COMMUTATORS AND ANALYTIC DEPENDENCE
OF FOURIER-BESSEL SERIES ON \((0, \infty)\)

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Abstract. In this paper we study the boundedness of the commutators \([b, S_n]\) where \(b\) is a BMO function and \(S_n\) denotes the \(n\)-th partial sum of the Fourier-Bessel series on \((0, \infty)\). Perturbing the measure by \(\exp(2b)\) we obtain that certain operators related to \(S_n\) depend analytically on the functional parameter \(b\).

0. Introduction.

Let \(J_\alpha\) be the Bessel function of order \(\alpha > -1\). The formula

\[
\int_0^\infty J_{\alpha+2n+1}(x)J_{\alpha+2m+1}(x) \frac{dx}{x} = \begin{cases} 0, & \text{if } n \neq m \\ 2^{-1}(\alpha + 2n + 1)^{-1}, & \text{if } n = m \end{cases}
\]

(see [14, XIII.13.41 (7), p. 404] and [14, XIII.13.42 (1), p. 405]) provides an orthonormal system \((j_n^\alpha)_{n \geq 0}\) in \(L^2((0, \infty), x^\alpha \, dx)\) \([L^2(x^\alpha), \text{from now on}],\) given by

\[
j_n^\alpha(x) = \sqrt{\alpha + 2n + 1} J_{\alpha+2n+1}(\sqrt{x}) x^{-\alpha/2-1/2}.
\]

In this paper we consider the Fourier expansion associated with this orthonormal system, which is usually referred to as the Fourier-Bessel series on \((0, \infty)\). For any suitable function \(f\) and any \(n \geq 0\), the \(n\)-th partial sum of this expansion is given by

\[
S_n f = \sum_{k=0}^n c_k(f) j_k^\alpha, \quad c_k(f) = \int_0^\infty f(t) j_k^\alpha(t) t^\alpha \, dt.
\]

We also consider the commutator of the Fourier-Bessel series on \((0, \infty)\) and the multiplication operator associated to a BMO function; this is defined, for any given \(b \in \text{BMO}\) and \(n \geq 0\), as

\[
[b, S_n]f = bS_n(f) - S_n(bf).
\]

In the case \(\alpha \geq -1/2\), one of the authors proved in [13] that the Fourier-Bessel series is bounded in \(L^p(x^\alpha)\), i.e., there exists some constant \(C > 0\) (depending on \(\alpha\) and \(p\)) such that for every \(n \geq 0\) and every \(f \in L^p(x^\alpha)\),

\[
\|S_n f\|_{L^p(x^\alpha)} \leq C \|f\|_{L^p(x^\alpha)},
\]

1991 Mathematics Subject Classification. Primary 42C10.
Key words and phrases. Fourier-Bessel series, commutators, BMO, \(A_p\) weights.
Research supported by DGICYT and by UR.
if and only if \( \max\{4/3, 4(\alpha + 1)/(2\alpha + 3)\} < p < \min\{4, 4(\alpha + 1)/(2\alpha + 1)\} \). In Theorem 1 we will extend this result to the case \( \alpha > -1 \) and prove the corresponding inequality for the commutator \([b, S_n]\), \( b \in \text{BMO} \).

Regarding the commutator \([b, S_n]\), results of this type are of independent interest and have been widely studied for many classical operators; see [2, 10, 11, 12, 4], for instance.

In our case, the commutator \([b, S_n]\) is closely related to the problem of perturbing the orthonormal system. Given an orthonormal system \((\varphi_n)_{n \geq 0} \) in some \(L^2(\nu)\) space and a suitable function \( b \) (in some sense close to 0), the classical Gram-Schmidt procedure can be applied to \((\varphi_n)_{n \geq 0} \) so as to obtain a new orthonormal system in \(L^2(e^{2b}d\nu)\), which we will refer to as a perturbed system. In this natural way a mapping can be defined that associates a perturbed system (and a perturbed orthogonal expansion) to each (small) function \( b \). For different compact perturbations of orthogonal polynomial systems and further references, see [7, 9, 1].

Let us take the system \((j_n^a)_{n \geq 0} \) in \(L^2(x^a)\) as our starting point. Let \( S_n(b) \) stand for the \( n \)-th partial sum operator of the Fourier series associated to the perturbed measure \( e^{2b}x^a \; dx \) in the aforementioned way. Once the boundedness properties of \( S_n = S_n(0) \) have been established, it is interesting to study the mapping \( b \mapsto S_n(b) \). This is not, however, a convenient setting, since each perturbed series \( S_n(b) \) acts on a different space \( L^2(e^{2b}x^a) \). Instead, we can consider the operators

\[
V_n(b) = e^b S_n(b) e^{-b}.
\]

Now, each \( V_n(b) \) acts on \( L^2(x^a) \) and its norm coincides with the operator norm of \( S_n(b) \) acting on \( L^2(e^{2b}x^a) \). The problem is further simplified if we take the operators

\[
T_n(b) = e^b S_n(0) e^{-b},
\]

i.e., \( T_n(b) f = e^b S_n(e^{-b} f) \). Indeed, it has been proved in [3] that the family \( (V_n(b))_{n \geq 0} \) depends analytically on \( b \) belonging to a neighbourhood of 0 in the complexification of BMO whenever the family \( (T_n(b))_{n \geq 0} \) does too.

We will prove in Theorem 2 that the family of operators \((T_n(b))_{n \geq 0} \) acting on \( L^2(x^a) \) is uniformly bounded for \( b \) belonging to some neighbourhood of 0 in the complexification of BMO. As a consequence (see [3, Propositions 2.1 and 2.3]), the operator-valued mappings \((T_n)_{n \geq 0} \) are uniformly analytic in a neighbourhood of 0 in the complexification of BMO and so are \((V_n)_{n \geq 0} \).

Now, the connection between \([b, S_n]\) and the perturbated Fourier series comes via the Gâteaux differential of \( T_n \) at 0 in the direction \( b \):

\[
\frac{d}{dz} T_n(zb)|_{z=0} = [b, S_n].
\]

In this way, the uniform analyticity of \( T_n \) in a neighbourhood of 0 gives the \( L^2 \)-boundedness of \([b, S_n]\).

1. Main results.

If \( b \) is a locally Lebesgue-integrable function on \((0, \infty)\), the mean of \( b \) on an interval \( I \subseteq (0, \infty) \) is

\[
b_I = \frac{1}{|I|} \int_I b(x) \; dx.
\]
The function $b$ is said to have bounded mean oscillation on $(0, \infty)$ if
\[
\|b\|_{\text{BMO}} = \sup_I \frac{1}{|I|} \int_I |b(x) - b_I| \, dx
\]
is finite, where the supremum is taken over all the intervals $I \subseteq (0, \infty)$. The space BMO of real-valued functions (modulo constants) having bounded mean oscillation on $(0, \infty)$ is a real Banach space with $\| \cdot \|_{\text{BMO}}$ as its norm.

**Theorem 1.** Let $1 < p < \infty$, $-1 < \alpha$ such that
\[
\begin{cases} 
4/3 < p < 4, & \text{if } -1 < \alpha < 0; \\
4(\alpha + 1) \quad 2\alpha + 3 < p < \frac{4(\alpha + 1)}{2\alpha + 1}, & \text{if } 0 \leq \alpha.
\end{cases}
\]

(a) There exists some constant $C > 0$ such that, for every $f \in L^p(x^\alpha)$ and $n \geq 0$,
\[
\|S_n f\|_{L^p(x^\alpha)} \leq C\|f\|_{L^p(x^\alpha)}.
\]
(b) If $b \in \text{BMO}$, then there exists some constant $C > 0$ such that, for every $f \in L^p(x^\alpha)$ and $n \geq 0$,
\[
\|[S_n, b] f\|_{L^p(x^\alpha)} \leq C\|f\|_{L^p(x^\alpha)}.
\]

Throughout this paper, we will denote by $C$ a positive constant which is independent of $n$ and $f$, but may be different in each occurrence, even within the same formula.

**Theorem 2.** Let $1 < p < \infty$, $-1 < \alpha$ such that
\[
\begin{cases} 
4/3 < p < 4, & \text{if } -1 < \alpha < 0; \\
4(\alpha + 1) \quad 2\alpha + 3 < p < \frac{4(\alpha + 1)}{2\alpha + 1}, & \text{if } 0 \leq \alpha.
\end{cases}
\]

Then there exist some $C, \delta > 0$ such that, for all $b \in \text{BMO}$ with $\|b\|_{\text{BMO}} < \delta$,
\[
\sup_n \|T_n(b)\|_{L^p(x^\alpha) \to L^p(x^\alpha)} \leq C.
\]

The next corollary is just a consequence of Theorem 2 and [3, Prop. 2.3].

**Corollary.** The sequences of operators $(T_n(b))_{n \geq 0}$ and $(V_n(b))_{n \geq 0}$, acting on the space $L^2(x^\alpha)$, are uniformly analytic in a neighbourhood of 0 in the complexification of BMO.

Some notation and previous results will be necessary. For $1 < p < \infty$, we write $p' = p/(p-1)$, i.e., $1/p + 1/p' = 1$. A weight is a nonnegative Lebesgue-measurable function on $(0, \infty)$. The class $A_p(0, \infty)$ [$A_{p'}$, for short] consists of those pairs of weights $(u, v)$ such that, for every subinterval $I \subseteq (0, \infty)$,
\[
\frac{1}{|I|} \int_I u \left( \frac{1}{|I|} \int_I v^{-p'/p} \right)^{p/p'} \leq C,
\]
Lemma 1. Let \( L \) and \( H \) be bounded linear operators on \( L^p(w) \), for any weight \( w \in A_p \). The norm of \( H : L^p(w) \rightarrow L^p(w) \) and the \( A_p \) constant of \( w \) depend only one on another, in the sense that given some constant \( C \) which verifies the \( A_p \) condition for \( w \), another constant \( C' \) depending only on \( A_p \) can be chosen so that \( \|H\| \leq C \), and viceversa. Therefore, for a sequence \( (w_n)_{n \in \mathbb{N}} \) uniformly in \( A_p \), i.e., with some constant \( C \) verifying the \( A_p \) condition for every \( w_n \), the Hilbert transform is uniformly bounded on \( L^p(w_n) \), \( n \in \mathbb{N} \). We refer the reader again to [6] for further details.

The Hilbert transform on \((0, \infty)\) will be denoted by \( H \). Fix \( 1 < p < \infty \); then \( H \) is a bounded linear operator on \( L^p(w) \), for any weight \( w \in A_p \). The norm of \( H : L^p(w) \rightarrow L^p(w) \) and the \( A_p \) constant of \( w \) depend only one on another, in the sense that given some constant \( C \) which verifies the \( A_p \) condition for \( w \), another constant \( C' \) depending only on \( A_p \) can be chosen so that \( \|H\| \leq C \), and viceversa.

Also, if \( (u, v) \) is a pair of weights such that \( C_1 u \leq w \leq C_2 v \) for some \( w \in A_p \), we deduce that \( H \) is a bounded operator from \( L^p(v) \) into \( L^p(u) \). The existence of such a weight \( w \) is equivalent to \((u^\delta, v^\delta) \in A_p \) for some \( \delta > 1 \) (see [8]). For short, this is written as \((u, v) \in A^\delta_p \).

Analogous results hold also with the commutator \([b, H]\), for any \( b \in \text{BMO} \) (see [2], for instance). Namely, given \( b \in \text{BMO} \) and \( w \in A_p \), \([b, H]\) is a bounded operator on \( L^p(w) \) with a norm that depends only on the \( \text{BMO} \)-norm of \( b \) and the \( A_p \) constant of \( w \), in the sense above.

2. Proofs.

Let us start with some auxiliary results:

**Lemma 1.** Let \( u, v, w \) be weights on \((0, +\infty)\), \( \lambda > 0 \).

(a) \( w(x) \in A_p \) if and only if \( w(\lambda x) \in A_p \); both weights have the same \( A_p \) constant.

(b) \( w \in A_p \) if and only if \( \lambda w \in A_p \); both weights have also the same \( A_p \) constant.

(c) If \( u, v \in A_p \), then \( u + v \in A_p \) and \( A_p(u + v) \leq A_p(u) + A_p(v) \).

(d) If \( u, v \in A_p \) and \( 1/w = 1/u + 1/v \), then \( w \in A_p \) and \( A_p(w) \leq C[A_p(u) + A_p(v)] \).

**Proof.** Parts (a) and (b) are trivial. Part (c) follows easily from the inequality
\[
\left( \frac{1}{|I|} \int_I (u + v)^{-p'/p} \right)^{p/p'} \leq \min \left\{ \left( \frac{1}{|I|} \int_I u^{-p'/p} \right)^{p/p'}, \left( \frac{1}{|I|} \int_I v^{-p'/p} \right)^{p/p'} \right\}.
\]

Part (d) is a consequence of (c) and the fact that \( u \in A_p \iff u^{-p'/p} \in A_{p'} \), with \( A_{p'}(u^{-p'/p}) = [A_p(u)]^{p'/p} \).

The proof of the next lemma is not difficult, but cumbersome, so we omit it. For the weight in (c), observe that \( x^r|x^{1/2} - 1|^s \sim x^r \) near 0, \( x^r|x^{1/2} - 1|^s \sim |x - 1|^s \) near 1 and \( x^r|x^{1/2} - 1|^s \sim x^r + s/2 \) near \( \infty \), whence the three conditions follow.

**Lemma 2.** Let \( r, s \in \mathbb{R} \).

(a) \( x^r \in A_p \iff -1 < r < p - 1 \).

(b) Set \( \Phi(x) = x^r \) if \( x \in (0, 1) \) and \( \Phi(x) = x^s \) if \( x \in (1, \infty) \). Then, \( \Phi \in A_p \) if and only if \( -1 < r < p - 1 \) and \( -1 < s < p - 1 \).

(c) \( x^r|x^{1/2} - 1|^s \in A_p \iff -1 < r < p - 1, -1 < s < p - 1 \) and \( -1 < r + s/2 < p - 1 \).
Lemma 3. Let \( n \in \mathbb{N}, \alpha > -1 \). Then
\[
\sum_{k=0}^{n} 2(\alpha + 2k + 1)J_{\alpha+2k+1}(x)J_{\alpha+2k+1}(t) = \frac{xt}{x^2 - t^2} [xJ_{\alpha+1}(x)J_{\alpha}(t) - tJ_{\alpha}(x)J_{\alpha+1}(t) + xJ'_{\alpha+2n+2}(x)J_{\alpha+2n+2}(t) - tJ_{\alpha+2n+2}(x)J'_{\alpha+2n+2}(t)].
\]

Proof. Using the equality \( J_{\nu-1}(z) + J_{\nu+1}(z) = \frac{2\nu}{z} J_{\nu}(z) \) (see [14, III.3.2, p. 45]) to express \( J_{\mu-1} \) and \( J_{\mu+2} \) in terms of \( J_{\mu} \) and \( J_{\mu+1} \) proves the formula
\[
\frac{xt}{x^2 - t^2} [xJ_{\mu}(x)J_{\mu-1}(t) - tJ_{\mu-1}(x)J_{\mu}(t) - xJ_{\mu+2}(x)J_{\mu+1}(t) + tJ_{\mu+1}(x)J_{\mu+2}(t)] = 2\mu J_{\mu}(x)J_{\mu}(t).
\]
This gives now
\[
\sum_{k=0}^{n} 2(\alpha + 2k + 1)J_{\alpha+2k+1}(x)J_{\alpha+2k+1}(t) = \frac{xt}{x^2 - t^2} [xJ_{\alpha+1}(x)J_{\alpha}(t) - tJ_{\alpha}(x)J_{\alpha+1}(t) - xJ_{\alpha+2n+3}(x)J_{\alpha+2n+2}(t) + tJ_{\alpha+2n+2}(x)J_{\alpha+2n+3}(t)].
\]
Finally, use the formula \( zJ_{\nu+1}(z) = \nu J_{\nu}(z) - J_{\nu}'(z) \) (see [14, III.3.2, p. 45]) to take out \( J_{\alpha+2n+3}. \)

Proof of Theorem 1. From the definition,
\[
S_n f(x) = x^{-\frac{\alpha}{2} - \frac{1}{2}} \int_{0}^{\infty} \left[ \sum_{k=0}^{n} (\alpha + 2k + 1)J_{\alpha+2k+1}(x^\frac{1}{2})J_{\alpha+2k+1}(t^\frac{1}{2}) \right] t^\frac{\alpha}{2} - \frac{1}{2} f(t) dt
\]
so that Lemma 3 leads to
\[
S_n f = W_1 f - W_2 f + W_{3,n} f - W_{4,n} f,
\]
where
\[
W_1 f(x) = 2^{-1} x^{-\alpha/2+1/2} J_{\alpha+1}(x^{1/2}) H(t^{\alpha/2} J_{\alpha}(t^{1/2}) f(t))(x),
\]
\[
W_2 f(x) = 2^{-1} x^{-\alpha/2} J_{\alpha}(x^{1/2}) H(t^{\alpha/2+1/2} J_{\alpha+1}(t^{1/2}) f(t))(x),
\]
\[
W_{3,n} f(x) = 2^{-1} x^{-\alpha/2+1/2} J_{\nu}'(x^{1/2}) H(t^{\alpha/2} J_{\nu}(t^{1/2}) f(t))(x),
\]
\[
W_{4,n} f(x) = 2^{-1} x^{-\alpha/2} J_{\nu}(x^{1/2}) H(t^{\alpha/2+1/2} J_{\nu}'(t^{1/2}) f(t))(x)
\]
and \( \nu = \alpha + 2n + 2. \) Thus, we will show that the operators \( W_1, W_2 \) are bounded and the operators \( W_{3,n}, W_{4,n} \) are uniformly bounded for \( n \geq 0. \) The proof for the commutator \([b, S_n]\) is the same: just put \([b, H]\) instead of \( H.\)
(I) **Boundedness of the operator** $W_1$. From the definition, it follows that

$$
\|W_1f\|_{L^p(x^\alpha)} \leq C\|f\|_{L^p(x^\alpha)}
$$

if and only if

$$
\|Hg\|_{L^p(x^{\alpha-\alpha p/2+p/2}|J_{\alpha+1}(x^{1/2})|^{-p})} \leq C\|g\|_{L^p(x^{\alpha-\alpha p/2+p/2}|J_{\alpha}(x^{1/2})|^{-p})}.
$$

Proving that there is a weight $\Phi \in A_p$ with

$$
(1) \quad Cx^{\alpha-\alpha p/2+p/2}|J_{\alpha+1}(x^{1/2})|^{-p} \leq \Phi(x) \leq Cx^{\alpha-\alpha p/2}|J_{\alpha}(x^{1/2})|^{-p}
$$

will be enough. According to the bounds

$$
|J_{\alpha}(x)| \leq C_\alpha x^\alpha, \quad x \in (0,1),
$$

$$
|J_{\alpha}(x)| \leq C_\alpha x^{-1/2}, \quad x \in (1,\infty)
$$

(see, e.g., [14, III.3.1 (8), p. 40] and [14, VII.7.21 (1), p. 199]), we have

$$
x^{\alpha-\alpha p/2+p/2}|J_{\alpha+1}(x^{1/2})|^{-p} \leq \begin{cases}
C_\alpha x^{\alpha+p}, & \text{if } x \in (0,1), \\
C_\alpha x^{\alpha-\alpha p/2+p/4}, & \text{if } x \in (1,\infty),
\end{cases}
$$

$$
x^{\alpha-\alpha p/2}|J_{\alpha}(x^{1/2})|^{-p} \geq \begin{cases}
C_\alpha x^{\alpha-p}, & \text{if } x \in (0,1), \\
C_\alpha x^{\alpha-\alpha p/2+p/4}, & \text{if } x \in (1,\infty).
\end{cases}
$$

Let us try

$$
\Phi(x) = \begin{cases}
x^{r}, & \text{if } x \in (0,1), \\
x^{\alpha-\alpha p/2+p/4}, & \text{if } x \in (1,\infty).
\end{cases}
$$

By (b) in Lemma 2, conditions (1) and $\Phi \in A_p$ will hold if

$$
\begin{cases}
\alpha - \alpha p \leq r \leq \alpha + p, \\
-1 < r < p - 1, \\
-1 < \alpha - \alpha p/2 + p/4 < p - 1.
\end{cases}
$$

The third line is equivalent to

$$
\frac{2\alpha - 1}{4} p < \alpha + 1, \quad \alpha + 1 < \frac{2\alpha + 3}{4} p,
$$

and these hold, by the hypothesis. For the inequalities involving $r$ it suffices

$$
\max\{-1, \alpha - \alpha p\} < \min\{p - 1, \alpha + p\}.
$$

It is easy to check that this also holds, whenever $\alpha > -1$ and $p > 1$.

(II) **Boundedness of the operator** $W_2$. The proof is entirely similar: we have

$$
\|W_2f\|_{L^p(x^\alpha)} \leq C\|f\|_{L^p(x^\alpha)}
$$

if and only if

$$
\|Hg\|_{L^p(x^{\alpha-\alpha p/2}|J_{\alpha}(x^{1/2})|^{-p})} \leq C\|g\|_{L^p(x^{\alpha-\alpha p/2-p/2}|J_{\alpha+1}(x^{1/2})|^{-p})}.
$$
so that we can prove that there is a weight \( \Psi \in A_p \) with
\[(2) \quad C x^{\alpha - \alpha p/2} |J_\alpha(x^{1/2})|^p \leq \Psi(x) \leq C x^{\alpha - \alpha p/2 - p/2} |J_{\alpha + 1}(x^{1/2})|^{-p}.
\]
Now we have
\[
x^{\alpha - \alpha p/2} |J_\alpha(x^{1/2})|^p \leq \begin{cases} 
  C x^\alpha, & \text{if } x \in (0, 1), \\
  C x^{\alpha - \alpha p/2 - p/4}, & \text{if } x \in (1, \infty),
\end{cases}
\]
\[
x^{\alpha - \alpha p/2 - p/2} |J_{\alpha + 1}(x^{1/2})|^{-p} \geq \begin{cases} 
  C x^{\alpha - \alpha p - p}, & \text{if } x \in (0, 1), \\
  C x^{\alpha - \alpha p/2 - p/4}, & \text{if } x \in (1, \infty).
\end{cases}
\]
Setting
\[
\Psi(x) = \begin{cases} 
  x^r, & \text{if } x \in (0, 1), \\
  x^{\alpha - \alpha p/2 - p/4}, & \text{if } x \in (1, \infty),
\end{cases}
\]
conditions (2) and \( \Psi \in A_p \) will hold if
\[
\begin{cases} 
  \alpha - \alpha p - p \leq r \leq \alpha, \\
  -1 < r < p - 1, \\
  -1 < \alpha - \alpha p/2 - p/4 < p - 1.
\end{cases}
\]
The third line is equivalent to
\[
\frac{2\alpha + 1}{4} \ p < \alpha + 1, \quad \alpha + 1 < \frac{2\alpha + 5}{4} \ p,
\]
and these hold, by the hypothesis. For the inequalities involving \( r \) we only need
\[
\max\{-1, \alpha - \alpha p - p\} < \min\{p - 1, \alpha\}.
\]
It is easy to check that this also holds, whenever \( \alpha > -1 \) and \( p > 1 \).

(III) Uniform boundedness of the operators \( W_{3,n} \). Here,
\[
\|W_{3,n} f\|_{L^p(x^\alpha)} \leq C \|f\|_{L^p(x^\alpha)}
\]
if and only if
\[
\|Hg\|_{L^p(x^\alpha - \alpha p/2 + p/2, J_{\nu}(x^{1/2})|^{-p})} \leq C \|g\|_{L^p(x^\alpha - \alpha p/2, J_{\nu}(x^{1/2})|^{-p})}.
\]
We make now use of the bounds
\[
|J_\nu(x)| \leq C x^{-1/4} \left[ |x - \nu| + \nu^{1/3} \right]^{-1/4}, \quad \nu = \alpha + 2n + 2, \ x \in (0, \infty),
\]
\[
|J_\nu'(x)| \leq C x^{-3/4} \left[ |x - \nu| + \nu^{1/3} \right]^{1/4}, \quad \nu = \alpha + 2n + 2, \ x \in (0, \infty),
\]
with some universal constant \( C \). They follow from those of [5], for instance. Therefore,
\[
x^{\alpha - \alpha p/2 + p/2} |J_\nu'(x^{1/2})|^p \leq C x^{\alpha - \alpha p/2 + p/8} \left[ |x^{1/2} - \nu| + \nu^{1/3} \right]^{p/4},
\]
\[
x^{\alpha - \alpha p/2} |J_\nu(x^{1/2})|^{-p} \geq C x^{\alpha - \alpha p/2 + p/8} \left[ |x^{1/2} - \nu| + \nu^{1/3} \right]^{p/4}.
\]
It will suffice to prove that $\varphi_\nu \in A_p$ uniformly in $n$, with

$$\varphi_\nu(x) = x^{\alpha - \alpha p/2 + p/8} \left[ |x^{1/2} - \nu| + \nu^{1/3} \right]^{-p/4}. \quad (3)$$

From Lemma 1, we have

$$\varphi_\nu(x) \in A_p \text{ unif. } \iff \varphi_\nu(\nu^2 x) \in A_p \text{ unif.}$$

$$\iff x^{\alpha - \alpha p/2 + p/8} \left| |x^{1/2} - 1| + \nu^{-2/3} \right|^{-p/4} \in A_p \text{ unif.}$$

$$\iff x^{\alpha - \alpha p/2 + p/8} |x^{1/2} - 1|^{p/4} + \nu^{-p/6} x^{\alpha - \alpha p/2 + p/8} \in A_p \text{ unif.,}$$

where the last equivalence follows from

$$\left| |x^{1/2} - 1| + \nu^{-2/3} \right|^{-p/4} \sim |x^{1/2} - 1|^{p/4} + \nu^{-p/6},$$

i.e., the ratio of both terms is bounded below and above by two positive constants not depending on $n$ or $x$. Now, again by Lemma 1, proving that $x^{\alpha - \alpha p/2 + p/8} \in A_p$ and $x^{\alpha - \alpha p/2 + p/8} |x^{1/2} - 1|^{p/4} \in A_p$ will suffice. According to Lemma 2,

$$\begin{cases}
  x^{\alpha - \alpha p/2 + p/8} \in A_p \\
  x^{\alpha - \alpha p/2 + p/8} |x^{1/2} - 1|^{p/4} \in A_p
\end{cases} \iff \begin{cases}
  -1 < \alpha - \alpha p/2 + p/8 < p - 1 \\
  -1 < \alpha - \alpha p/2 + p/4 < p - 1 \\
  -1 < \alpha - \alpha p/2 + p/8 \\
  4/3 < p \\
 \frac{2\alpha - 1}{4} p < \alpha + 1 < \frac{2\alpha + 3}{4} p \\
  4/3 < p
\end{cases}$$

and these inequalities follow from the initial conditions.

(IV) Uniform boundedness of the operators $W_{4,n}$. Finally,

$$\|W_{4,n}f\|_{L^p(x^\alpha)} \leq C \|f\|_{L^p(x^\alpha)}$$

if and only if

$$\|Hg\|_{L^p(x^{\alpha - \alpha p/2} |J_\nu(x^{1/2})|^p)} \leq C \|g\|_{L^p(x^{\alpha - \alpha p/2 - p/2} |J_\nu'(x^{1/2})|^{-p})}.$$  

Also,

$$x^{\alpha - \alpha p/2} |J_\nu(x^{1/2})|^p \leq C x^{\alpha - \alpha p/2 - p/8} \left[ |x^{1/2} - \nu| + \nu^{1/3} \right]^{-p/4},$$

$$x^{\alpha - \alpha p/2 - p/2} |J_\nu'(x^{1/2})|^{-p} \geq C x^{\alpha - \alpha p/2 - p/8} \left[ |x^{1/2} - \nu| + \nu^{1/3} \right]^{-p/4}$$

so let us put

$$\psi_\nu(x) = x^{\alpha - \alpha p/2 - p/8} \left[ |x^{1/2} - \nu| + \nu^{1/3} \right]^{-p/4} \quad (4)$$
and show that $\psi_\nu \in A_p$ uniformly in $n$. Indeed,

$$\psi_\nu(x) \in A_p \text{ unif. } \iff \psi_\nu(\nu^2x) \in A_p \text{ unif.}$$

$$\iff x^{\alpha - \alpha p/2 - p/8} \left[|x^{1/2} - 1| + \nu^{-2/3}\right]^{-p/4} \in A_p \text{ unif.}$$

and

$$\left( x^{\alpha - \alpha p/2 - p/8} \left[|x^{1/2} - 1| + \nu^{-2/3}\right]^{-p/4} \right)^{-1} \sim x^{-\alpha + \alpha p/2 + p/8} \left[|x^{1/2} - 1|^{p/4} + \nu^{-p/6}\right]$$

$$= \left[x^{\alpha - \alpha p/2 - p/8} |x^{1/2} - 1|^{-p/4}\right]^{-1} + \left[\nu^{p/6} x^{\alpha - \alpha p/2 - p/8}\right]^{-1}$$

so that proving that $x^{\alpha - \alpha p/2 - p/8} |x^{1/2} - 1|^{-p/4} \in A_p$ and $x^{\alpha - \alpha p/2 - p/8} \in A_p$ will suffice. But

$$\left\{\begin{array}{l} x^{\alpha - \alpha p/2 - p/8} \in A_p \\ x^{\alpha - \alpha p/2 - p/8} |x^{1/2} - 1|^{-p/4} \in A_p \end{array}\right. \iff \left\{\begin{array}{l} -1 < \alpha - \alpha p/2 - p/8 < p - 1 \\ -1 < -p/4 < p - 1 \\ -1 < \alpha - \alpha p/2 - p/4 < p - 1 \\ \alpha - \alpha p/2 - p/8 < p - 1 \\ p < 4 \\ -1 < \alpha - \alpha p/2 - p/4 \\ 2a+1/4, p < \alpha + 1 < \frac{2a+9/2}{4} \end{array}\right. \iff \left\{\begin{array}{l} 2a+1/4, p < \alpha \end{array}\right. \text{ and these inequalities hold by the hypothesis. The proof of Theorem 1 is now complete.} \quad \square$$

**Proof of Theorem 2.** For each $n \geq 0$ and $b \in \text{BMO}$, $T_n(b) : L^p(x^\alpha) \rightarrow L^p(x^\alpha)$ is bounded if and only if $S_n : L^p(e^{pb}x^\alpha) \rightarrow L^p(e^{pb}x^\alpha)$ is bounded, and both operators have the same norm. Thus, we can follow the proof of Theorem 1 and conclude that conditions (1), (2), (3) and (4), i.e.,

$$C x^{\alpha - \alpha p/2 + p/2} |J_{\alpha+1}(x^{1/2})|^p \leq \Phi(x) \leq C x^{\alpha - \alpha p/2} |J_\alpha(x^{1/2})|^{-p},$$

$$C x^{\alpha - \alpha p/2} |J_\alpha(x^{1/2})|^p \leq \Phi(x) \leq C x^{\alpha - \alpha p/2 - p/2} |J_{\alpha+1}(x^{1/2})|^{-p},$$

$$\varphi_\nu(x) = x^{\alpha - \alpha p/2 + p/8} \left[|x^{1/2} - \nu| + \nu^{1/3}\right]^{p/4},$$

$$\psi_\nu(x) = x^{\alpha - \alpha p/2 - p/8} \left[|x^{1/2} - \nu| + \nu^{1/3}\right]^{-p/4},$$

are still sufficient, if we require now $e^{pb}\Phi, e^{pb}\Psi, e^{pb}\varphi_\nu, e^{pb}\psi_\nu \in A_p$ uniformly in $\nu$. The proof of Theorem 1, together with next lemma, finish the proof of Theorem 2. \quad \square
Lemma 4. Let $1 < p < \infty$. For each $\phi \in A_p$, there exists some $\delta > 0$ such that $e^{pb}\phi \in A_p$ whenever $b \in \text{BMO}$ with $\|b\|_{\text{BMO}} < \delta$. Moreover, $\delta$ and the $A_p$ constant of $e^{pb}\phi$ depend only on the $A_p$ constant of $\phi$.

Remark. Again, statements like “$\delta$ depends only on the $A_p$ constant of $\phi$” should be understood as: given a constant $C > 0$ which verifies the $A_p$ condition for $\phi$, some $\delta$ can be chosen depending only on $C$.

Proof. If $\phi \in A_p$, there exists some $\varepsilon > 1$ such that $\phi^\varepsilon \in A_p$; moreover, $\varepsilon$ and the $A_p$ constant of $\phi^\varepsilon$ depend only on the $A_p$ constant of $\phi$ [6, Theorem IV.2.7, p. 399]. Take now $1/\varepsilon + 1/\varepsilon' = 1$. There exists some $\delta > 0$ such that

$$\|b\|_{\text{BMO}} < \delta \implies e^{p\varepsilon'b} \in A_p;$$

here, $\delta$ and the $A_p$ constant of $e^{p\varepsilon'b}$ depend only on $\varepsilon'$ [6, p. 409]. This, together with $\phi^\varepsilon \in A_p$ and Hölder’s inequality, imply $e^{pb}\phi \in A_p$ with an $A_p$ constant depending only on the $A_p$ constant of $\phi$. □

References

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