

A COMPLEX FUNCTION THAT INTERPOLATES THE DUNKL FACTORIAL

ANTONIO J. DURÁN , ALEJANDRO GIL ASENSI , AND JUAN L. VARONA 

ABSTRACT. In the Dunkl context (on the real line) with parameter $\nu > -1$, the role of the ordinary exponential function $\exp(z) = \sum_{n=0}^{\infty} z^n/n!$ is played by the Dunkl exponential function $E_\nu(z) = \sum_{n=0}^{\infty} z^n/\gamma_{n,\nu}$, for certain sequence $\gamma_{n,\nu}$ (depending on ν), in such a way that $\gamma_{n,-1/2} = n!$ and $E_{-1/2}(z) = \exp(z)$. The gamma function is a complex function that interpolates the factorial, i.e., it satisfies $\Gamma(n+1) = n!$. In this paper, we look for a suitable complex function $\Gamma_\nu(z)$ that satisfies $\Gamma_\nu(n+1) = \gamma_{n,\nu}$, and study its properties.

Our starting point is the generalization of the exponential function $\exp(z) = \sum_{n=0}^{\infty} z^n/n!$ and the factorial sequence $n!$ which appear in the Dunkl context on the real line with parameter $\nu > -1$ (see, for instance, [1–4]). More precisely, the Dunkl exponential function $E_\nu(z)$ (also known as Dunkl kernel) is defined by

$$E_\nu(z) = \sum_{n=0}^{\infty} \frac{z^n}{\gamma_{n,\nu}}, \quad z \in \mathbb{C},$$

with

$$(1) \quad \gamma_{n,\nu} = \frac{2^n \Gamma(1 + \lfloor n/2 \rfloor) \Gamma(\nu + 1 + \lfloor (n+1)/2 \rfloor)}{\Gamma(\nu + 1)} = \begin{cases} 2^{2k} k! (\nu + 1)_k, & \text{if } n = 2k, \\ 2^{2k+1} k! (\nu + 1)_{k+1}, & \text{if } n = 2k + 1, \end{cases}$$

where $(a)_n$ denotes the Pochhammer symbol: if a is not a negative integer,

$$(a)_n = a(a+1)(a+2) \cdots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

In the case $\nu = -1/2$ we have $E_{-1/2}(z) = \exp(z)$ and $\gamma_{n,-1/2} = n!$, so $\gamma_{n,\nu}$ is known as the Dunkl factorial. Moreover, in the same way that $n! = n \cdot (n-1)!$, we have now an expression of the type $\gamma_{n,\nu} = \theta_{n,\nu} \gamma_{n-1,\nu}$, where

$$(2) \quad \theta_{n,\nu} = n + (1 + (-1)^n)(\nu + 1/2), \quad n = 1, 2, 3, \dots$$

In the Dunkl context, the operator

$$\Lambda_\nu f(x) = f'(x) + \frac{2\nu + 1}{2} \frac{f(x) - f(-x)}{x}$$

extends the ordinary derivative (in the sense that $\Lambda_{-1/2} = \frac{d}{dx}$). Then, the function $E_\nu(x)$ plays the role of $\exp(x)$ in the sense that $E_\nu(\lambda x)$ is an eigenfunction of the Dunkl operator, namely $\Lambda_\nu E_\nu(\lambda x) = \lambda E_\nu(\lambda x)$, and $E_\nu(0) = 1$ (in the same way as $\frac{d}{dx} \exp(\lambda x) = \lambda \exp(\lambda x)$, and $\exp(0) = 1$).

In the mathematical literature, many classical results have been extended to the Dunkl context. For instance, there is a Dunkl transform that generalizes the Fourier transform using the kernel $E_\nu(-ixy)$ instead of e^{-ixy} [2, 3], Paley-Wiener theorems [5–7], sampling theorems [8], Fourier-Dunkl series instead of ordinary Fourier series [8, 9], Appell-Dunkl sequences of polynomials instead of Appell sequences [9–12], zeta functions [13], umbral calculus [14], maximal functions [6],

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Riesz transforms [15], as well as some other mathematical concepts that have been studied in the Dunkl world.

It is well known that

$$(3) \quad n! = \int_0^\infty t^n e^{-t} dt, \quad n = 0, 1, 2, \dots$$

In the recent paper [16], we study the moment problem

$$(4) \quad \gamma_{n,\nu} = \int_{-\infty}^\infty t^n d\Omega(t), \quad n = 0, 1, 2, \dots,$$

that consists of finding a (positive) measure $d\Omega$ that satisfies (4).

In that paper we prove that, for $\nu \in \mathbb{R} \setminus \mathbb{Z}$, the measure $w_\nu(t) dt$, where

$$(5) \quad w_\nu(t) = \frac{2^{-\nu-1}|t|^{\nu+1}(\operatorname{sgn}(t)K_{\nu+1}(|t|) + K_\nu(|t|))}{\Gamma(\nu+1)}$$

and K_μ is the so-called modified Bessel function of the second kind and order μ (see [17] or [18]), satisfies

$$(6) \quad \int_{-\infty}^\infty t^n w_\nu(t) dt = \gamma_{n,\nu}$$

for every $n \in \mathbb{N}$ with $n > -2\nu - 2$. The condition $n > -2\nu - 2$ is necessary to make the integral (6) convergent, and it includes every $n = 0, 1, 2, \dots$ when $\nu > -1$ (when necessary, and as it is usual in the Dunkl context, we will assume $\nu > -1$).

In [16], we also prove that the moment problem (4) has a solution $d\Omega$ which is a positive measure if and only if $-1 < \nu \leq -1/2$. In that case, the solution is unique and given by the measure w_ν . The limit case $\nu = -1/2$ is the moment problem for the factorial sequence $n!$ (see (3)) and then $w_{-1/2}(t) = e^{-t}\chi_{[0,\infty)}(t)$. If $\nu > -1/2$, $w_\nu(t) > 0$ for $t > 0$ and the measure w_ν also solves the moment problem (4), although it is no longer a positive measure because $w_\nu(t) < 0$ for $t < 0$. However, the function w_ν enjoys some positivity conditions:

$$(7) \quad \begin{cases} w_\nu(t) + w_\nu(-t) = \frac{1}{2^\nu \Gamma(\nu+1)} |t|^{\nu+1} K_\nu(|t|) > 0, & t \neq 0, \\ w_\nu(t) - w_\nu(-t) = \frac{1}{2^\nu \Gamma(\nu+1)} |t|^{\nu+1} K_{\nu+1}(|t|) > 0, & t \neq 0. \end{cases}$$

The gamma function is defined for $\operatorname{Re}(s) > 0$ by changing in (3) the function t^n to t^{s-1} :

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt.$$

Apart from interpolating the factorial ($\Gamma(n+1) = n!$), the gamma function has a huge amount of properties (see, for instance, [19]), and also some alternative definitions (using, for instance, an infinite product). There are also ways to define a kind of gamma function Γ_f associated to a function f satisfying a functional equation of the type $\phi(s+1) = f(s)\phi(s)$, [20], and other kinds of generalizations of the gamma function have been proposed in the mathematical literature, with different purposes and using different ideas; see, for instance, [21–24].

In this paper, we extend the gamma function to the Dunkl context by changing in (6) the function t^n to t^{s-1} :

$$(8) \quad \Gamma_\nu(s) = \int_{-\infty}^\infty t^{s-1} w_\nu(t) dt, \quad \operatorname{Re}(s) > 0.$$

From the definition, we have that $\Gamma_\nu(n+1) = \gamma_{n,\nu}$ (recall (6)), so that Γ_ν interpolates the Dunkl factorial. One of the more apparent differences between $\Gamma(s)$ and $\Gamma_\nu(s)$ with $\nu \neq -1/2$ is the following. The integral in (8) includes the negative real axes, and then t^{s-1} is not a real number when $s > 0$ is not an integer, but a complex number. As a consequence, we have $\Gamma : (0, \infty) \rightarrow (0, \infty)$, but $\Gamma_\nu : (0, \infty) \rightarrow \mathbb{C}$ for any $\nu \neq -1/2$.

The content of this paper is as follows. In Section 2, we extend the function Γ_ν to the whole complex plane (except for some isolated poles) using analytic continuation. By doing that, we

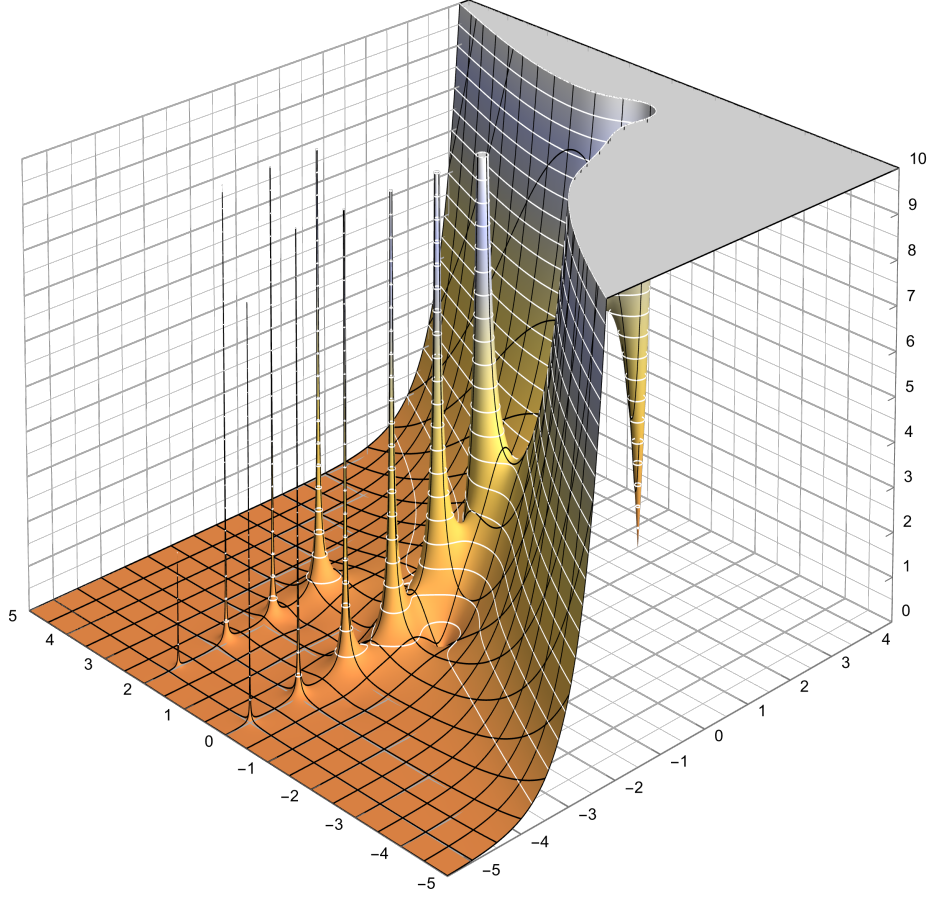


FIGURE 1. Graphic representation of $|\Gamma_{2/5-i}(x+iy)|$ for $(x, y) \in (-5.5, 4.5) \times (-5, 5)$.

get the following explicit expression valid for $s \in \mathbb{C}$ (except for some isolated poles)

$$(9) \quad \Gamma_\nu(s) = \frac{2^{s-2}}{\Gamma(\nu+1)} \left((1 - e^{i\pi s}) \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s+2\nu+1}{2}\right) + (1 + e^{i\pi s}) \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+2\nu+2}{2}\right) \right)$$

or the equivalent expression

$$(10) \quad \Gamma_\nu(s) = \frac{\sqrt{\pi}}{2\Gamma(\nu+1)} \Gamma(s) \left((1 - e^{i\pi s}) \frac{\Gamma\left(\frac{s+2\nu+1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + (1 + e^{i\pi s}) \frac{\Gamma\left(\frac{s+2\nu+2}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} \right).$$

This allows to consider the function Γ_ν also for $\nu \in \mathbb{C} \setminus \{-1, -2, -3, \dots\}$.

In Section 3, we study the poles of the function Γ_ν , showing that they form a double sequence $-n, -n-1-2\nu$, $n = 0, 1, \dots$, located in the negative real axis when ν is a real number, or in the half-lines $\text{Im}(s) = 0$ and $\text{Re}(s) \leq 0$, $\text{Im}(s) = -2\text{Im}(\nu)$ and $\text{Re}(s) \leq 2\text{Re}(\nu)$, when ν is not real. They are simple, except for some exceptions when $\nu \in \mathbb{N}$. We also prove that the function Γ_ν has infinitely many zeros except for $\nu = -1/2$ (that is, the gamma function). Figure 1, for the function $|\Gamma_\nu(s)|$ with $\nu = 2/5 - i$, shows some poles located in the half-lines $\text{Im}(s) = 0$ and $\text{Re}(s) \leq 0$, $\text{Im}(s) = 2$ and $\text{Re}(s) \leq 4/5$, and a zero at $s \sim 3.14616 - 0.555327i$.

In Section 4, we study the functions $\Gamma_\nu^{(1)}$ and $\Gamma_\nu^{(2)}$ defined by

$$\Gamma_\nu^{(1)}(s) = 2^{2s-1} \frac{\Gamma(s)\Gamma(s+\nu+1)}{\Gamma(\nu+1)}, \quad \Gamma_\nu^{(2)}(s) = 2^{2s} \frac{\Gamma(s+1)\Gamma(s+\nu+1)}{\Gamma(\nu+1)},$$

respectively. They can be defined from the measure w_ν (5) as follows ($\text{Re}(s) > 0$):

$$\Gamma_\nu^{(1)}(s) = \frac{1}{2^\nu \Gamma(\nu+1)} \int_0^{+\infty} K_{\nu+1}(t) t^{\nu+2s} dt,$$

$$\Gamma_\nu^{(2)}(s) = \frac{1}{2^\nu \Gamma(\nu+1)} \int_0^{+\infty} K_\nu(t) t^{\nu+2s+1} dt$$

(the connection with w_ν is even clearer in view of (7)). Let us note that (9) can be rewritten as

$$(11) \quad \Gamma_\nu(s) = \frac{1}{2} \left((1 - e^{i\pi s}) \Gamma_\nu^{(2)}((s-1)/2) + (1 + e^{i\pi s}) \Gamma_\nu^{(1)}(s/2) \right).$$

In Section 4 we prove that the function $\Gamma_\nu^{(1)}$ interpolates the odd terms of the Dunkl factorial

$$\Gamma_\nu^{(1)}(n) = \gamma_{2n-1, \nu}, \quad n = 1, 2, \dots,$$

it satisfies the functional equation

$$\Gamma_\nu^{(1)}(s+1) = 4s(s+1+\nu) \Gamma_\nu^{(1)}(s),$$

and it can be characterized by a Bohr-Mollerup type theorem. We also prove similar results for the $\Gamma_\nu^{(2)}$ function.

The gamma function is the only function defined in $(0, +\infty)$ interpolating the factorial sequence $n!$, satisfying the functional equation $f(s+1) = sf(s)$, $f(0) = 1$ and being log-convex (there are several more characterizations for the gamma function, see [23, 25–27]). Taking all this into account, the identity (11), the properties of the functions $\Gamma_\nu^{(1)}$ and $\Gamma_\nu^{(2)}$ and the key role of the measure w_ν (5) allow us to consider the function Γ_ν (8) a quite suitable extension of the gamma function to the Dunkl context.

1. PRELIMINARIES: THE FUNCTION $K_\mu(t)$

We include here some results on the function K_μ that we will need in the next sections.

By definition (see [18, § 3.70, (6), p. 78]) the modified Bessel function of the second kind is

$$(12) \quad K_\mu(t) = \frac{\pi}{2} \frac{I_{-\mu}(t) - I_\mu(t)}{\sin(\mu\pi)}, \quad \mu \in \mathbb{R} \setminus \mathbb{Z}$$

(this can be extended to $\mu \in \mathbb{Z}$ by taking limits in a standard way), where $I_\mu(t)$ are the so-called modified Bessel function (of the first kind), namely,

$$(13) \quad I_\mu(t) = i^{-\mu} J_\mu(it) = \sum_{k=0}^{\infty} \frac{t^{2k+\mu}}{2^{2k+\mu} k! \Gamma(\mu+k+1)}.$$

In particular, this implies that $K_{-\mu}(t) = K_\mu(t)$.

From (12) and (13) we have, for $\mu \in \mathbb{R} \setminus \mathbb{Z}$,

$$K_\mu(t) \sim \frac{\pi}{2 \sin(\mu)} \left(\frac{t^{-\mu}}{2^{-\mu} \Gamma(1-\mu)} - \frac{t^\mu}{2^\mu \Gamma(1+\mu)} \right), \quad t \rightarrow 0^+$$

(the significant term is the first if $\mu > 0$ and the second if $\mu < 0$; in both cases, the other term is negligible). Then, except by a multiplicative constant,

$$(14) \quad K_\mu(t) \sim t^{-|\mu|}, \quad t \rightarrow 0^+.$$

We also have (see [18, § 7.23, (1), p. 202])

$$(15) \quad K_\mu(t) \sim \left(\frac{\pi}{2t} \right)^{1/2} e^{-t}, \quad t \rightarrow \infty.$$

Moreover, by using [28, § 2.16.2, (2), p. 343] we have

$$(16) \quad \int_0^\infty t^{r-1} K_\mu(t) dt = 2^{r-2} \Gamma\left(\frac{r+\mu}{2}\right) \Gamma\left(\frac{r-\mu}{2}\right), \quad \operatorname{Re}(r) > |\operatorname{Re}(\mu)|.$$

2. THE FUNCTION $\Gamma_\nu(s)$

As we wrote in the introduction, we define the function $\Gamma_\nu(s)$, for $\nu > -1$ and $\operatorname{Re}(s) > 0$, using the integral representation (8).

We next find a more convenient expression of the function $\Gamma_\nu(s)$ which will allow us to extend it to the whole complex plane (with isolated poles), and also when ν is a complex number.

In order to do that, we introduce the functions $\gamma_{\text{even},\nu}(s)$ and $\gamma_{\text{odd},\nu}(s)$ defined by

$$(17) \quad \gamma_{\text{even},\nu}(s) = 2^s \Gamma\left(\frac{s}{2} + 1\right) \frac{\Gamma\left(\frac{s+2\nu+2}{2}\right)}{\Gamma(\nu+1)}$$

and

$$(18) \quad \gamma_{\text{odd},\nu}(s) = 2^s \Gamma\left(\frac{s+1}{2}\right) \frac{\Gamma\left(\frac{s+2\nu+3}{2}\right)}{\Gamma(\nu+1)}.$$

We have the following:

Lemma 1. *Assume that the complex number s satisfies*

$$\operatorname{Re}(s) > \begin{cases} -2\nu - 1, & \text{if } -1 < \nu < -1/2, \\ 0, & \text{if } \nu \geq -1/2. \end{cases}$$

Then the function $\Gamma_\nu(s)$ (defined in (8) for $\nu > -1$) can be written as

$$(19) \quad \Gamma_\nu(s) = \frac{1}{2} \left((1 - (-1)^s) \gamma_{\text{even},\nu}(s-1) + (1 + (-1)^s) \gamma_{\text{odd},\nu}(s-1) \right),$$

where we are setting $(-1)^s = e^{i\pi s}$.

Proof. We decompose $\Gamma_\nu(s)$ in the following way:

$$\begin{aligned} \Gamma_\nu(s) &= \int_0^\infty t^{s-1} \frac{t^{\nu+1}}{2^{\nu+1} \Gamma(\nu+1)} (K_\nu(t) + K_{\nu+1}(t)) dt \\ &\quad + \int_{-\infty}^0 t^{s-1} \frac{(-t)^{\nu+1}}{2^{\nu+1} \Gamma(\nu+1)} (K_\nu(-t) - K_{\nu+1}(-t)) dt =: I_1(s) + I_2(s). \end{aligned}$$

Moreover, with the change $t \mapsto -t$, we can write

$$I_2(s) = \int_0^\infty (-1)^{s-1} t^{s-1} \frac{t^{\nu+1}}{2^{\nu+1} \Gamma(\nu+1)} (K_\nu(t) - K_{\nu+1}(t)) dt.$$

By using (16), we have

$$\int_0^\infty t^{s+\nu} K_\nu(t) dt = 2^{s+\nu-1} \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s+2\nu+1}{2}\right) = 2^\nu \Gamma(\nu+1) \gamma_{\text{even},\nu}(s-1)$$

when $\operatorname{Re}(s) + \nu + 1 > |\nu|$, and

$$\int_0^\infty t^{s+\nu} K_{\nu+1}(t) dt = 2^{s+\nu-1} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+2\nu+2}{2}\right) = 2^\nu \Gamma(\nu+1) \gamma_{\text{odd},\nu}(s-1)$$

when $\operatorname{Re}(s) + \nu + 1 > |\nu+1| = \nu+1$. This inequality is equivalent to $\operatorname{Re}(s) > 0$. In turn, the inequality $\operatorname{Re}(s) + \nu + 1 > |\nu|$ above adds the request $\operatorname{Re}(s) > -2\nu - 1$. Then,

$$\begin{aligned} I_1(s) &= \frac{1}{2} \gamma_{\text{even},\nu}(s-1) + \frac{1}{2} \gamma_{\text{odd},\nu}(s-1), \\ I_2(s) &= (-1)^{s-1} \left(\frac{1}{2} \gamma_{\text{even},\nu}(s-1) - \frac{1}{2} \gamma_{\text{even},\nu}(s-1) \right), \end{aligned}$$

and so we have (19). $\square$

The lemma allows us to extend the function $\Gamma_\nu(s)$ to the complex plane (with isolated poles) and for complex values of ν . We will study the poles of the function Γ_ν in the next section.

Definition 2. For $\nu \in \mathbb{C} \setminus \{-1, -2, \dots\}$ and $s \in \mathbb{C}$, we define $\Gamma_\nu(s)$ explicitly by the expression given in (9).

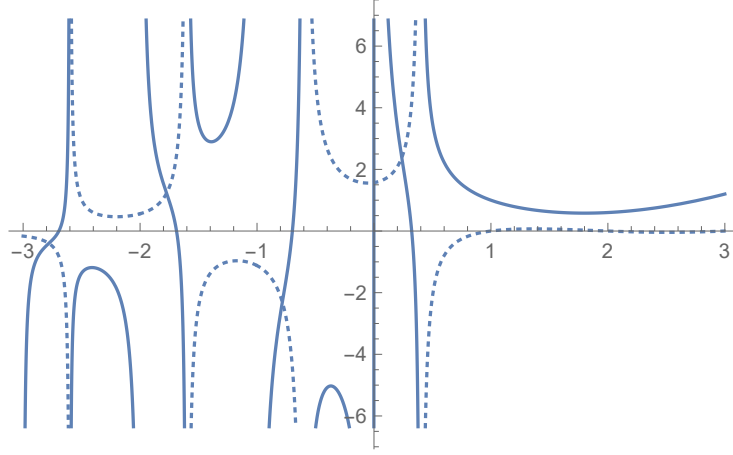


FIGURE 2. Real and imaginary parts (solid line and dashed line, respectively) of (9) for $\nu = -7/10$ and $-3 \leq s \leq 3$.

The identity (10) for Γ_ν follows easily from (9) by using the duplication formula $\Gamma(z)\Gamma(z + 1/2) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$ (with $z = s/2$).

For $\nu = -1/2$, we recover the gamma function $\Gamma_{-1/2}(s) = \Gamma(s)$ (the easier way to check it is to take $\nu = -1/2$ in (10) and use $\Gamma(1/2) = \sqrt{\pi}$). As explained in the introduction, for $\nu > -1/2$ the function $\Gamma_\nu(s)$ is not real for $s \in \mathbb{R}$. We can see it in Figure 2 for the particular value $\nu = -7/10$.

Because of the definition (1), and analogously to how the iterative function of the Collatz problem can be extended to the complex plane, there are many ways to make an extension of a function that has an expression for even n and a different expression for odd n . The Collatz problem uses a function $T : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ defined as $T(n) = n/2$ for even n and $T(n) = 3n + 1$ for odd n , and the Collatz conjecture says that the iteration of $n \mapsto T(n)$ always arrives at 1. The original iterative approach is not related to our problem. However, the extension of T to the complex plane has also been considered, and for them different ideas have been used, as can be seen in [29, 30]; this is what we want to do with $\gamma_{n,\nu}$. In our case, to interpolate the Dunkl factorial $\gamma_{n,\nu}$, one could have taken any function of the form

$$G_\nu(s) = \frac{\sqrt{\pi}}{\Gamma(\nu+1)}\Gamma(s) \left(g_1(s) \frac{\Gamma\left(\frac{s+2\nu+1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} + g_2(s) \frac{\Gamma\left(\frac{s+2\nu+2}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} \right),$$

where $g_1(s)$ and $g_2(s)$ are entire functions that satisfy

$$g_1(s) = \begin{cases} 0, & \text{for } s = k \in \mathbb{N}, \text{ even,} \\ 1, & \text{for } s = k \in \mathbb{N}, \text{ odd,} \end{cases} \quad g_2(s) = \begin{cases} 1, & \text{for } s = k \in \mathbb{N}, \text{ even,} \\ 0, & \text{for } s = k \in \mathbb{N}, \text{ odd.} \end{cases}$$

This happens, for instance, with $g_1(s) = \sin^2(s\pi/2)$ and $g_2(s) = \cos^2(s\pi/2)$. However, due to the role played for the measure (5) in the construction of the function Γ_ν , we have considered that the choice $g_1(s) = (1 - e^{i\pi s})/2$ and $g_2(s) = (1 + e^{i\pi s})/2$, that gives (9), is the best one in this scenario.

The rest of this section will be devoted to finding some more properties and identities for the function $\Gamma_\nu(s)$.

Starting from (9) and applying $i \cot((\pi s)/2) = (1 + e^{i\pi s})/(1 - e^{i\pi s})$, we get the following:

$$\begin{aligned} \Gamma_\nu(s) &= \frac{2^{s-2}}{\Gamma(\nu+1)} \left((1 - e^{i\pi s}) \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s+2\nu+1}{2}\right) + (1 + e^{i\pi s}) \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+2\nu+2}{2}\right) \right) \\ &= \frac{2^{s-2}(1 - e^{i\pi s})}{\Gamma(\nu+1)} \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s+2\nu+1}{2}\right) \left(1 + i \cot\left(\frac{\pi s}{2}\right) \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} \frac{\Gamma\left(\frac{s+2\nu+2}{2}\right)}{\Gamma\left(\frac{s+2\nu+1}{2}\right)} \right). \end{aligned}$$

Then, if we denote

$$(20) \quad \text{dpc}_\nu(s) = \frac{\Gamma(s/2)\Gamma((s+2\nu+2)/2)}{\Gamma((s+1)/2)\Gamma((s+2\nu+1)/2)},$$

we can write

$$\Gamma_\nu(s) = \frac{2^{s-2}(1 - e^{i\pi s})}{\Gamma(\nu+1)} \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s+2\nu+1}{2}\right) \left(1 + i \cot\left(\frac{\pi s}{2}\right) \text{dpc}_\nu(s)\right).$$

Remark 1. Observe that $\text{dpc}_\nu(s)$ is a rational function if $\nu = -1/2 + k$ with $k \in \mathbb{Z}$ (actually, a quotient of two polynomials in s of degree $|k|$). For $k = 0$, we have $\text{dpc}_{-1/2}(s) = 1$. For $k \neq 0$,

$$\text{dpc}_{k-1/2}(s) = \frac{(s+1)(s+3) \cdots (s+2k-1)}{s(s+2) \cdots (s+2k-2)}, \quad k = 1, 2, 3, \dots$$

and

$$\text{dpc}_{k-1/2}(s) = \frac{(s-2)(s-4) \cdots (s+2k)}{(s-1)(s-3) \cdots (s+2k+1)}, \quad k = -1, -2, -3, \dots$$

Remark 2. In the case of (10), and in a similar way, when $\nu = -1/2 + k$ with $k \in \mathbb{Z}$, by using $\Gamma(z+1) = z\Gamma(z)$ we easily get that

$$\frac{\Gamma\left(\frac{s+2\nu+1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} = \begin{cases} 1, & \text{if } k = 0, \\ 2^{-k}s(s+2) \cdots (s+2k-2), & \text{if } k = 1, 2, 3, \dots, \\ 2^{-k}/((s-2)(s-4) \cdots (s+2k)), & \text{if } k = -1, -2, -3, \dots, \end{cases}$$

and

$$\frac{\Gamma\left(\frac{s+2\nu+2}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} = \begin{cases} 1, & \text{if } k = 0, \\ 2^{-k}(s+1)(s+3) \cdots (s+2k-1), & \text{if } k = 1, 2, 3, \dots, \\ 2^{-k}/((s-1)(s-3) \cdots (s+2k+1)), & \text{if } k = -1, -2, -3, \dots \end{cases}$$

The functions $\gamma_{\text{even},\nu}(s)$ and $\gamma_{\text{odd},\nu}(s)$ defined in (17) and (18) satisfy a number of identities. The more basic ones are the following:

$$(21) \quad \gamma_{\text{even},\nu}(s) = s\gamma_{\text{odd},\nu}(s-1),$$

$$(22) \quad \gamma_{\text{odd},\nu}(s) = (s+2\nu+1)\gamma_{\text{even},\nu}(s-1).$$

But there are many other relations that can be easily checked from the definition using the reflection formula and/or the duplication formula for the gamma function:

$$\begin{aligned} \gamma_{\text{even},\nu}(s)\gamma_{\text{even},\nu}(-s) &= \pi s \csc\left(\frac{\pi s}{2}\right) \frac{\Gamma\left(\frac{s+2\nu+2}{2}\right) \Gamma\left(\frac{-s+2\nu+2}{2}\right)}{2\Gamma(\nu+1)^2}, \\ \gamma_{\text{odd},\nu}(s)\gamma_{\text{odd},\nu}(-s) &= \pi \sec\left(\frac{\pi s}{2}\right) \frac{\Gamma\left(\frac{s+2\nu+3}{2}\right) \Gamma\left(\frac{-s+2\nu+3}{2}\right)}{\Gamma(\nu+1)^2}, \end{aligned}$$

as well as expressions for products of $\gamma_{\text{even},\nu}(s-1)$ times $\gamma_{\text{even},\nu}(1-s)$, and $\gamma_{\text{even},\nu}(s)$ times $\gamma_{\text{even},\nu}(s+1)$ (and with “odd” in the place of “even”).

We also display some other identities which seem to be of a different nature from the previous ones:

$$\begin{aligned} \gamma_{\text{even},\nu}(s)\gamma_{\text{even},-1-\nu}(-s) &= \frac{\pi s \sin(\pi\nu)}{\cos(\pi\nu) - \cos(\pi(s+\nu))}, \\ \gamma_{\text{odd},\nu}(s)\gamma_{\text{odd},-1-\nu}(-s) &= -\frac{\pi(s+2\nu+1) \sin(\pi\nu)}{\cos(\pi\nu) + \cos(\pi(s+\nu))}; \end{aligned}$$

similar expressions can be found for $\gamma_{\text{even},\nu}(s)\gamma_{\text{odd},-1-\nu}(1-s)$ and $\gamma_{\text{odd},\nu}(s)\gamma_{\text{even},-1-\nu}(1-s)$.

In the next theorem, we extend the functional equation $\Gamma(s+1) = s\Gamma(s)$ to the function Γ_ν .

Theorem 3. *The function Γ_ν defined in (9) satisfies the functional equation*

$$(23) \quad \Gamma_\nu(s+1) = \theta_\nu(s)\Gamma_\nu(s),$$

with

$$(24) \quad \theta_\nu(s) = s + (2\nu+1) \left(\frac{\gamma_{\text{even},\nu}(s-1)}{\gamma_{\text{even},\nu}(s-1) + i \cot\left(\frac{\pi s}{2}\right) \gamma_{\text{odd},\nu}(s-1)} \right).$$

Moreover, when $s = n \in \mathbb{Z}^+$ and with the notation of (2), we have $\theta_\nu(n) = \theta_{n,\nu} = n + (1 + (-1)^n)(\nu + 1/2)$, so the function $\theta_\nu(s)$ interpolates $\{\theta_{n,\nu}\}_{n=1}^\infty$ (just as the function $\Gamma_\nu(s)$ interpolates the Dunkl factorial).

Proof. Using (9), (21) and (22), we get

$$\begin{aligned}\Gamma_\nu(s+1) &= \frac{1}{2} \left((1 + e^{i\pi s})\gamma_{\text{even},\nu}(s) + (1 - e^{i\pi s})\gamma_{\text{odd},\nu}(s) \right) \\ &= \frac{1}{2} \left((1 + e^{i\pi s})s\gamma_{\text{odd},\nu}(s) + (1 - e^{i\pi s})(s + 2\nu + 1)\gamma_{\text{even},\nu}(s-1) \right) \\ &= \frac{s}{2} \left((1 - e^{i\pi s})\gamma_{\text{even},\nu}(s-1) + (1 + e^{i\pi s})\gamma_{\text{odd},\nu}(s-1) \right) \\ &\quad + \frac{1}{2}(2\nu + 1)(1 - e^{i\pi s})\gamma_{\text{even},\nu}(s-1) \\ &= s\Gamma_\nu(s) + (\nu + 1/2)(1 - e^{i\pi s})\gamma_{\text{even},\nu}(s-1).\end{aligned}$$

Then, to find (23) it is enough to take

$$\begin{aligned}\theta_\nu(s) &= \frac{\Gamma_\nu(s+1)}{\Gamma_\nu(s)} = \frac{s\Gamma_\nu(s) + (\nu + 1/2)(1 - e^{i\pi s})\gamma_{\text{even},\nu}(s-1)}{\Gamma_\nu(s)} \\ &= s + (\nu + 1/2) \frac{(1 - e^{i\pi s})\gamma_{\text{even},\nu}(s-1)}{\Gamma_\nu(s)}.\end{aligned}$$

From this expression, multiplying the numerator and the denominator by $1/(1 - e^{i\pi s})$, using (9), and applying $i \cot((\pi s)/2) = (1 + e^{i\pi s})/(1 - e^{i\pi s})$, we obtain (24). Finally, it is a simple exercise to check that $\theta_\nu(n) = \theta_{n,\nu} = n + (1 + (-1)^n)(\nu + 1/2)$ when $n \in \mathbb{Z}^+$. $\square$

Remark 3. An alternative way to express (24) is

$$\theta_\nu(s) = s + (2\nu + 1) \left(\frac{1}{1 + i \cot\left(\frac{\pi s}{2}\right) \text{dpc}_\nu(s)} \right),$$

where the function $\text{dpc}_\nu(s)$ is defined in (20). Recall that $\text{dpc}_\nu(s)$ is a rational function if $\nu = -1/2 + k$ with $k \in \mathbb{Z}$ (see Remark 1).

3. POLES AND ZEROS OF $\Gamma_\nu(s)$

Recall that the function $\Gamma(s)$ does not have zeros, and the singularities are $s = 0, -1, -2, \dots$ (poles of order 1). In this section, we study the poles and the zeros of $\Gamma_\nu(s)$. In Figure 3, following the ideas of [31], we present a drawing, on the complex plane, of the lines $\text{Re}(\Gamma_\nu(s)) = 0$ and $\text{Im}(\Gamma_\nu(s)) = 0$ for $\nu = 6/5$. The poles and the zeros of $\Gamma_\nu(s)$ are in the intersection of these lines.

We start studying the poles of Γ_ν . In what follows, we are using $\mathbb{N} = \mathbb{Z}^+ \cup \{0\}$.

Theorem 4. *Let $\nu \in \mathbb{C} \setminus \{-1, -2, -3, \dots\}$. Then the poles of $\Gamma_\nu(s)$ are the following:*

- (i) *The non-positive integers $s = -m$, with $m \in \mathbb{N}$, are simple poles, except when $\nu = k \in \mathbb{N}$, in which case the poles at $s = -m - 2k$, with $m \in \mathbb{Z}^+$, have order 2.*
- (ii) *When $\nu \neq k - 1/2$, $k \in \mathbb{Z}$, the numbers $s = -m - (2\nu + 1)$, with $m \in \mathbb{N}$, are also simple poles.*

Proof. Recall that the function $\Gamma(s)$ does not have zeros, and the singularities are $s = 0, -1, -2, \dots$ (poles of order 1). From (9), the possible poles of the function $\Gamma_\nu(s)$ are the poles of $\gamma_{\text{even},\nu}(s-1)$ and $\gamma_{\text{odd},\nu}(s-1)$; i.e., the poles of $\Gamma\left(\frac{s+1}{2}\right)$, $\Gamma\left(\frac{s}{2}\right)$, $\Gamma\left(\frac{s+2\nu+1}{2}\right)$ and $\Gamma\left(\frac{s+2\nu+2}{2}\right)$. More precisely, they are (for $n = 0, 1, 2, \dots$):

$$(25) \quad \frac{s+1}{2} = -n \Rightarrow s = -2n - 1,$$

$$(26) \quad \frac{s}{2} = -n \Rightarrow s = -2n,$$

$$(27) \quad \frac{s+2\nu+1}{2} = -n \Rightarrow s = -2n - 1 - 2\nu,$$

$$(28) \quad \frac{s+2\nu+2}{2} = -n \Rightarrow s = -2n - 2 - 2\nu.$$

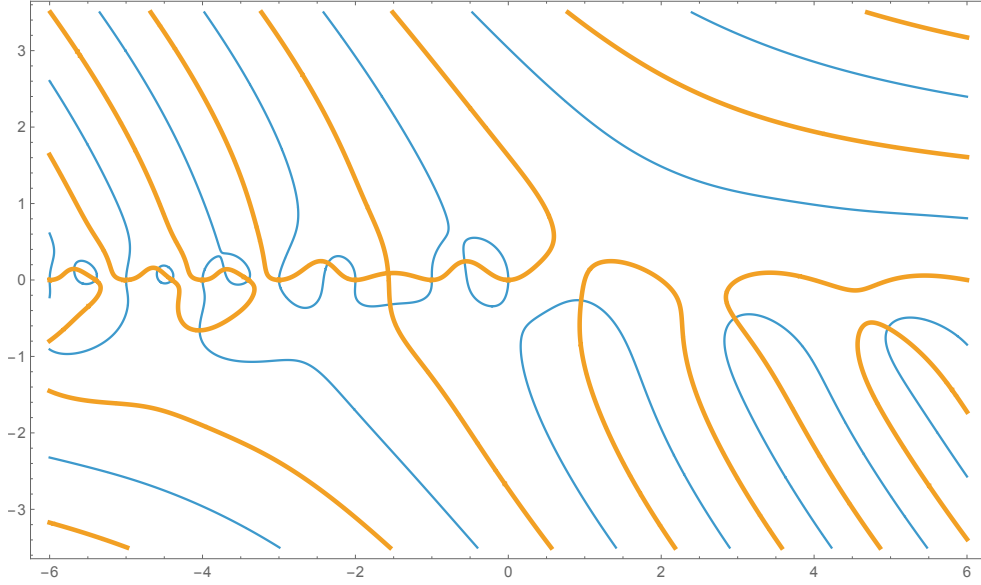


FIGURE 3. The lines $\operatorname{Re}(\Gamma_\nu(s)) = 0$ (blue) and $\operatorname{Im}(\Gamma_\nu(s)) = 0$ (orange, thicker) for $\nu = 6/5$ on a rectangle of the complex plane.

Consequently, $\Gamma_\nu(s)$ has possible poles at $s = -m$ and $s = -m - (2\nu + 1)$, with $m = 0, 1, 2, \dots$. Then, apart from the poles of the function $\Gamma(s)$, there are also possible poles at $s = -m - (2\nu + 1)$.

In order to decide if a “possible pole” (25)–(28) of $\Gamma_\nu(s)$ is a pole, we must check if it is canceled by a zero of $(1 - e^{i\pi s})$ (i.e.: $s = 2j$, $j \in \mathbb{Z}$) in the case of (25) and (27), or the zeros of $(1 + e^{i\pi s})$ (i.e.: $s = 2j - 1$, $j \in \mathbb{Z}$) in the case of (26) and (28).

In the case of (25) and (26), this cancellation never happens, so we have (i). In the case of (27), the cancellation occurs when $-2n - 1 - 2\nu = 2j$, i.e., $\nu = -n - j - 1/2$ (a half-integer). With more details, when $\nu = k - 1/2$, $k \in \mathbb{Z}$, the values $s = -2n - 2k$ of (27) are not poles of the first summand of (9) for $n = 0, 1, 2, \dots$; but, when $-2n - 2k \leq 0$, they are poles of the second summand (by (26)). In the case of (28), the cancellation occurs when $-2n - 2 - 2\nu = 2j - 1$, again $\nu = -n - j - 1/2$. More precisely, when $\nu = k - 1/2$, $k \in \mathbb{Z}$, the values $s = -2n - 2k - 1$ of (28) are not poles of the second summand of (9) for $n = 0, 1, 2, \dots$; but, when $-2n - 2k - 1 \leq 0$, they are poles of the first summand (by (25)). This, together with the following observation, again gives (i) in the case $\nu = k - 1/2$, $k \in \mathbb{Z}$.

Observe that another way in which a possible pole would not be a pole would occur if that pole were in the two summands of (9), and canceled when summing. But this does not happen in our case. If in one of the factors of the first summand there is a pole for a certain s , and also in another of those of the second summand, then $1 - e^{i\pi s} = 0$ or $1 + e^{i\pi s} = 0$, and when multiplying the pole of the corresponding summand is canceled with the zero. So the possible poles never cancel when adding two of them.

Finally, note that some of the possible poles (25)–(28) may actually be poles of order 2; this occurs if they are poles of the two factors of one of the summands of (9).

In the first summand, for the two factors to have poles (i.e., for (25) and (27) to be satisfied, although for different values of $n \in \mathbb{N}$), it must be $\nu = k \in \mathbb{N}$ (recall that $\nu \notin \mathbb{Z}^-$). In this case, there will be a pole of order 2 at $s = -2n - 1 - 2k$, with $n \in \mathbb{N}$. In the second summand, for the two factors to have poles (i.e., for (26) and (28) to be satisfied), it must again be $\nu = k \in \mathbb{N}$. In this case, there will be a pole of order 2 at $s = -2n - 2 - 2k$, with $n \in \mathbb{N}$. This completes (i) for $\nu = k \in \mathbb{N}$. $\square$

Regarding the zeros, what happens with $\Gamma_\nu(s)$ is completely different from what happens with $\Gamma(s)$. Even though $\Gamma(s)$ has no zeros, we next prove that the function $\Gamma_\nu(s)$ has always an infinite number of complex zeros except, precisely, when $\nu = -1/2$, i.e., the gamma function (for numerical purposes, some of these zeros can be located using FindRoot of Mathematica).

Theorem 5. *If $\nu \in \mathbb{C} \setminus \{-1, -2, \dots\}$ and $\nu \neq -1/2$, the function Γ_ν has an infinite number of zeros in $\mathbb{C}$.*

Proof. Consider the entire functions defined by

$$A_\nu(s) = \frac{1}{\Gamma(s/2)\Gamma(s/2 + \nu + 1)}, \quad B_\nu(s) = \frac{1}{\Gamma(s/2 + 1/2)\Gamma(s/2 + \nu + 1/2)}.$$

We can rewrite (10) in the form

$$\Gamma_\nu(s) = \frac{\sqrt{\pi}}{2\Gamma(\nu + 1)} \Gamma(s)\Gamma(s/2 + \nu + 1/2)\Gamma(s/2 + \nu + 1) \left(A_\nu(s) + B_\nu(s) + (B_\nu(s) - A_\nu(s))e^{i\pi s} \right).$$

Define, finally,

$$(29) \quad \Phi_\nu(s) = A_\nu(s) + B_\nu(s) + (B_\nu(s) - A_\nu(s))e^{i\pi s}.$$

Assume first that $\nu \neq k - 1/2$, $k \in \mathbb{Z}$. Then the poles of

$$(30) \quad \Gamma(s)\Gamma(s/2 + \nu + 1/2)\Gamma(s/2 + \nu + 1)$$

are either zeros of $A_\nu(s)$ or zeros of $B_\nu(s)$. Moreover, if they are zeros of $A_\nu(s)$, they are not zeros of $(1 + e^{i\pi s})B_\nu(s)$; and, if they are zeros of $B_\nu(s)$, they are not zeros of $(1 - e^{i\pi s})A_\nu(s)$. Then, the poles of the product (30) (which are the poles of Γ_ν , see the previous theorem) can not be zeros of $\Phi_\nu(s)$. Hence we deduce that the zeros of Γ_ν and Φ_ν are the same, and so Γ_ν has a finite number of zeros in $\mathbb{C}$ if and only if the entire function Φ_ν has a finite number of zeros in $\mathbb{C}$. We next proceed by *reduction ad absurdum*, assuming that Φ_ν has a finite number of complex zeros.

Since the entire functions A_ν , B_ν and $e^{i\pi s}$ have order 1, then Φ_ν has order at most 1. Since we are assuming that it has a finite number of complex zeros, we deduce that

$$(31) \quad \Phi_\nu(s) = P(s)e^{as+b},$$

where P is a polynomial and $a, b \in \mathbb{C}$.

Using that, for fixed $u > 0$ and $v \in \mathbb{C}$ (see [19, formula 5.11.7]),

$$\lim_{x \rightarrow +\infty} \frac{\Gamma(ux + v)}{\sqrt{2\pi}e^{-ux}(ux)^{ux+v-1/2}} = 1, \quad x \in \mathbb{R},$$

we conclude that

$$\lim_{x \rightarrow +\infty} \frac{A_\nu(x)}{\varphi_\nu(x)} = 1, \quad \lim_{x \rightarrow +\infty} \frac{B_\nu(x)}{\varphi_\nu(x)} = 1, \quad x \in \mathbb{R},$$

where

$$\varphi_\nu(x) = \frac{e^x(x/2)^{-x-\nu}}{2\pi}.$$

On the one hand, using (29) we have

$$\lim_{x \rightarrow +\infty} \frac{\Phi_\nu(x)}{\varphi_\nu(x)} = 2.$$

But, on the other, using (31) we have

$$\lim_{x \rightarrow +\infty} \left| \frac{\Phi_\nu(x)}{\varphi_\nu(x)} \right| = +\infty,$$

which is a contradiction. Hence, the function Γ_ν must have an infinite number of complex zeros.

If $\nu = k - 1/2$, $k \in \mathbb{Z} \setminus \{0\}$, we can write

$$A_\nu(s) = \frac{\frac{s}{2} \cdots (\frac{s}{2} + k - 1)}{\Gamma(\frac{s}{2} + k)\Gamma(\frac{s}{2} + k + \frac{1}{2})}, \quad B_\nu(s) = \frac{(\frac{s}{2} + \frac{1}{2}) \cdots (\frac{s}{2} + k - \frac{1}{2})}{\Gamma(\frac{s}{2} + k)\Gamma(\frac{s}{2} + k + \frac{1}{2})}.$$

Hence, by setting

$$P(s) = \frac{s}{2} \cdots \left(\frac{s}{2} + k - 1 \right), \quad Q(s) = \left(\frac{s}{2} + \frac{1}{2} \right) \cdots \left(\frac{s}{2} + k - \frac{1}{2} \right),$$

we have

$$\Gamma_\nu(s) = \frac{\sqrt{\pi}}{2\Gamma(\nu + 1)} \Gamma(s) \left(P(s) + Q(s) + (Q(s) - P(s))e^{i\pi s} \right).$$

Proceeding similarly as before, we can prove that also for $\nu = k - 1/2$, $k \in \mathbb{Z} \setminus \{0\}$, the function Γ_ν must have an infinite number of complex zeros. $\square$

4. BOHR-MOLLERUP TYPE THEOREM FOR A DECOMPOSITION OF THE GAMMA-DUNKL FUNCTION

In this section we prove some interesting results for the functions $\gamma_{\text{even},\nu}(s)$ (17) and $\gamma_{\text{odd},\nu}(s)$ (18), including a Bohr-Mollerup type theorem.

In order to simplify the expression, we consider the functions

$$(32) \quad \Gamma_\nu^{(1)}(s) = \gamma_{\text{odd},\nu}(2s-1) = 2^{2s-1} \Gamma(s) \frac{\Gamma(s+\nu+1)}{\Gamma(\nu+1)},$$

$$(33) \quad \Gamma_\nu^{(2)}(s) = \gamma_{\text{even},\nu}(2s) = 2^{2s} \Gamma(s+1) \frac{\Gamma(s+\nu+1)}{\Gamma(\nu+1)} = 2s \Gamma_\nu^{(1)}(s).$$

It is easy to check that

$$\Gamma_{-1/2}^{(1)}(s) = \gamma_{\text{odd},-1/2}(2s-1) = 2^{2s-1} \Gamma(s) \frac{\Gamma(s+1/2)}{\Gamma(1/2)} = \Gamma(2s)$$

and

$$\begin{aligned} \Gamma_{-1/2}^{(2)}(s) &= \gamma_{\text{even},-1/2}(2s) = 2^{2s} \Gamma(s+1) \frac{\Gamma(s+1/2)}{\Gamma(1/2)} \\ &= (2s) 2^{2s-1} \Gamma(s) \frac{\Gamma(s+1/2)}{\Gamma(1/2)} = (2s) \Gamma(2s) = \Gamma(2s+1). \end{aligned}$$

This shows that both functions $\Gamma_\nu^{(1)}(\frac{s}{2})$ and $\Gamma_\nu^{(2)}(\frac{s-1}{2})$ extend the $\Gamma(s)$ to the Dunkl context. However, each one of them only interpolates half of the Dunkl factorials. More precisely $\Gamma_\nu^{(1)}(n) = \gamma_{2n-1,\nu}$ and $\Gamma_\nu^{(2)}(n) = \gamma_{2n,\nu}$, $n = 1, 2, \dots$

The functions $\Gamma_\nu^{(1)}$ and $\Gamma_\nu^{(2)}$ satisfy very simple functional equations:

$$\begin{aligned} \Gamma_\nu^{(1)}(s+1) &= \theta_\nu^{(1)}(s) \Gamma_\nu^{(1)}(s) = 4s(s+1+\nu) \Gamma_\nu^{(1)}(s), \\ \Gamma_\nu^{(2)}(s+1) &= \theta_\nu^{(2)}(s) \Gamma_\nu^{(2)}(s) = 4(s+1)(s+1+\nu) \Gamma_\nu^{(2)}(s) \end{aligned}$$

(they follow easily using (32) and (33)). These expressions are, essentially, a generalization of the functional equation for the gamma function, $\Gamma(s+1) = s\Gamma(s)$: in both cases, with $\nu = -1/2$ and the change $s \mapsto s/2$, we recover the iterated functional equation $\Gamma(s+2) = (s+1)s\Gamma(s)$.

The following theorem is a Bohr-Mollerup type theorem for $\Gamma_\nu^{(1)}(s)$ (the proof is based in the Bohr-Mollerup theorem for the gamma function; see, for example, [25, Theorem 2.1]).

Theorem 6. *Let $\nu > -1$. There exists an unique function $f_\nu(x)$ defined on the interval $(0, \infty)$ such that*

- $f_\nu(1) = 2(\nu+1)$,
- $f_\nu(s+1) = 4s(s+1+\nu)f_\nu(s)$,
- the logarithm of $f_\nu(s)$ is a convex function.

Furthermore, such a function is $\Gamma_\nu^{(1)}(s)$.

Proof. It is obvious that $\Gamma_\nu^{(1)}(s)$ verifies the first two conditions (the first one is due to $\gamma_{1,\nu} = 2(\nu+1)$). Checking that $\Gamma_\nu^{(1)}(s)$ is log-convex is a consequence of [25, Theorem 1.9], since

$$\begin{aligned} \Gamma_\nu^{(1)}(s) &= \gamma_{\text{odd},\nu}(2s-1) = \frac{1}{2^\nu \Gamma(\nu+1)} \int_0^\infty K_{\nu+1}(t) t^{\nu+1} t^{2s-1} dt \\ &= \frac{1}{2^{\nu+1} \Gamma(\nu+1)} \int_0^\infty K_{\nu+1}(\sqrt{u}) u^{(\nu+1)/2} u^{s-1} du \end{aligned}$$

and the weight $K_{\nu+1}(\sqrt{u}) u^{(\nu+1)/2}$ is positive for $u > 0$. Notice that, by (14) and (15) (and recalling $\nu > -1$), the integral converges for $s > 0$.

We next prove that the function $f_\nu(s)$ that satisfies the three conditions is unique. To do that, it is enough to prove the uniqueness in the interval $(0, 1)$, since $f_\nu(s+1)$ is completely determined

by $f_\nu(s)$. Moreover, the proof would follow if we show that any function f_ν that satisfies the three conditions also satisfies

$$(34) \quad f_\nu(s) = \lim_{n \rightarrow \infty} \frac{(4n(\nu + n + 1))^s n! (\nu + 1)_{n+1}}{2(s)_{n+1}(s + 1 + \nu)_{n+1}}, \quad 0 < s \leq 1.$$

Notice that the first two conditions imply that $f_\nu(s)$ interpolates $\gamma_{2n+1,\nu}$, that is

$$\begin{aligned} f_\nu(1) &= 2(\nu + 1) = \gamma_{1,\nu}, \\ f_\nu(2) &= 2 \cdot 4 \cdot 1 \cdot (2 + \nu)(\nu + 1) = \gamma_{3,\nu}, \\ f_\nu(3) &= 2 \cdot 4^2 \cdot 1 \cdot 2 \cdot (3 + \nu)(2 + \nu)(\nu + 1) = \gamma_{5,\nu}, \\ &\vdots = \vdots \\ f_\nu(n+1) &= 2^{2n+1} n! (\nu + 1)_{n+1} = \gamma_{2(n+1)-1,\nu} = \gamma_{2n+1,\nu}. \end{aligned}$$

We also have, from the second condition,

$$(35) \quad f_\nu(s+n) = \theta_\nu^{(1)}(s+n-1) f_\nu(s+n-1) = \cdots = \theta_\nu^{(1)}(s+n-1) \cdots \theta_\nu^{(1)}(s) f_\nu(s).$$

Then, if we define

$$\Theta_{n,\nu}^{(1)}(s) := \theta_\nu^{(1)}(s+n-1) \cdots \theta_\nu^{(1)}(s) = 4^n(s)_n(s+1+\nu)_n,$$

we can write

$$f_\nu(s+n) = \Theta_{n,\nu}^{(1)}(s) f_\nu(s) = 4^n(s)_n(s+1+\nu)_n f_\nu(s).$$

Then, let $f_\nu(s)$ be a function that satisfies the three conditions. Since $f_\nu(s)$ is log-convex, for $0 < s \leq 1$ and $n \geq 2$, by definition we get

$$\frac{\log f_\nu(-1+n) - \log f_\nu(n)}{(-1+n) - n} \leq \frac{\log f_\nu(s+n) - \log f_\nu(n)}{(s+n) - n} \leq \frac{\log f_\nu(1+n) - \log f_\nu(n)}{(1+n) - n},$$

i.e.,

$$-\log \gamma_{2n-3,\nu} + \log \gamma_{2n-1,\nu} \leq \frac{\log f_\nu(s+n) - \log \gamma_{2n-1,\nu}}{s} \leq \log \gamma_{2n+1,\nu} - \log \gamma_{2n-1,\nu},$$

that it can also be written as

$$\left(\frac{\gamma_{2n-1,\nu}}{\gamma_{2n-3,\nu}} \right)^s \gamma_{2n-1,\nu} \leq f_\nu(s+n) \leq \left(\frac{\gamma_{2n+1,\nu}}{\gamma_{2n-1,\nu}} \right)^s \gamma_{2n-1,\nu}.$$

Then, using (35) we get

$$\left(\frac{\gamma_{2n-1,\nu}}{\gamma_{2n-3,\nu}} \right)^s \frac{\gamma_{2n-1,\nu}}{\Theta_{n,\nu}^{(1)}(s)} \leq f_\nu(s) \leq \left(\frac{\gamma_{2n+1,\nu}}{\gamma_{2n-1,\nu}} \right)^s \frac{\gamma_{2n-1,\nu}}{\Theta_{n,\nu}^{(1)}(s)}.$$

Since the first inequality is true for $n \geq 2$, we can take $n \mapsto n+1$ on the left side. Therefore,

$$\left(\frac{\gamma_{2n+1,\nu}}{\gamma_{2n-1,\nu}} \right)^s \frac{\gamma_{2n+1,\nu}}{\Theta_{n+1,\nu}^{(1)}(s)} \leq f_\nu(s) \leq \left(\frac{\gamma_{2n+1,\nu}}{\gamma_{2n-1,\nu}} \right)^s \frac{\gamma_{2n+1,\nu}}{\Theta_{n+1,\nu}^{(1)}(s)} \frac{\gamma_{2n-1,\nu}}{\gamma_{2n+1,\nu}} \theta_\nu^{(1)}(s+n).$$

This means that

$$\frac{\gamma_{2n+1,\nu}}{\gamma_{2n-1,\nu} \theta_\nu^{(1)}(s+n)} f_\nu(s) \leq \left(\frac{\gamma_{2n+1,\nu}}{\gamma_{2n-1,\nu}} \right)^s \frac{\gamma_{2n+1,\nu}}{\Theta_{n+1,\nu}^{(1)}(s)} \leq f_\nu(s).$$

But, since

$$\begin{aligned} \frac{\gamma_{2n+1,\nu}}{\gamma_{2n-1,\nu} \theta_\nu^{(1)}(s+n)} &= \frac{2^{2n+1} n! (\nu + 1)_{n+1}}{2^{2n-1} (n-1)! (\nu + 1)_n 4(s+n)(s+n+\nu+1)} \\ &= \frac{n(\nu + n + 1)}{(s+n)(s+n+\nu+1)} \rightarrow 1, \quad n \rightarrow \infty, \end{aligned}$$

we have

$$\begin{aligned} f_\nu(s) &= \lim_{n \rightarrow \infty} \left(\frac{\gamma_{2n+1,\nu}}{\gamma_{2n-1,\nu}} \right)^s \frac{\gamma_{2n+1,\nu}}{\Theta_{n+1,\nu}^{(1)}(s)} = \lim_{n \rightarrow \infty} \frac{(4n(\nu+n+1))^s \gamma_{2n+1,\nu}}{4^{n+1}(s)_{n+1}(s+1+\nu)_{n+1}} \\ &= \lim_{n \rightarrow \infty} \frac{(4n(\nu+n+1))^s 2^{2n+1} n! (\nu+1)_{n+1}}{4^{n+1}(s)_{n+1}(s+1+\nu)_{n+1}} \\ &= \lim_{n \rightarrow \infty} \frac{(4n(\nu+n+1))^s n! (\nu+1)_{n+1}}{2(s)_{n+1}(s+1+\nu)_{n+1}}, \end{aligned}$$

so $f_\nu(s)$ satisfies (34). $\square$

For $\Gamma_\nu^{(2)}(s)$, we have the following theorem analogous to the previous one (the proof follows taking into account that $\Gamma_\nu^{(2)}(s) = 2s\Gamma_\nu^{(1)}(s)$ and it is omitted).

Theorem 7. *Let $\nu > -1$. Then, there exists a unique function $f_\nu(x)$ defined on the interval $(0, \infty)$ such that*

- $f_\nu(1) = 4(\nu+1)$,
- $f_\nu(s+1) = 4(s+1)(s+1+\nu)f_\nu(s)$,
- *the logarithm of $f_\nu(s)$ is convex.*

Furthermore, such a function is $\Gamma_\nu^{(2)}(s)$.

The gamma function has a very well-know definition using the following infinite product (by Euler)

$$\Gamma(s) = \frac{1}{s} \prod_{n=1}^{\infty} \frac{n}{s+n} \left(\frac{n+1}{n} \right)^s$$

or as the following limit (by Gauss)

$$\Gamma(s) = \lim_{n \rightarrow \infty} \frac{1 \cdot 2 \cdot 3 \cdots n}{s(s+1)(s+2) \cdots (s+n)} n^s.$$

This kind of representations also exists for $\Gamma_\nu^{(1)}(s)$ and $\Gamma_\nu^{(2)}(s)$. We show them only for $\Gamma_\nu^{(1)}(s)$; the corresponding formulas for $\Gamma_\nu^{(2)}(s)$ follow straightforwardly using $\Gamma_\nu^{(2)}(s) = 2s\Gamma_\nu^{(1)}(s)$.

Corollary 8. *For $\nu > -1$, $s \neq 0, -1, -2, \dots$ and $s \neq -\nu-1, -\nu-2, -\nu-3, \dots$, the function $\Gamma_\nu^{(1)}(s)$ can be written as*

$$(36) \quad \Gamma_\nu^{(1)}(s) = \lim_{n \rightarrow \infty} \frac{(4n(\nu+n+1))^s n! (\nu+1)_{n+1}}{2(s)_{n+1}(s+1+\nu)_{n+1}}.$$

Proof. The proof for $s \in (0, 1]$ is (34). Otherwise, we can write $s = s_0 + k$ or $s = s_0 - k$ with $0 < s_0 \leq 1$ and $k = 1, 2, 3, \dots$, and use (36) for s_0 . For the first case, we have

$$\begin{aligned} \Gamma_\nu^{(1)}(s+k) &= 4^k (s(s+1) \cdots (s+k-1)) ((s+1+\nu)(s+2+\nu) \cdots (s+k+\nu)) \Gamma_\nu^{(1)}(s) \\ &= 4^k (s(s+1) \cdots (s+k-1)) ((s+1+\nu)(s+2+\nu) \cdots (s+k+\nu)) \\ &\quad \times \lim_{n \rightarrow \infty} \frac{(4n(\nu+n+1))^s n! (\nu+1)_{n+1}}{2(s)_{n+1}(s+1+\nu)_{n+1}} \\ &= \lim_{n \rightarrow \infty} \frac{(4n(\nu+n+1))^{s+k} n! (\nu+1)_{n+1}}{2(s+k)_{n+1}(s+k+1+\nu)_{n+1}}. \end{aligned}$$

The second case is similar by using

$$\Gamma_\nu^{(1)}(s-k) = \frac{\Gamma_\nu^{(1)}(s)}{4^k ((s-k)(s-k+1) \cdots (s-1)) ((s-k+1+\nu)(s-k+2+\nu) \cdots (s+\nu))}. \quad \square$$

Moreover, a simple consequence of (36) is the following infinite product representation.

Corollary 9. For $\nu > -1$, $s \neq 0, -1, -2, \dots$ and $s \neq -\nu - 1, -\nu - 2, -\nu - 3, \dots$, the function $\Gamma_\nu^{(1)}(s)$ can be written as

$$\Gamma_\nu^{(1)}(s) = \frac{2^{2s-1}(\nu+1)(\nu+2)^s}{s(s+\nu+1)} \prod_{k=1}^{\infty} \frac{k(k+\nu+1)}{(s+k)(s+k+\nu+1)} \left(\frac{(k+1)(k+\nu+2)}{k(k+\nu+1)} \right)^s.$$

Proof. It follows easily expanding the Pochhammer symbols in (36) and writing $\prod_{k=1}^{\infty}$ as $\lim_{n \rightarrow \infty} \prod_{k=1}^n$ (where we can use telescopic cancelling between the consecutive terms). $\square$

Two relevant functional equations satisfied by the gamma function are the duplication formula

$$(37) \quad \Gamma(2s) = \pi^{-1/2} 2^{2s-1} \Gamma(s) \Gamma\left(s + \frac{1}{2}\right)$$

and the reflection formula

$$(38) \quad \Gamma(s) \Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

Both can be extended to the functions $\Gamma_\nu^{(1)}(s)$ and $\Gamma_\nu^{(2)}(s)$. In what follows, we can assume $\nu \in \mathbb{C} \setminus \{-1, -2, \dots\}$ and $s \in \mathbb{C}$, and the identities have some isolated poles.

Theorem 10 (Duplication formula). *The functions $\Gamma_\nu^{(1)}(s)$ and $\Gamma_\nu^{(2)}(s)$ satisfy the following duplication formulas:*

$$(39) \quad \begin{aligned} \Gamma_\nu^{(1)}(2s) &= 2^{4s+2\nu} \frac{\Gamma(\nu+1)}{\pi} \frac{\Gamma(2s+\nu+1)}{\Gamma(2s+2\nu+2)} \Gamma_\nu^{(1)}(s) \Gamma_\nu^{(1)}\left(s + \frac{1}{2}\right), \\ \Gamma_\nu^{(2)}(2s) &= 2^{4s+2\nu+1} \frac{\Gamma(\nu+1)}{\pi} \frac{\Gamma(2s+\nu+1)}{\Gamma(2s+2\nu+2)(2s+1)} \Gamma_\nu^{(2)}(s) \Gamma_\nu^{(2)}\left(s + \frac{1}{2}\right). \end{aligned}$$

Proof. Using (37), we have

$$\begin{aligned} \Gamma_\nu^{(1)}(s) \Gamma_\nu^{(1)}\left(s + \frac{1}{2}\right) &= 2^{2s-1} 2^{2s} \frac{\Gamma(s) \Gamma(s+\nu+1) \Gamma\left(s + \frac{1}{2}\right) \Gamma\left(s + \nu + \frac{3}{2}\right)}{\Gamma(\nu+1)^2} \\ &= 2^{4s-1} \frac{\pi^{1/2} 2^{1-2s} \Gamma(2s) \pi^{1/2} 2^{1-2(s+\nu+1)} \Gamma(2s+2\nu+2)}{\Gamma(\nu+1)^2} \\ &= 2^{-2\nu-1} \frac{\pi \Gamma(2s) \Gamma(2s+2\nu+2)}{\Gamma(\nu+1)^2} \\ &= \pi 2^{-2\nu-1} 2^{1-4s} \frac{2^{4s-1} \Gamma(2s) \Gamma(2s+\nu+1)}{\Gamma(\nu+1)} \frac{\Gamma(2s+2\nu+2)}{\Gamma(2s+\nu+1) \Gamma(\nu+1)} \\ &= \pi 2^{-2\nu-4s} \Gamma_\nu^{(1)}(2s) \frac{\Gamma(2s+2\nu+2)}{\Gamma(2s+\nu+1) \Gamma(\nu+1)}, \end{aligned}$$

which implies (39). The proof for $\Gamma_\nu^{(2)}$ is similar (or a consequence of the relation $\Gamma_\nu^{(2)}(s) = 2s \Gamma_\nu^{(1)}(s)$). $\square$

It is a matter of calculation to check that the duplication formula (39) recovers the classical duplication formula (37) when $\nu = -1/2$ and $s \mapsto s/2$.

Theorem 11 (Reflection formula). *The functions $\Gamma_\nu^{(1)}(s)$ and $\Gamma_\nu^{(2)}(s)$ satisfy the following reflection formulas:*

$$(40) \quad \begin{aligned} \Gamma_\nu^{(1)}(s) \Gamma_\nu^{(1)}(1-s) &= \frac{\pi}{\sin(\pi s)} \frac{\Gamma(\nu+1+s) \Gamma(2+\nu-s)}{\Gamma(\nu+1)^2}, \\ \Gamma_\nu^{(2)}(s) \Gamma_\nu^{(2)}(1-s) &= 4s(1-s) \frac{\pi}{\sin(\pi s)} \frac{\Gamma(\nu+1+s) \Gamma(2+\nu-s)}{\Gamma(\nu+1)^2}. \end{aligned}$$

Proof. It is enough to prove the formula for $\Gamma_\nu^{(1)}$. Using the classical reflection formula (38), we have

$$\begin{aligned}\Gamma_\nu^{(1)}(s)\Gamma_\nu^{(1)}(1-s) &= 2^{2s-1}\Gamma(s)\frac{\Gamma(s+\nu+1)}{\Gamma(\nu+1)}2^{1-2s}\Gamma(1-s)\frac{\Gamma(2+\nu-s)}{\Gamma(\nu+1)} \\ &= \frac{\pi}{\sin(\pi s)}\frac{\Gamma(s+\nu+1)\Gamma(2+\nu-s)}{\Gamma(\nu+1)^2}.\end{aligned}\quad \square$$

Let us finally show what happens with (40) in the case $\nu = -1/2$. Making the change $s \mapsto s/2$ and using $\Gamma(1/2) = \sqrt{\pi}$, (40) becomes

$$\Gamma_{-1/2}^{(1)}(s/2)\Gamma_{-1/2}^{(1)}\left(\frac{2-s}{2}\right) = \frac{1}{\sin(\pi s/2)}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{3-s}{2}\right),$$

i.e. (by using $\Gamma_{-1/2}^{(1)}(s) = \Gamma(2s)$),

$$\Gamma(s)\Gamma(2-s) = \frac{1}{\sin(\pi s/2)}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{3-s}{2}\right).$$

Now, use $\Gamma(2-s) = (1-s)\Gamma(1-s)$ and $\Gamma\left(\frac{3-s}{2}\right) = \left(\frac{1-s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)$, so we can write

$$(41) \quad \Gamma(s)\Gamma(1-s) = \frac{1}{2\sin(\pi s/2)}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1-\frac{s+1}{2}\right).$$

Note that this expression can also be found by using (38) twice. Indeed, take

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}, \quad \Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1-\frac{s+1}{2}\right) = \frac{\pi}{\sin(\pi(s+1)/2)};$$

then, (41) follows from (38) by using $\sin(\pi s) = 2\sin(\pi s/2)\sin(\pi(s+1)/2)$.

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ORCID

Antonio J. Durán  <https://orcid.org/0000-0002-8351-7392>

Alejandro Gil Asensi  <https://orcid.org/0009-0007-9667-0744>

Juan L. Varona  <https://orcid.org/0000-0002-2023-9946>

REFERENCES

- [1] Dunkl CF. Differential-difference operators associated to reflection groups. Trans. Amer. Math. Soc. 1989;311: 167–183. doi: [10.2307/2001022](https://doi.org/10.2307/2001022)
- [2] de Jeu MFE. The Dunkl transform. Invent. Math. 1993;113:147–162. doi: [10.1007/BF01244305](https://doi.org/10.1007/BF01244305)
- [3] Rosenblum M. Generalized Hermite polynomials and the Bose-like oscillator calculus. Oper. Theory Adv. Appl. 1994;73:369–396. doi: [10.1007/978-3-0348-8522-5_15](https://doi.org/10.1007/978-3-0348-8522-5_15)
- [4] Rösler M. Dunkl operators. Theory and applications. Orthogonal Polynomials and Special Functions. In: Koelink E, Van Assche W, editors. Berlin: Springer, Lecture Notes in Mathematics, vol. 1817; 2003, p. 93–135. doi: [10.1007/3-540-44945-0_3](https://doi.org/10.1007/3-540-44945-0_3)
- [5] Andersen NB, de Jeu M. Elementary proofs of Paley-Wiener theorems for the Dunkl transform on the real line. Int. Math. Res. Not. 2005;30:1817–1831. doi: [10.1155/IMRN.2005.1817](https://doi.org/10.1155/IMRN.2005.1817)
- [6] Thangavelu S, Xu Y. Convolution operator and maximal function for the Dunkl transform. J. Anal. Math. 2005;97:25–55. doi: [10.1007/BF02807401](https://doi.org/10.1007/BF02807401)
- [7] Trimèche K. Paley-Wiener theorems for the Dunkl transform and Dunkl translation operators. Integral Transforms Spec. Funct. 2002;13:17–38. doi: [10.1080/10652460212888](https://doi.org/10.1080/10652460212888)
- [8] Ciaurri Ó, Varona JL. A Whittaker-Shannon-Kotel'nikov sampling theorem related to the Dunkl transform. Proc. Amer. Math. Soc. 2007;135:2939–2947. doi: [10.1090/S0002-9939-07-08831-4](https://doi.org/10.1090/S0002-9939-07-08831-4)
- [9] Ciaurri Ó, Durán AJ, Pérez M, Varona JL. Bernoulli-Dunkl and Apostol-Euler-Dunkl polynomials with applications to series involving zeros of Bessel functions. J. Approx. Theory. 2018;235:20–45. doi: [10.1016/j.jat.2018.06.001](https://doi.org/10.1016/j.jat.2018.06.001)
- [10] Ben Cheikh Y, Gaiad M. Dunkl-Appell d -orthogonal polynomials. Integral Transforms Spec. Funct. 2007;18: 581–597. doi: [10.1080/10652460701445302](https://doi.org/10.1080/10652460701445302)

- [11] Ciaurri Ó, Mínguez Cenicerós J, Varona JL. Bernoulli-Dunkl and Euler-Dunkl polynomials and their generalizations. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*. 2019;113:2853–2876. doi: [10.1007/s13398-019-00662-z](https://doi.org/10.1007/s13398-019-00662-z)
- [12] Dimovski IH, Hristov VZ. Nonlocal operational calculi for Dunkl operators. *SIGMA Symmetry Integrability Geom. Methods Appl.* 2009;5:Paper 030, 16 p. doi: [10.3842/SIGMA.2009.030](https://doi.org/10.3842/SIGMA.2009.030)
- [13] Gil Asensi A, Varona JL. Appell-Dunkl sequences and Hurwitz-Dunkl zeta functions. *J. Math. Anal. Appl.* 2023;520:126870, 40 p. doi: [10.1016/j.jmaa.2022.126870](https://doi.org/10.1016/j.jmaa.2022.126870)
- [14] Gil Asensi A, Mínguez Cenicerós J, Varona JL. Sheffer-Dunkl sequences via umbral calculus in the Dunkl context. *Bull. Malays. Math. Sci. Soc.* 2024;47:Paper No. 172, 28 p. doi: [10.1007/s40840-024-01768-3](https://doi.org/10.1007/s40840-024-01768-3)
- [15] Thangavelu S, Xu Y. Riesz transform and Riesz potentials for Dunkl transform. *J. Comput. Appl. Math.* 2007; 199:181–195. doi: [10.1016/j.cam.2005.02.022](https://doi.org/10.1016/j.cam.2005.02.022)
- [16] Durán AJ, Gil Asensi A, Varona JL. Moment problems associated to the Dunkl factorial. *Proc. Amer. Math. Soc.* to appear. doi: [10.1090/proc/16852](https://doi.org/10.1090/proc/16852)
- [17] Olver FWJ, Maximon LC. Bessel Functions. In: Olver FWJ, Lozier DW, Boisvert RF, et al., editors. *NIST handbook of mathematical functions*. New York: Cambridge University Press; 2010. Chapter 10. <http://dlmf.nist.gov/10>
- [18] Watson GN. *A Treatise on the Theory of Bessel Functions* (2nd edition). Cambridge: Cambridge University Press; 1944.
- [19] Askey RA, Roy R. Gamma Function. In: Olver FWJ, Lozier DW, Boisvert RF, et al., editors. *NIST handbook of mathematical functions*. New York: Cambridge University Press; 2010. Chapter 5. <http://dlmf.nist.gov/5>
- [20] Post EL. The generalized gamma functions. *Annals of Math.* 1919;20:202–217. doi: [10.2307/1967871](https://doi.org/10.2307/1967871)
- [21] Jackson FH. A generalisation of the functions $\Gamma(n)$ and x^n . *Proc. R. Soc. Lond.* 1904–1905;74:64–72.
- [22] Sebastian N. A generalized gamma model associated with a Bessel function. *Integral Transforms Spec. Funct.* 2011;22:631–645. doi: [10.1080/10652469.2010.536413](https://doi.org/10.1080/10652469.2010.536413)
- [23] Pérez-Marco R. On the definition of Euler gamma function. *Enseign. Math.* 2022;68:135–160. doi: [10.4171/lem/1023](https://doi.org/10.4171/lem/1023)
- [24] Pérez-Marco R. On the definition of higher Euler gamma function. *Constr. Approx.* 2025;61:413–443. doi: [10.1007/s00365-023-09674-w](https://doi.org/10.1007/s00365-023-09674-w)
- [25] Artin E. *The Gamma Function*. Athena series: selected topics in mathematics. New York: Holt, Rinehart and Winston; 1964.
- [26] Shen YY. On characterizations of the gamma function. *Math. Mag.* 1995;68:301–305. doi: [10.2307/2690585](https://doi.org/10.2307/2690585)
- [27] Borwein JM, Corless RM. Gamma and factorial in the Monthly. *Amer. Math. Monthly.* 2018;125:400–424. doi: [10.1080/00029890.2018.1420983](https://doi.org/10.1080/00029890.2018.1420983)
- [28] Prudnikov AP, Brychkov AY, Marichev OI. *Integrals and series. Vol. 2: Special functions*. New York: Gordon and Breach Science Publishers; 1986.
- [29] Chamberland M. A continuous extension of the $3x + 1$ problem to the real line. *Dynam. Contin. Discrete Impuls. Systems.* 1996;2:495–509.
- [30] Letherman S, Schleicher D, Wood R. The $3n + 1$ -problem and holomorphic dynamics, *Experiment. Math.* 1999; 8:241–251. doi: [10.1080/10586458.1999.10504402](https://doi.org/10.1080/10586458.1999.10504402)
- [31] Arias-de-Reyna J. X-ray of Riemann zeta-function. *arXiv:math/0309433 [math.NT]*. doi: [10.48550/arXiv.math/0309433](https://doi.org/10.48550/arXiv.math/0309433)

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO AND IMUS, UNIVERSIDAD DE SEVILLA, SEVILLA, SPAIN
Email address: duran@us.es

DEPARTAMENTO DE MATEMÁTICAS Y COMPUTACIÓN, UNIVERSIDAD DE LA RIOJA, LOGROÑO, SPAIN
Email address: algila@unirioja.es

DEPARTAMENTO DE MATEMÁTICAS Y COMPUTACIÓN, UNIVERSIDAD DE LA RIOJA, LOGROÑO, SPAIN
Email address: jvarona@unirioja.es
URL: <http://www.unirioja.es/cu/jvarona/>