Rational values of the arccosine function*

Juan L. Varona†

Departamento de Matemáticas y Computación,
Universidad de La Rioja,
Edificio Vives, Calle Luis de Ulloa s/n, 26004 Logroño, Spain

Received 7 November 2005; accepted 8 February 2006

Abstract

We give a short proof to characterize the cases when $\arccos(\sqrt{r})$, the arccosine of the square-root of a rational number $r \in [0,1]$, is a rational multiple of π: This happens exactly if r is an integer multiple of $1/4$. The proof relies on the well-known recurrence relation for the Chebyshev polynomials of the first kind.

Keywords: Arccosine, cosine, rational, irrational.
MSC (2000): 11J72, 33B10.

The arithmetic properties of trigonometric functions has been a recurring topic in the mathematical literature. In 1933, D. H. Lehmer [2] proved that if $d > 2$ and k/d is an irreducible fraction, then $2\cos(2\pi k/d)$ is an algebraic integer of degree $\varphi(d)/2$ (with $\varphi(d)$ being Euler's totient function); the proof can be also found in [3, Theorem 3.9]. As a consequence, it can be shown that, for $t \in \mathbb{Q}$, the only rational values of $\cos(\pi t)$ are $\cos(\pi t) = 0, \pm 1, \pm 1/2$. But this can be proved independently of Lehmer's result; see also [3, Chapter 3] for historical references. Another nice and self-contained proof appears in [4, § 6.3, Theorem 6.16].

In [1, Chapter 6], as a key step for the construction of Dehn's counterexamples to Hilbert's third problem about decomposing polyhedra, it is established that

$$\frac{1}{\pi}\arccos\left(\frac{1}{\sqrt{n}}\right) \notin \mathbb{Q} \quad \text{when } n \in \mathbb{N}, \quad n \text{ odd}, \quad n \geq 3. \tag{1}$$

The aim of this paper is to give a direct and simple proof of a much more general result: the complete characterization of the $r \in \mathbb{Q}$ such that

$$\frac{1}{\pi}\arccos\left(\sqrt{r}\right) \in \mathbb{Q}.$$

*Research partially supported by grant BFM2003-06335-C03-03 of the DGI (Spain).
†Email: `jvarona@dmc.unirioja.es`
This paper has been published in: *Cent. Eur. J. Math.* **4** (2006), no. 2, 319–322.

Let us explain the idea of our proof. The elegant proof of (1) given in [1, Chapter 6] is based in the trigonometric identity

$$\cos((k+1)\theta) = 2\cos(\theta)\cos(k\theta) - \cos((k-1)\theta), \tag{2}$$

which is an immediate consequence of

$$\cos(\alpha) + \cos(\beta) = 2\cos(\tfrac{\alpha+\beta}{2})\cos(\tfrac{\alpha-\beta}{2}).$$

For even n, a different method is suggested in that book, distinguishing between the cases $n = 2^j$ and n not a power of 2. Thus, it is obtained that $(1/\pi)\arccos(1/\sqrt{n})$ is rational if and only if $n \in \{1, 2, 4\}$.

The relation (2) can be read in term of Chebyshev polynomials of the first kind. These polynomials are well known in the mathematical literature (see, for instance, [6] or [5]), mainly for their importance in approximation theory (they are orthogonal polynomials, and are also used in least squares fit and in quatrature formulas for numerical integration). At least for this author, polynomial relations are easier to handle than trigonometric relations and, thus, the proof given in [1, Chapter 6] seems to be clearer when written in terms of polynomials. Moreover, this allows a useful generalization of (1) whose proof does not lose the simplicity of [1]; concretely, in this way we show that

$$\frac{1}{\pi}\arccos\left(\frac{m}{2\sqrt{nM}}\right) \notin \mathbb{Q}$$

when $n, m, M \in \mathbb{N}$, $n \geq 2$, $\gcd(n, m) = 1$ and $m/(2\sqrt{nM}) < 1$.

Finally, the square root of every positive rational number $r < 1$ can be written as $\sqrt{r} = m/(2\sqrt{nM})$ with $\gcd(n, m) = 1$, with the exceptions of $\sqrt{r} = 1$, $1/2$, $1/\sqrt{2}$ and $\sqrt{3}/2$. Thus, we conclude that, for $r \in \mathbb{Q}$ with $0 \leq r \leq 1$, the number $(1/\pi)\arccos(\sqrt{r})$ is irrational except in the cases arising from these values of the cosine function:

$$\begin{aligned} &\cos(0) = 1, \quad \cos(\pi/6) = \sqrt{3}/2, \quad \cos(\pi/4) = 1/\sqrt{2}, \\ &\cos(\pi/3) = 1/2, \quad \text{and} \quad \cos(\pi/2) = 0. \end{aligned} \tag{3}$$

Remark. As an easy consequence, for $t \in \mathbb{Q}$, the only possible rational values of $\cos^2(\pi t)$ are given by $\cos(\pi t) = \pm 1$, $\pm\sqrt{3}/2$, $\pm 1/\sqrt{2}$, $\pm 1/2$ and 0. Actually, this can also be proved by using that the only rational values of $\cos(\pi t)$ are 0, ± 1, and $\pm 1/2$, and the relation $\cos^2(\theta) = (1 + \cos(2\theta))/2$; or can be derived from Lehmer's result by searching algebraic integers of degree at most 2 of $\cos(\pi t)$. But it seems that none of these facts about $\cos^2$ have been noticed in the literature. In any case, we are giving a direct and new proof of this result. Of course, using the elementary trigonometric relations $\cos^2(\theta) = 1 - \sin^2(\theta)$ and $\cos^2(\theta) = 1/(1 + \tan^2(\theta))$, similar results for the rational values of $\sin^2(\pi t)$ and $\tan^2(\pi t)$ can be obtained.

Thus, let us state

Theorem. *Let $r \in \mathbb{Q}$ with $0 \leq r \leq 1$. Then, the number*

$$\frac{1}{\pi}\arccos\left(\sqrt{r}\right)$$

is rational if and only if r is 0, 1/4, 1/2, 3/4, or 1; and the same holds for $(1/\pi)\arcsin(\sqrt{r})$.

Proof. Noticing that $\arccos(x) + \arcsin(x) = \pi/2$, it is enough to analyze the case arccos. Even more, the "if" part is clear from the trigonometric values shown in (3), so let us study the "only if" part.

We claim that every $r \in \mathbb{Q} \setminus \{0, 1/4, 2/4, 3/4, 1\}$, $r \geq 0$, can be written as

$$r = \frac{m^2}{4nM} \tag{4}$$

with conditions $n, m, M \in \mathbb{N}$, $n \geq 2$ and $\gcd(n, m) = 1$. This is true because given $r = p/q$, with p and q co-prime and q not a divisor of 4, there are only two possibilities:

- if q has an odd divisor n, say $q = nM'$, we can write $r = (2p)^2/(4nM)$ (with $M = pM'$);
- if q is a power of 2, then $q = 2^j$ with $j \geq 3$, p is odd and we have $r = p^2/(4 \cdot 2^{j-2}p)$ (with $n = 2^{j-2} \geq 2$).

In both cases we have found the decomposition (4). Then, we only need to prove that

$$A(n, m, M) = \frac{1}{\pi} \arccos\left(\frac{m}{2\sqrt{nM}}\right) \notin \mathbb{Q}$$

when $n, m, M \in \mathbb{N}$, $n \geq 2$, $\gcd(n, m) = 1$ and $m/(2\sqrt{nM}) < 1$.

For $x \in [-1, 1]$ and $k \in \mathbb{N} \cup \{0\}$, let $T_k(x)$ be

$$T_k(\cos(\theta)) = \cos(k\theta), \qquad x = \cos(\theta).$$

It is immediate that $T_0(x) = 1$ and $T_1(x) = x$. Moreover, the trigonometric relation

$$\cos((k+1)\theta) = 2\cos(\theta)\cos(k\theta) - \cos((k-1)\theta)$$

proves the recurrence formula

$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x), \quad k \geq 1.$$

In particular, this implies that $T_k(x)$ is a polynomial of degree k, and so we get the so-called Chebyshev polynomials of the first kind. Making the substitution $g_k(x) = 2T_k(mx/2)$, we get

$$\begin{cases} g_0(x) = 2, \quad g_1(x) = mx, \\ g_{k+1}(x) = mxg_k(x) - g_{k-1}(x), \quad k \geq 1. \end{cases} \tag{5}$$

Then, $g_k(x)$ is a polynomial of degree k and coefficients in $\mathbb{Z}$, and it verifies

$$2\cos(k\theta) = 2T_k(\cos(\theta)) = g_k(2\cos(\theta)/m). \tag{6}$$

Now, let us take

$$\theta = \arccos\left(\frac{m}{2\sqrt{nM}}\right), \quad x = \frac{1}{2\sqrt{nM}}, \quad \cos(\theta) = mx.$$

By (6), we have

$$2\cos(k\theta) = g_k(2x) = g_k\left(\frac{1}{\sqrt{nM}}\right) = \frac{B_k}{(\sqrt{nM})^k} \tag{7}$$

for some $B_k \in \mathbb{Z}$. From (5) it is easy to check that $B_0 = 2$, $B_1 = m$, and $B_{k+1} = mB_k - nMB_{k-1}$. Now, let us recall that $\gcd(n, m) = 1$. Then, by induction on k (starting with $k = 1$), it follows that n does not divide B_k for any $k \geq 1$.

To conclude the proof, let us suppose that $A(n, m, M) = (1/\pi)\theta = h/k \in \mathbb{Q}$. Then, $k\theta = h\pi$ and, by (7),

$$\pm 2 = 2\cos(h\pi) = 2\cos(k\theta) = \frac{B_k}{(\sqrt{nM})^k}, \quad B_k \in \mathbb{Z}.$$

This implies that n divides B_k, which is a contradiction. □

Acknowledgment

Thanks to Professors Günter M. Ziegler, Jaime Vinuesa, and the referee, for their interest and their valuable comments, that allowed to improve this paper.

References

[1] M. Aigner and G. M. Ziegler, *Proofs from THE BOOK*, 3rd edition, Springer, 2004.

[2] D. H. Lehmer, A note on trigonometric algebraic numbers, *Amer. Math. Monthly* **40** (1933), 165–166.

[3] I. Niven, *Irrational numbers*, Carus Monographs no. 11, The Mathematical Association of America (distributed by John Wiley and Sons), 1956.

[4] I. Niven, H. S. Zuckerman, and H. L. Montgomery, *An introduction to the theory of numbers*, 5th edition, Wiley, 1991.

[5] N. M. Temme, *Special functions: an introduction to the classical functions of mathematical physics*, John Wiley and Sons, 1996.

[6] E. W. Weisstein, Chebyshev polynomial of the first kind. From *MathWorld*–A Wolfram Web Resource, `http://mathworld.wolfram.com/ChebyshevPolynomialoftheFirstKind.html`.