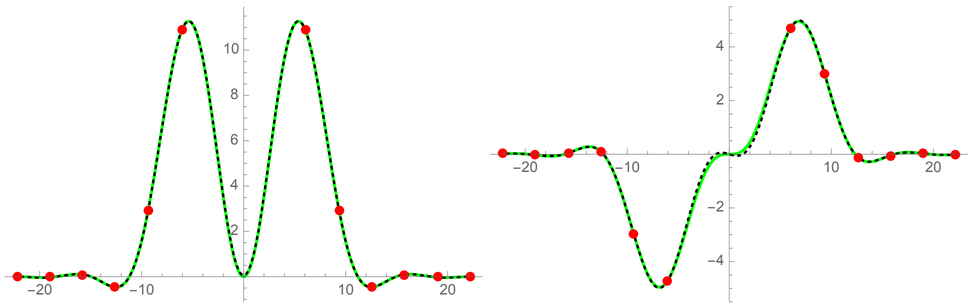

Figure 1: Real and imaginary parts (left and right, respectively) of $f(t) = t^{2n}E_\beta(it)$ (solid line) with $\beta = 7.1$ and $n = 1$, and its approximation (dotted line) according to Theorem 1.2 with $\alpha = 2.7$ (using m up to 6).

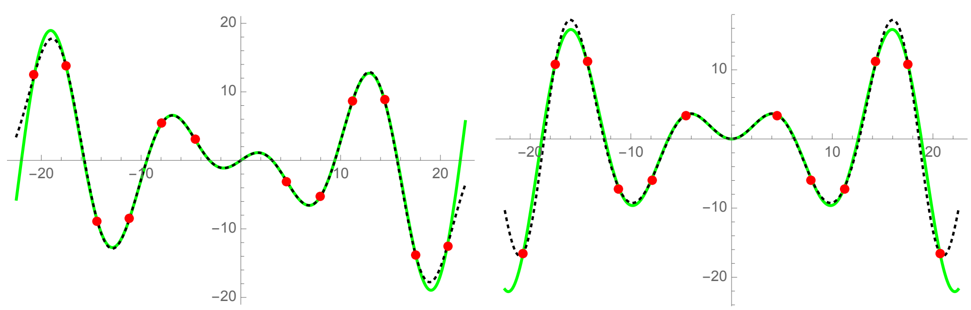

Figure 2: Real and imaginary parts (left and right, respectively) of $f(t) = te^{it/2}$ (solid line) and its approximation (dotted line) according to Theorem 4.1 with $\alpha = -9/4$ (using, in (4.4), six pairs of the real roots of $J_{-9/4}(t)$ and its four complex roots).

by the function $\mathcal{I}_\alpha(it)$. We have used in (4.4) six pairs of the real roots of $J_{-9/4}(t)$ and its four complex roots. The red points are the real and imaginary parts of the evaluation of $f(t)$ at the real roots; the complex roots $\pm 0.906543 + 1.41591i$ and $\pm 0.906543 - 1.41591i$ and its images by f (that is, $\pm 0.0960465 + 0.822684i$ and $\pm 2.9129 - 1.77804i$, respectively) cannot be represented in this way.

In Figure 3, we illustrate sampling Theorem 4.1 with the function $f(t) = \sin(t/2)/t$ and $\alpha = 1/4$. For $\alpha \geq -1/2$, it is easy to see that $f \in PW_\alpha$ if and only if $\alpha < 0$ (see (1.13)), so the function f does not satisfy the hypothesis of sampling Theorem 1.2 for $\alpha = 1/4$. However, according to Theorem 4.1 (with $N = -1$), the sampling can actually be applied to f under the assumption that $\alpha < 1 - 1/2 = 1/2$ which includes the case $\alpha = 1/4$. We have used m up to 10 in (1.12) and to simplify the picture, we have multiplied the identity (4.4) by the function $\mathcal{I}_\alpha(it)$. The imaginary parts

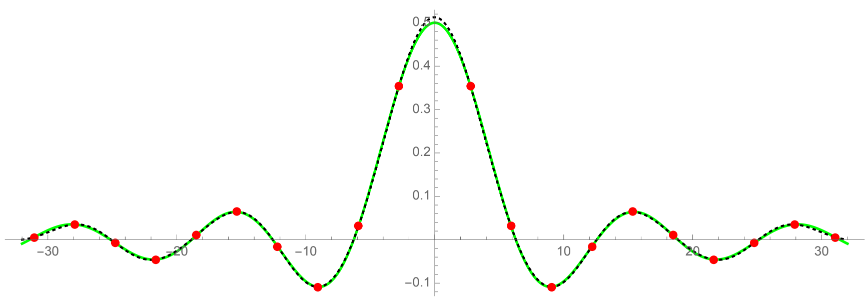

Figure 3: The function $f(t) = \sin(t/2)/t$ (solid line) and its approximation (dotted line) according to Theorem 4.1 with $\alpha = 1/4$ (using, in (4.4), ten pairs of the real roots of $J_{1/4}(t)$).

of both the function and its approximation are null, so the picture represents the function (solid lines) and its approximation (dotted lines), and the red points are the evaluation of $f(t)$ at the roots.

Remark. Notice that sampling Theorem 4.1 can also be applied to the function $g(t) = \sin(t)/t$ and $\alpha = 1/4$. However, the convergence for this example is extremely slow (using ten thousand zeros we still get an approximation for g poorer than that of Figure 3 for the function $f(t) = \sin(t/2)/t$ using ten zeros). The reason can be found in the proof of the Theorem 4.1. Indeed, on the one hand, since

$$|g(z)| \leq C\,\frac{e^{|\operatorname{Im}(z)|}}{1+|z|},$$

the proof of the Theorem 4.1 says that the error between the function g and its approximants is of the order $m^{\alpha-1+1/2} = m^{-1/4}$ (see (4.7)). On the other hand since

$$|f(z)| \leq C\,\frac{e^{|\operatorname{Im}(z)|/2}}{1+|z|},$$

the error between the function f and its approximants is now of the order of $m^{\alpha-3/2} = m^{-5/4}$.

The following sampling theorem for the Paley-Wiener space PW_α, $\alpha > -1$, can be found in [5].

Theorem 4.3 ([5, Theorem 2])**.** *If $\alpha > -1$ and $f \in PW_\alpha$, then*

$$\frac{f(t)}{\mathcal{I}_{\alpha+1}(it)} = f(0) + \sum_{m\in\mathbb{Z}\setminus\{0\}} f(j_{m,\alpha+1})\frac{t}{2(\alpha+1)\mathcal{I}_\alpha(ij_{m,\alpha+1})(t-j_{m,\alpha+1})}, \tag{4.8}$$

which converges uniformly on compact subsets of $\mathbb{R}\setminus\{j_{m,\alpha+1} : m \in \mathbb{Z}\setminus\{0\}\}$.

This result is proved in [5] using standard techniques of Hilbert spaces with reproducing kernel. We show now that Theorem 4.3 can also be derived from our Theorem 1.2. We need the following lemma, before.

Lemma 4.4. *Let $\alpha > -1$ and $f \in PW_\alpha$, with $f(0) = 0$. Then $f(t)/t \in PW_{\alpha+1}$.*

Proof. Using (2.8), we deduce that, for certain constant $c > 0$,

$$|\mathcal{I}_\alpha(iz)| \leq \begin{cases} c(1+|z|)e^{|\operatorname{Im}(z)|}, & z \in \mathbb{C}, \quad \text{for } -1 < \alpha, \\ ce^{|\operatorname{Im}(z)|}, & z \in \mathbb{C}, \quad \text{for } -1/2 \leq \alpha. \end{cases}$$

And then we also have

$$|E_\alpha(izw)| \leq c(1+|zw|)e^{|\operatorname{Im}(zw)|}, \quad z, w \in \mathbb{C}. \tag{4.9}$$

Using (4.9) it is straightforward to prove that if $f \in PW_\alpha$ (i.e., (1.11)), $\alpha > -1$, then

$$|f(z)| \leq c_f(1+|z|)e^{|\operatorname{Im} z|}, \quad z \in \mathbb{C}.$$

Since $f(0) = 0$, we also have

$$\left|\frac{f(z)}{z}\right| \leq \tilde{c}_f e^{|\operatorname{Im} z|}, \quad z \in \mathbb{C}.$$

On the other hand, since $f \in L^2(\mathbb{R}, d\mu_\alpha)$, it follows that $f(t)/t \in L^2(\mathbb{R}, d\mu_{\alpha+1})$. It is now enough to use Theorem 12 of [1] (for $k = \alpha + 3/2 \geq 1/2$). □

Proof of Theorem 4.3. Let $\alpha > -1$, $f \in PW_\alpha$, and assume that $f(0) = 0$. It follows from the previous lemma that $f(t)/t \in PW_{\alpha+1}$. Then, Theorem 1.2 with $\alpha + 1$ gives

$$\frac{f(t)}{t\mathcal{I}_{\alpha+1}(it)} = -2(\alpha+2) \sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{f(j_{m,\alpha+1})}{j_{m,\alpha+1}^2 \mathcal{I}_{\alpha+2}(ij_{m,\alpha+1})(t - j_{m,\alpha+1})}.$$

Using (3.6), we get

$$\frac{f(t)}{t\mathcal{I}_{\alpha+1}(it)} = \sum_{m \in \mathbb{Z}\setminus\{0\}} \frac{f(j_{m,\alpha+1})}{2(\alpha+1)\mathcal{I}_\alpha(ij_{m,\alpha+1})(t - j_{m,\alpha+1})}, \tag{4.10}$$

that is, the series (4.8).

Consider finally the case $f(0) \neq 0$. Then $\mathcal{I}_{\alpha+1}(it) = \operatorname{Re}(E_{\alpha+1}(it)) \in PW_\alpha$ (because of Proposition 4.2). Theorem 4.3 follows now by applying (4.10) to the function $g(t) = f(t) - f(0)\mathcal{I}_{\alpha+1}(it)$. □

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