Some conjectures on Wronskian and Casorati determinants of orthogonal polynomials*

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Abstract

In this paper we conjecture some regularity properties for the zeros of Wronskian and Casorati determinants whose entries are orthogonal polynomials. These determinants are formed by choosing orthogonal polynomials whose degrees run on a finite set F of nonnegative integers. The case when F is formed by consecutive integers was studied by Karlin and Szegő.

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1 Introduction

Wronskian and Casorati determinants whose entries are orthogonal polynomials have been considered in the literature from long time ago. For instance, Karlin and Szegő considered such determinants in their celebrated paper [Karlin and Szegő 60/61], devoted to extending Turán's inequality for Legendre polynomials [Turán 50] to Hankel determinants whose entries are ultraspherical, Laguerre and Hermite polynomials (see, also, [Karlin and McGregor 59], [Dimitrov 98], [Ismail 05], [Ismail and Laforgia 07], [Felder et al. 12], [Filipuk and Zhang 14]).

Wronskian and Casorati determinants whose entries are classical orthogonal polynomials are nowadays receiving increasing interest because of their role

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in the construction of the so-called exceptional and exceptional discrete polynomials. Exceptional polynomials p_n, $n \in X \subsetneq \mathbb{N}$, are complete orthogonal polynomial systems with respect to a positive measure which in addition are eigenfunctions of a second order differential operator. They extend the classical families of Hermite, Laguerre and Jacobi. The most apparent difference between classical polynomials and exceptional polynomials is that the exceptional families have gaps in their degrees, in the sense that not all degrees are present in the sequence of polynomials. In the same way, exceptional discrete polynomials are complete orthogonal polynomial systems with respect to a positive measure which in addition are eigenfunctions of a second order difference operator, extending the discrete classical families of Charlier, Meixner, Krawtchouk and Hahn. The last few years have seen a great deal of activity in the area of exceptional polynomials (see [Durán 14a], [Durán 14b], [Gómez-Ullate et al. 09], [Gómez-Ullate et al. 14a], [Sasaki et al. 10], and the references therein).

It turns out that the regularity of the second order differential operator associated to exceptional polynomials (no singularities in the operator's domain) is very much related with the zeros of certain Wronskians constructed from classical polynomials. And the same role is played by Casorati determinants constructed from classical discrete polynomials in relation with the regularity of the second order difference operator associated to exceptional discrete polynomials.

The purpose of this paper is to conjecture some regularity properties for the zeros of Wronskian and Casorati determinants whose entries are orthogonal polynomials. These conjectures are displayed in Sections 2 (Wronskian) and 3 (Casorati), while in Section 4 we discuss all the evidences we have found to give support to these conjectures.

2 Conjectures on Wronskians

Along this paper, $\mathfrak{M}_K$ denotes the set of positive measures μ on $\mathbb{R}$ with finite moments of order i, $0 \leq i \leq 2K$, so that we can integrate polynomials of degree less than or equal to $2K$ with respect to μ. We also assume that μ has at least $2K$ points in its support. Hence, each measure $\mu \in \mathfrak{M}_K$ has associated a sequence of polynomials $(p_n)_{n=0}^K$, p_n of degree n, orthogonal with respect to μ and with positive norm. We normalize the sequence $(p_n)_{n=0}^K$ by assuming that the leading coefficient of each p_n is equal to 1.

From now on, F will denote a finite set of nonnegative integers. We will write $F = \{f_1, \ldots, f_k\}$, with $f_i < f_{i+1}$. Hence k is the number of elements of F and f_k is the maximum element of F. A finite set of (nonnegative) consecutive integers will be called a segment.

Given a finite set F of nonnegative integers and a measure $\mu \in \mathfrak{M}_{\max F}$, we consider the $k \times k$ Wronskian W_F^μ defined by

$$W_F^\mu(x) = |p_{f_i}^{(j-1)}(x)|_{i,j=1,\ldots,k}. \tag{2.1}$$

To simplify the notation, we sometimes write $W_F = W_F^\mu$.

Karlin and Szegő studied the zeros of these Wronskian determinants when F is a segment. More precisely, they proved the following theorem (see [Karlin and Szegő 60/61]):

Theorem 2.1 (Theorems 1 and 2 of [Karlin and Szegő 60/61])**.** *Given the segment $F_n = \{n, n+1, \dots, n+k-1\}$, where n is a nonnegative integer, then:*

1. *If k is even the Wronskian $W^{\mu}_{F_n}$ has no zeros in $\mathbb{R}$.*
2. *If k is odd the Wronskian $W^{\mu}_{F_n}$ has exactly n simple zeros in $\mathbb{R}$. Moreover, the zeros lie in the convex hull of* $\operatorname{supp}\mu$, *and the zeros of W_{F_n} separate the zeros of $W_{F_{n+1}}$.*

Karlin and Szegő's proof for this result shows that, when F is a segment, this regularity for the zeros of the Wronskian W^{μ}_{F} is a relative of the usual regularity properties for the zeros of a sequence of orthogonal polynomials with respect to a positive measure.

In this section, we conjecture some regularity properties for the zeros of the Wronskian W^{μ}_{F} when F is not a segment.

Using standard determinantal techniques, it is not difficult to see that, for any set F, the Wronskian W^{μ}_{F} is a polynomial of degree

$$\deg W^{\mu}_{F} = \sum_{f\in F} f - \frac{k(k-1)}{2}, \tag{2.2}$$

and leading coefficient equal to $\prod_{1\le i<j\le k}(f_j - f_i)$. Hence, neither the degree nor the leading coefficient of the Wronskian W^{μ}_{F} depend on the measure μ. On the other hand, with the normalization of the set F, the leading coefficient of W^{μ}_{F} is always positive.

Our first conjecture is related to the location of the zeros of W^{μ}_{F}. We guess that Karlin and Szegő's result on the zero location for the Wronskian of segments is true for any finite set F of nonnegative integers.

Conjecture 2.2. *Given a finite set F of nonnegative integers and a measure $\mu \in \mathfrak{M}_{\max F}$, the real zeros of the Wronskian W^{μ}_{F} (if any) always lie in the convex hull of* $\operatorname{supp}\mu$.

To study the number of real zeros of the Wronskian W^{μ}_{F}, we have first to clarify how to count the zeros of a Wronskian. Denote by $\mathcal{W}^{\mu}_{F}$ the Wronskian matrix $\mathcal{W}^{\mu}_{F}(x) = (p^{(j-1)}_{f_i}(x))_{i,j=1,\dots,k}$. From this definition, it follows at once that x_0 is a zero of the Wronskian W^{μ}_{F} if and only if 0 is an eigenvalue of $\mathcal{W}^{\mu}_{F}(x_0)$.

Definition 2.3. We define the geometric multiplicity of a zero x_0 of the Wronskian W^{μ}_{F} as the geometric multiplicity of 0 as an eigenvalue of the $k\times k$ matrix $\mathcal{W}^{\mu}_{F}(x_0)$. That is, the geometric multiplicity of x_0 is $k - \operatorname{rank}\mathcal{W}^{\mu}_{F}(x_0)$.

We have the following straightforward lemma.

Lemma 2.4. *The geometric multiplicity of a zero x_0 of the Wronskian W_F^μ is less than or equal to its usual multiplicity. In particular, if x_0 is a simple zero of W_F^μ then its geometric multiplicity is* 1.

According to this lemma, for simple zeros there is no difference between the geometric and the usual multiplicity. For multiple zeros the difference can be quite large. For instance, take a symmetric measure μ. This means that each orthogonal polynomial $p_{2n+1}(x)$, $n \geq 0$, is an odd function, and hence 0 is a root of $p_{2n+1}^{(2j)}(x)$, $j \geq 0$. Take now a set F formed by k odd numbers. A simple computation shows that the multiplicity of 0 as a root of W_F^μ is at least $\binom{k+1}{2}$; on the other hand, its geometric multiplicity is at most $k-1$.

We now introduce the so-called Wronskian zero counting set for F.

Definition 2.5. Let F be a finite set of nonnegative integers. The Wronskian zero counting set for F, denoted by Z_F, is formed by all nonnegative integers l for which there exists a measure $\mu \in \mathfrak{M}_{\max F}$ such that the Wronskian W_F^μ has exactly l real zeros, counted according to their geometric multiplicities (see Definition 2.3).

Since $\deg W_F = \sum_{f\in F} f - \frac{k(k-1)}{2}$, we trivially have

$$Z_F \subset \left\{0, 1, \dots, \sum_{f\in F} f - \frac{k(k-1)}{2}\right\}.$$

The key concept to study the Wronskian zero counting set Z_F is that of admissible sets. In order to define the admissibility of a set F, we first need to consider the segment decomposition of F:

Definition 2.6. Given a finite set of nonnegative integers F, the segment $\{0, 1, 2, \dots, \max F\}$ has a unique decomposition of the following form:

$$\{0, 1, 2, \dots, \max F\} = X_0 \cup Y_1 \cup X_1 \cup Y_2 \cup X_2 \cup \cdots \cup Y_{g_F} \cup X_{g_F},$$

where X_0 and X_i, Y_i, $i = 1, \dots, g_F$, satisfy

1. $X_0 = \emptyset$ if $0 \notin F$; otherwise, $0 \in X_0$ and X_0 is a segment;
2. all X_i and Y_i, $i = 1, \dots g_F$, are nonempty segments (that is, they are formed by consecutive numbers);
3. $F = X_0 \cup X_1 \cup \cdots \cup X_{g_F}$;
4. $Y_i \cap F = \emptyset$, $i = 1, \dots, g_F$;
5. $1 + \max Y_i = \min X_i$, $1 + \max X_i = \min Y_{i+1}$.

The segments X_i, $i = 0, \dots, g_F$, will be called the segment decomposition of F, while the segments Y_i, $i = 1, \dots, g_F$, will be called the complementary segment decomposition of F. The number g_F will be called the genre of F.

In other words: Y_i are the gaps in F, X_i are the maximal segments in F, and we start at X_0 if $0 \in F$, at X_1 otherwise.

All the segments X_0, Y_1, X_1, …, Y_{g_F}, X_{g_F} are pairwise disjoint.

Definition 2.7. A finite set of nonnegative integers F is called admissible if each segment X_i, $i = 1, \dots, g_F$, in the segment decomposition of F has an even number of elements. Notice that X_0 is not required to have an even number of elements.

It is easy to see that admissible sets F are also characterized from the following property: $\prod_{f\in F}(x-f) \geq 0$ for any nonnegative integer x. This concept of admissibility has appeared several times in the literature. Relevant to this paper are [Krein 57] and [Adler 94], because of the relationship with zeros of certain Wronskians. Indeed, consider a second order differential operator T of the form $T = -d^2/dx^2 + U$. Write ϕ_n, $n \geq 0$, for a sequence of eigenfunctions for T, and form the Wronskian $\Omega_F^T(x) = |\phi_{f_l}^{(j-1)}(x)|_{l,j=1}^k$. For operators defined on a half-line, and under certain boundary conditions, Krein proved [Krein 57] that F is admissible if and only if Ω_F^T does not vanish on the real line. A similar result was proved by Adler [Adler 94] for operators defined on a bounded interval (although Adler's result extends easily to the whole real line and, in fact, he considered in [Adler 94] the case of Wronskian determinants of Hermite polynomials). Admissibility appeared also in [Karlin and Szegő 60/61] in connection with the sign of some Casorati determinants, though in a subordinate form.

Our next conjecture can be considered an improvement of part (1) in Theorem 2.1 of Karlin and Szegő above.

Conjecture 2.8. *A finite set F of nonnegative integers is admissible if and only if $Z_F = \{0\}$. In other words, if and only if the Wronskian W_F^μ has no zeros in $\mathbb{R}$ for any measure $\mu \in \mathfrak{M}_{\max F}$.*

When F is not admissible, our conjecture is that the structure of Z_F is somehow determined by the distance of F to the admissible sets. This distance is defined as follows.

We will write $\mathcal{A}$ for the set formed by all admissible sets of nonnegative integers.

Definition 2.9. For a finite set F of nonnegative integers we define the distance of F to $\mathcal{A}$ by

$$d(F, \mathcal{A}) = \min\big\{l : \exists x_1, \dots, x_l \notin F \text{ such that } F \cup \{x_1, \dots, x_l\} \in \mathcal{A}\big\}.$$

Some easy consequences of this definition are included in the next lemma.

Lemma 2.10. *Let F be a finite set of nonnegative integers. Then we have*

$$d(F, \mathcal{A}) = \min\big\{l : \exists x_1, \dots, x_l \in F \text{ such that } F \setminus \{x_1, \dots, x_l\} \in \mathcal{A}\big\}.$$

Moreover, $d(F, \mathcal{A}) = |\{l \geq 1 : X_l \text{ has an odd number of elements}\}|$, where $F = X_0 \cup X_1 \cup \dots \cup X_{g_F}$ is the segment decomposition of F.

According to this Lemma 2.10, if $d(F,\mathcal{A}) = m$, there are m positive integers $0 < l_1 < l_2 < \cdots < l_m \leq g_F$ such that X_{l_i} are the only odd segments in the segment decomposition of F (apart, possibly, from X_0). We set $l_0 = 0$ and write

$$\xi_j = \sum_{i=l_{j-1}+1}^{l_j} |Y_i|, \quad j = 1, \ldots, m, \tag{2.3}$$

where $\{Y_j\}$ is the complementary segment decomposition of F.

Definition 2.11. For a finite set of nonnegative integers F with $d(F,\mathcal{A}) = m$, the numbers ξ_i, $i = 1, \ldots, m$, defined by (2.3) are called the characteristic gap lengths of F.

The next conjecture characterizes the sets F with $Z_F = \{m\}$ for some $m \geq 1$ as those whose distance to the admissible sets is just 1.

Conjecture 2.12. *A finite set F of nonnegative integers satisfies $d(F,\mathcal{A}) = 1$ if and only if Z_F is a singleton other than $\{0\}$.*

When $d(F,\mathcal{A}) = 1$, we can also conjecture the value of the unique element of Z_F in terms of the characteristic gap lengths of F (see Definition 2.11).

Conjecture 2.13. *If $d(F,\mathcal{A}) = 1$ then*

$$Z_F = \{\xi_1\}.$$

We will prove in Section 4 (see Theorem 4.3) that conjecture 2.13 is a consequence of conjeture 2.8.

The structure of Z_F is more complicated when $d(F,\mathcal{A}) \geq 2$. Indeed, according to Conjectures 2.8 and 2.12, the number of zeros of W_F^μ does not depend on μ when $d(F,\mathcal{A})$ equals 0 or 1. This situation changes when $d(F,\mathcal{A}) \geq 2$. Anyway, there seems to be still some regularity in the structure of Z_F.

Conjecture 2.14. *Let F be a finite set of nonnegative integers satisfying $d(F,\mathcal{A}) = m \geq 2$. Then*

1. *Z_F is formed either by consecutive odd numbers or by consecutive even numbers, depending on whether $\deg W_F$ (see (2.2)) is odd or even, respectively;*
2. *$d(F,\mathcal{A}) \leq |Z_F| \leq \sum_{f\in F} f - \frac{k(k-1)}{2}$.*

When one counts multiple zeros according to the usual multiplicity, the number of real zeros of a polynomial with real coefficients has always the same parity as the degree of the polynomial, but this no longer holds when the geometric multiplicity is used. Since we are using geometric multiplicity, it is not clear at all that Z_F is formed by numbers with the same parity as $\deg W_F$, and this statement is hence a genuine part of the conjecture.

According to Conjecture 2.14, the set Z_F is completely described by its maximum and minimum values, $\max Z_F$ and $\min Z_F$.

For $m = 2$, our conjecture for these values is:

Conjecture 2.15. *If $d(F, \mathcal{A}) = 2$, then*

$$\begin{aligned} \max Z_F &= 2\xi_1 + \xi_2, \\ \min Z_F &= \xi_2. \end{aligned}$$

We will prove in Section 4 (see Theorem 4.4) that the bound $\max Z_F \leq 2\xi_1 + \xi_2$ in Conjecture 2.15 is a consequence of Conjecture 2.8.

Notice that in this case the set Z_F depends only on the characteristic gap lengths ξ_j, $j = 1, 2$, of F (see Definition 2.11). This is also the case when $d(F, \mathcal{A})$ equals 0 or 1. Is this dependence of Z_F only on the characteristic gap lengths of F true for all finite sets F? If the answer if yes, it would be quite a surprising result. Our numerical experiments are not conclusive enough neither to propose this result as a conjecture nor to forget it.

Our last conjecture is about the zeros of Wronskians whose entries belong to the most important families of orthogonal polynomials.

Conjecture 2.16. *Let F be a finite set of nonnegative integers with $d(F, \mathcal{A}) = m$ and write ξ_j, $j = 1, \ldots, m$, for its characteristic gap lengths (see Definition 2.11). If μ is any of the classical weights of Hermite, Laguerre or Jacobi, then the number of zeros of W_F^μ, counted according to their geometric multiplicities (see Definition 2.3) is always*

$$\sum_{j=1}^{[(m+1)/2]} \xi_{m-2j+2}.$$

In particular, Conjecture 2.16 is saying that the set W_F^μ does not depend on the parameters of the classical orthogonal polynomials. This is not true even if we change the classical families by any of the classical discrete families, which in many senses are so closely related. For instance, if we consider the Charlier measure

$$\mu = \sum_{n=0}^{\infty} \frac{a^n}{n!} \delta_n, \quad a > 0,$$

then the set W_F^μ depends on the parameter a. (For results concerning zeros of classical orthogonal polynomials see [Szegő 59, Ch. VI], [Driver el al. 11, Driver and Jordaan 11]).

After this paper was accepted for publication, we earned that this conjecture has been partially proved in [García-Ferrero and Gómez-Ullate 14b].

3 Conjectures on Casorati determinants

In this section we denote by $\tilde{\mathfrak{M}}_K$ the subset of $\mathfrak{M}_K$ formed by all discrete measures of the form

$$\mu = \sum_{n=0}^{\Upsilon} \alpha_n \delta_{a_n}, \tag{3.1}$$

where Υ is either a positive integer or infinity and, for every n, $\alpha_n > 0$, $a_n \in \mathbb{R}$ and $a_n < a_{n+1}$.

Given two finite sets F and G of nonnegative integers with k-elements satisfying

- $F = \{f_1, \dots, f_k\}$, $f_i < f_{i+1}$,
- $G = \{g_1, \dots, g_k\} \subseteq \{0, 1, \dots, \Upsilon\}$, $g_i < g_{i+1}$,

and a discrete measure $\mu \in \tilde{\mathfrak{M}}_{\max F}$ as in (3.1), we consider the $k \times k$ Casorati determinant $C_F^{G,\mu}$ defined by

$$C_F^{G,\mu} = |p_{f_i}(a_{g_j})|_{i,j=1,\dots,k}.$$

Notice that the Casorati determinant $C_F^{G,\mu}$ is a number, while the Wronskian W_F^μ (2.1) is a polynomial in x.

Karlin and Szegő studied the sign of these Casorati determinants when F is a segment. More precisely, they proved the following theorem (see [Karlin and Szegő 60/61]):

Theorem 3.1 (Theorem 3' of [Karlin and Szegő 60/61])**.** *Given the segment $F = \{m, m+1, \dots, m+k-1\}$, where m is a nonnegative integer, and k is an even positive integer, then the Casorati determinant $C_F^{G,\mu}$ is positive for any admissible set G and any discrete measure $\mu \in \tilde{\mathfrak{M}}_{\max F}$.*

We have found examples of admissible sets F and G for which there exist discrete measures $\mu \in \tilde{\mathfrak{M}}_{\max F}$ such that $C_F^{G,\mu}$ is negative. However, we can conjecture what can be considered a (strong) dual version of the previous theorem.

Conjecture 3.2. *A finite set F of nonnegative integers is admissible if and only if the Casorati determinant $C_F^{G,\mu}$ is positive for any segment G and any discrete measure $\mu \in \tilde{\mathfrak{M}}_{\max F}$.*

4 Discussion

4.1 "Logical" evidences for the conjectures on Wronskians

For any set F and any measure $\mu \in \mathfrak{M}_{\max F}$, let $Z(W_F^\mu)$ be the number of real zeros of the Wronskian W_F^μ counted according to their usual multiplicity. Notice that for $Z(W_F^\mu)$ we are counting the zeros in a different way that for Z_F, where the geometric multiplicity is used (see Definition 2.3).

Lemma 4.1. *Let F_1 and F_2 be two sets such that $F_1 = (F_1 \cap F_2) \cup \{p\}$, $F_2 = (F_1 \cap F_2) \cup \{q\}$ for some integers $p < q$, $p \notin F_2$, $q \notin F_1$. Then, for any measure $\mu \in \mathfrak{M}_{\max F_2}$,*

$$\left| Z(W_{F_2}^\mu) - Z(W_{F_1}^\mu) - 1 \right| \leq Z(W_{F_1 \cup F_2}^\mu) + Z(W_{F_1 \cap F_2}^\mu). \tag{4.1}$$

Moreover, every zero of $W_{F_1}^\mu W_{F_2}^\mu$ with multiplicity m is a zero of $W_{F_1 \cup F_2}^\mu W_{F_1 \cap F_2}^\mu$ with multiplicity at least $m - 1$.

Proof. The proof is based in the proof of Theorem 2 in [Karlin and Szegő 60/61, p. 6]. We write W_F instead of W_F^μ throughout the proof.

Consider the Wronskian $W_{F_1\cup F_2}$. Applying Sylvester's identity (see, for instance, [Karlin and Szegő 60/61, p. 26]) to the rows corresponding to the elements p and q of $F_1 \cup F_2$ and the last two columns gives

$$W_{F_1\cup F_2} W_{F_1\cap F_2} = W_{F_1} W'_{F_2} - W'_{F_1} W_{F_2}. \tag{4.2}$$

This proves the last part of the lemma: every zero of $W_{F_1} W_{F_2}$ with multiplicity m is a zero of $W_{F_1\cup F_2} W_{F_1\cap F_2}$ with multiplicity at least $m-1$. Furthermore, equation (4.2) gives

$$\left(\frac{W_{F_1}}{W_{F_2}}\right)' = -\frac{W_{F_1\cup F_2} W_{F_1\cap F_2}}{W_{F_2}^2}.$$

Let φ be the greatest common divisor of W_{F_1} and W_{F_2}, so that $W_{F_1} = \varphi g_1$, $W_{F_2} = \varphi g_2$, and g_1 and g_2 have no common zeros. Then

$$\left(\frac{g_1}{g_2}\right)' = -\frac{W_{F_1\cup F_2} W_{F_1\cap F_2}}{W_{F_2}^2}.$$

If $a_1 < a_2 < \cdots < a_m$ are the distinct poles of the rational function g_1/g_2, then in each interval (a_j, a_{j+1}) there must be at least one zero or one critical point of g_1/g_2. In other words, in each interval (a_j, a_{j+1}) there is at least one zero of g_1, $W_{F_1\cup F_2}$ or $W_{F_1\cap F_2}$. This proves that

$$\begin{aligned} m - 1 \le Z(g_1) &+ |\{\text{zeros of } W_{F_1\cup F_2} \text{ which are not zeros of } g_2\}| \\ &+ |\{\text{zeros of } W_{F_1\cap F_2} \text{ which are not zeros of } g_2\}|. \end{aligned} \tag{4.3}$$

Now, for each $j = 1, \ldots, m$, it follows from (4.2) that if a_j is a zero of g_2 with multiplicity r_j, then it is a zero of $W_{F_1\cup F_2} W_{F_1\cap F_2}$ with multiplicity at least $r_j - 1$. In other words,

$$\begin{aligned} Z(g_2) - m \le &|\{\text{zeros of } W_{F_1\cup F_2} \text{ which are zeros of } g_2\}| \\ &+ |\{\text{zeros of } W_{F_1\cap F_2} \text{ which are zeros of } g_2\}|. \end{aligned} \tag{4.4}$$

Adding (4.3) and (4.4) gives

$$Z(g_2) - 1 \le Z(g_1) + Z(W_{F_1\cup F_2}) + Z(W_{F_1\cap F_2})$$

and

$$Z(W_{F_2}) - 1 \le Z(W_{F_1}) + Z(W_{F_1\cup F_2}) + Z(W_{F_1\cap F_2}). \tag{4.5}$$

On the other hand, if $W_{F_1\cup F_2} W_{F_1\cap F_2}$ has k zeros in any interval (a_j, a_{j+1}), then g_1 has at most $k+1$ zeros there. In the intervals $(-\infty, a_1)$ and $(a_m, +\infty)$ the situation is slightly different: if $W_{F_1\cup F_2} W_{F_1\cap F_2}$ has k zeros in one of the intervals $(-\infty, a_1)$ and $(a_m, +\infty)$, then g_1 has at most k zeros there: the reason is that $\lim_{x\to\pm\infty} \frac{g_1}{g_2} = 0$. Therefore,

$$\begin{aligned} Z(g_1) \le &|\{\text{zeros of } W_{F_1\cup F_2} \text{ which are not zeros of } g_2\}| \\ &+ |\{\text{zeros of } W_{F_1\cap F_2} \text{ which are not zeros of } g_2\}| + m - 1. \end{aligned}$$

Adding (4.4) gives now

$$Z(g_1) + Z(g_2) - m \le Z(W_{F_1\cup F_2}) + Z(W_{F_1\cap F_2}) + m - 1.$$

Since $m \le Z(g_2)$, this leads to

$$Z(g_1) - Z(g_2) + 1 \le Z(W_{F_1\cup F_2}) + Z(W_{F_1\cap F_2})$$

and

$$Z(W_{F_1}) - Z(W_{F_2}) + 1 \le Z(W_{F_1\cup F_2}) + Z(W_{F_1\cap F_2}),$$

which, together with (4.5), proves (4.1). □

As a particular case, the following result states that if $d(F, \mathcal{A}) = 1$, moving the unique odd segment one place to the right increases the number of zeros of the Wronskian in one unit, for each measure. For each integer k, let $\tau_k(n) = n + k$.

Lemma 4.2. *Assume that every admissible set F has $Z_F = \{0\}$. Let F_1 be such that $d(F_1, \mathcal{A}) = 1$, with a segment decomposition*

$$F_1 = X_0 \cup X_1 \cup \cdots \cup X_l \cup \cdots \cup X_{g_{F_1}},$$

where for some l ($1 \le l \le g_{F_1}$) X_l is the unique odd segment apart, possibly, from X_0. Let

$$F_2 = X_0 \cup X_1 \cup \cdots \cup \tau_1(X_l) \cup \cdots \cup X_{g_{F_1}}.$$

All zeros of $W^{\mu}_{F_1}$ are simple. Moreover, if $W^{\mu}_{F_1}$ has m zeros for some measure $\mu \in \mathfrak{M}_{\max F_2}$, then $W^{\mu}_{F_2}$ has $m + 1$ zeros.

Proof. The sets F_1 and F_2 are under the hypothesis of Lemma 4.1, and both $F_1 \cap F_2$ and $F_1 \cup F_2$ are admissible sets. Therefore, (4.1) proves that $Z(W^{\mu}_{F_2}) = Z(W^{\mu}_{F_1}) + 1$. Since $W^{\mu}_{F_1\cup F_2} W^{\mu}_{F_1\cap F_2}$ has no zeros, the last part of Lemma 4.1 proves that all zeros of $W^{\mu}_{F_1}$ are simple. □

Then, the assumption that admissible sets F have $Z_F = \{0\}$ proves Conjecture 2.13:

Theorem 4.3. *Assume that every admissible set F has $Z_F = \{0\}$. Then, for every set F of nonnegative integers with $d(F, \mathcal{A}) = 1$, the Wronskian zero counting set is*

$$Z_F = \{\xi_1\}. \tag{4.6}$$

Proof. According to Lemma 4.2 the zeros of W^{μ}_F are simple, so the geometric and the usual multiplicity coincides, and hence we count the zeros in Z_F and $Z(W^{\mu}_F)$ in the same way.

Since $d(F, \mathcal{A}) = 1$, the segment decomposition of F is of the form

$$F = X_0 \cup X_1 \cup \cdots \cup X_l \cup \cdots \cup X_{g_F},$$

where for some l $(1 \leq l \leq g_F)$ X_l is the unique odd segment apart, possibly, from X_0.

Lemma 4.2 shows that $Z_F = \{m\}$ for some $m \geq 1$ if and only if the set

$$F_{\pm 1} = X_0 \cup X_1 \cup \cdots \cup \tau_{\pm 1}(X_l) \cup \cdots \cup X_{g_F}$$

has Wronskian zero counting set $Z_{F_{\pm 1}} = \{m \pm 1\}$. Moreover, m is given by (4.6) if and only if $m \pm 1$ corresponds to $F_{\pm 1}$ by the same rule. In other words, we can move the odd segment X_l to the left and to the right one position.

We proceed by induction on the number g_F of segments. If $g_F = 1$, then either

$$F = X_1 = \{m, m+1, \ldots, m+k\}$$

for some $m \geq 1$ and an even integer k, or

$$F = X_0 \cup X_1 = \{0, \ldots, n\} \cup \{m, m+1, \ldots, m+k\}$$

with $n + 1 < m$ and an even integer k. In the first case, Theorem 2 of [Karlin and Szegő 60/61, p. 6] proves that the Wronskian W_F has exactly m zeros. That is, $Z_F = \{m\}$, which is consistent with (4.6). In the second case, the set

$$F_{n+1-m} = X_0 \cup \tau_{n+1-m}(X_1) = \{0, 1, \ldots, n+1+k\}$$

has Wronskian zero counting set $Z_{F_{n+1-m}} = \{0\}$. An iterated application of Lemma 4.2 shows that $Z_F = \{m - n - 1\}$, which is consistent with (4.6).

Let us now suppose that $g_F > 1$. Assume, for instance, that X_l is not the rightmost segment of F; if

$$k + 1 + \max X_l = \min X_{l+1},$$

then repeating the moves we have that $Z_F = \{m\}$ if and only if the set

$$F_k = X_0 \cup X_1 \cup \cdots \cup \tau_k(X_l) \cup X_{l+1} \cup \cdots \cup X_{g_F}$$

has Wronskian zero counting set $Z_{F_k} = \{m + k\}$. Now, $\tau_k(X_l) \cup X_{l+1}$ is a consecutive segment, so that F_k decomposes in $g_F - 1$ segments, exactly one of them being odd. Thus, Z_{F_k} is given by the analogous to formula (4.6) and the same holds for Z_F.

A similar procedure works if the odd segment X_l is not X_1: in this case we arrive at a set

$$F_{-k} = X_0 \cup X_1 \cup \cdots \cup X_{l-1} \cup \tau_{-k}(X_l) \cup \cdots \cup X_{g_F}$$

which decomposes in $g_F - 1$ segments, exactly one of them being odd. □

The assumptions that admissible sets F have $Z_F = \{0\}$ proves also part of Conjecture 2.15:

Theorem 4.4. *Assume that every admissible set F has $Z_F = \{0\}$. Then, for every set F of nonnegative integers with $d(F,\mathcal{A}) = 2$ the Wronskian zero counting set Z_F satisfies the inequality*

$$\max Z_F \le 2\xi_1 + \xi_2, \tag{4.7}$$

where ξ_1 and ξ_2 are the characteristic gap lengths of F (see Definition 2.11).

Proof. Let

$$F = X_0 \cup X_1 \cup \cdots \cup X_{l_1} \cup \cdots \cup X_{l_2} \cup \cdots \cup X_n$$

be the segment decomposition of F, where X_{l_1} and X_{l_2} are the unique odd segments apart, possibly, from X_0. We choose F_1 by removing the first element of X_{l_2} and adding the first element after X_{l_1}:

$$\begin{aligned} \tilde{X}_{l_1} &= X_{l_1} \cup \{1 + \max X_{l_1}\}, \\ \tilde{X}_{l_2} &= X_{l_2} \setminus \{\min X_{l_2}\}, \\ F_1 &= X_0 \cup X_1 \cup \cdots \cup \tilde{X}_{l_1} \cup \cdots \cup \tilde{X}_{l_2} \cup \cdots \cup X_n. \end{aligned}$$

Observe that this decomposition of F_1 might not be the segment decomposition, for $\tilde{X}_{l_1} \cup X_{l_1+1}$ might be a joint segment and $\tilde{X}_{l_2}$ might be the empty set. Anyway, both $\tilde{X}_{l_1}$ and $\tilde{X}_{l_2}$ are even segments, so that F_1 is admissible and $Z_{F_1} = \{0\}$.

The sets $F_1 \cup F$ and $F_1 \cap F$ are as follow:

$$\begin{aligned} F_1 \cup F &= X_0 \cup X_1 \cup \cdots \cup \tilde{X}_{l_1} \cup \cdots \cup X_{l_2} \cup \cdots \cup X_n, \\ F_1 \cap F &= X_0 \cup X_1 \cup \cdots \cup X_{l_1} \cup \cdots \cup \tilde{X}_{l_2} \cup \cdots \cup X_n. \end{aligned}$$

Apart from X_0, only X_{l_2} is odd in the first decomposition; only X_{l_1} is odd in the second one. Again, they might not be the segment decompositions, but it is easy to check that this does not affect the following calculations. Firstly, this proves that $d(F_1 \cup F, \mathcal{A}) = d(F_1 \cap F, \mathcal{A}) = 1$. Secondly, according to Theorem 4.3,

$$\begin{aligned} Z_{F_1 \cap F} &= \Big\{ \sum_{j \le l_1} |Y_j| + (|Y_{l_1+1}| - 1) + \sum_{l_1+2 \le j \le l_2} |Y_j| \Big\} = \Big\{ -1 + \sum_{j \le l_2} |Y_j| \Big\}, \\ Z_{F_1 \cap F} &= \Big\{ \sum_{j \le l_1} |Y_j| \Big\}, \end{aligned}$$

even in the cases where the above decompositions are not the segment decompositions. By Lemma 4.1,

$$Z(W_F^\mu) \le 1 + Z(W_{F_1 \cup F}^\mu) + Z(W_{F_1 \cap F}^\mu) = 2 \sum_{j \le l_1} |Y_j| + \sum_{l_1 < j \le l_2} |Y_j|$$

for every measure $\mu \in \mathfrak{M}_{\max F}$, and this proves (4.7). □

Remark 4.5. Since all zeros of $Z(W_{F_1 \cup F}^\mu)$ or $Z(W_{F_1 \cap F}^\mu)$ are simple, Lemma 4.1 proves that all zeros of $Z(W_F^\mu)$ have multiplicity at most 3, if $d(F,\mathcal{A}) = 2$.

Remark 4.6. For any set F with $d(F, \mathcal{A}) = 2$, it is worthwhile mentioning that $\xi_2 = \sum\limits_{l_1 < j \leq l_2} |Y_j|$ has the same parity as the degree of the Wronskian W_F^μ, which is

$$\deg W_F^\mu = \sum_{f \in F} f - \frac{|F|(|F|-1)}{2}.$$

Indeed, it is easy to see that both quantities keep their parity if two consecutive numbers are removed from F. Thus, it is enough to consider the obvious case $F = \{p, q\}$, with $p + 1 < q$.

Remark 4.7. As we mentioned before, for any set F and any measure $\mu \in \mathfrak{M}_{\max F}$, the Wronskian W_F^μ is a polynomial of degree

$$\sum_{f \in F} f - \frac{|F|(|F|-1)}{2},$$

so obviously $\max Z_F \leq \sum_{f \in F} f - \frac{|F|(|F|-1)}{2}$. Let $F = \{f_1, f_2, \dots\}$ with a segment decomposition

$$F = X_0 \cup X_1 \cup \cdots \cup X_n$$

and complementary segments $Y_1, \dots, Y_n$. One can see with no much effort that $f_j = j - 1 + |Y_1| + |Y_2| + \cdots + |Y_l|$, if $f_j \in X_l$. Therefore,

$$\begin{aligned}
\sum_{f \in F} f - \frac{|F|(|F|-1)}{2} &= \sum_{j=1}^{|F|} (f_j - j + 1) = \sum_{l=1}^{n} |X_l| \sum_{j=1}^{l} |Y_j| \\
&= \sum_{j=1}^{n} |Y_j| \sum_{l=j}^{n} |X_l|.
\end{aligned}$$

In particular,

$$\sum_{f \in F} f - \frac{|F|(|F|-1)}{2} \geq \sum_{j=1}^{n} (n + 1 - j) |Y_j|$$

and the equality holds when all segments $X_1, \dots, X_n$ consist of just one element. Under the conditions of Theorem 4.4, it follows that

$$2\xi_1 + \xi_2 = 2 \sum_{j \leq l_1} |Y_j| + \sum_{l_1 < j \leq l_2} |Y_j| \leq \sum_{f \in F} f - \frac{|F|(|F|-1)}{2},$$

and again the equality holds when all segments $X_1, \dots, X_n$ consist of just one element (and $n = 2$). That is, (4.7) gives a non trivial bound on $\max Z_F$.

Remark 4.8. A common impression after all the conjectures and results is that the segment X_0 plays no role. This is reinforced by the fact that the contribution of X_0 to the Wronskian matrix $\mathcal{W}_F^\mu$ is an upper diagonal block with non zero constant entries in the diagonal. For instance, it is easy to conclude that the Wronskians W_F^μ associated to $F = [0, 1, \dots, n, n+k]$ have $k-1$ simple real zeros. For the admissible sets $F = [0, 1, \dots, n, n+k, n+k+1]$, the Wronskians have no real zeros, as predicted by Conjecture 2.8. This is consequence of the fact that, given two orthogonal polynomials, their derivatives belong to another sequence of orthogonal polynomials: the interlacing of zeros of orthogonal polynomials is preserved under differentiation (see, for instance, [Fisk 08, Theorem 1.47]), and the corresponding interlacing property for the derivatives guarantee that the derivatives of the two polynomials belong to a sequence of orthogonal polynomials (see [Wendroff 61]).

4.2 Computational evidences for the conjectures on Wronskians

In the process of establishing and fine tuning our conjectures, we have done billions of symbolic and numerical experiments. We have used Maple, Mathematica and Sage.

The main goal of this paper is to describe the set Z_F for every finite set F of nonnegative integers. A way to establish that some $k \in Z_F$ is to find a positive measure μ whose Wronskian W_F^μ has k zeros in $\mathbb{R}$. Let us also note that, by Favard's Theorem, for each three term recurrence relation

$$p_{n+1}(x) = (x - a_n)p_n(x) - b_n p_{n-1}(x), \qquad p_0(x) = 1, \quad p_1(x) = x - a_0, \tag{4.8}$$

where $a_n \in \mathbb{R}$ and $b_n > 0$ for every n, there exists a measure μ such that $(p_n)_{n\geq 0}$ are the monic orthogonal polynomials with respect to μ. Consequently, producing sequences a_n and b_n as in (4.8) is a way to compute orthogonal polynomial sequences and, then, the Wronskians W_F^μ. Actually, we only need to compute p_n up to $\max F$, so it is enough to have finite sequences a_n and b_n.

In our numerical experiments, for a fixed F we have generated many random finite sequences a_n and b_n and their corresponding Wronskians. To avoid numerical rounding errors, we have always taken a_n and b_n to be rational numbers generated by mean of pseudorandom procedures. Then, we use (4.8) to compute the polynomials and the Wronskian by mean of computer algebra systems. Finally, we find the number of real roots of the Wronskian using appropriate procedures. For this kind of random experiments, it is virtually impossible to obtain examples with multiple roots, so there is no need to worry about the multiplicity.

After repeating this procedure a huge number of times, it becomes apparent that Z_F has the form $\{j, j+2, \dots, j+2t\}$, but the "central elements" of Z_F appear in the experiments really much more often than the extremes, which sometimes are very unusual. Whatever the number of experiments we perform, we can not ensure that we have found $\max Z_F$ and $\min Z_F$, except when a Wronskian appears with exactly 0 or 1 roots, what clearly implies that $\min Z_F = 0$ or 1, respectively.

Then, in the search for Z_F for a given F (in particular, to identify $\min Z_F$ and $\max Z_F$) we typically do millions of experiments. Unfortunately, the computer time increases very fast with the size of the determinant and the degree of the polynomials, so it is not possible to do many experiments for big sets F. Then, we have performed a reasonable number of experiments for sets F with distance $d(F, \mathcal{A})$ up to 6, and only with $\max F < 25$. Although, the corresponding conjectures are best experimentally checked for small values of $d(F, \mathcal{A})$. With this kind of experiments we have found many computational evidences of conjectures 2.8, 2.12, 2.13, 2.14 and 2.15.

For instance, the set $F = [5, 7, 10]$ has $d(F, \mathcal{A}) = 3$ and characteristic gap lengths $\xi_1 = 5$, $\xi_2 = 1$, $\xi_3 = 2$, and the same happens with the sets

$$[5, 6, 7, 9, 12], \quad [5, 7, 8, 9, 12], \quad [5, 7, 10, 11, 12], \quad [5, 7, 9, 10, 12, 13, 14].$$

With any of these sets F we have found the number of roots of W_F^μ for at least one million measures randomly generated as explained above, and we have always found Wronskians with $3, 5, 7, 9$ and 11 roots (and no other number of roots). Strictly speaking, we can only ensure that $\{3, 5, 7, 9, 11\} \subset Z_F$, but it seems reasonable to conjecture that $Z_F = \{3, 5, 7, 9, 11\}$ for all those sets F.

Another kind of experiments have been done to come across conjecture 2.16. Now, there is no need to use the three term recurrence relation, for the Jacobi, Laguerre and Hermite polynomials are already defined in the software. Then, given F, we have randomly generated the parameters for the orthogonal polynomials and, again, computed the number of roots of the corresponding Wronskian. For Laguerre polynomials and Jacobi polynomials with $\alpha - \beta$ not an even integer, the geometric multiplicity of the roots (as explained in Definition 2.3) is involved only in a few cases, which can be detected checking if $\gcd(W_F(x), W_F'(x))$ is a polynomial of degree greater than 0. But computing the geometric multiplicity without really having the roots is not easy (and, moreover, it is slow), so our routines do not consider it; when one of those few cases occurs, it must be checked apart.

For instance, for Laguerre polynomials with $\alpha = 7/5$ and $F = [1, 5]$ (so that $\xi_1 = 1$, $\xi_2 = 3$) the Wronskian $W_F(x)$ has the roots $12/5$ (triple) and $(99 \pm 5\sqrt{33})/10$. The Wronskian matrix is $\mathcal{W}_F(12/5) = \left(\begin{smallmatrix} 0 & 0 \\ -1 & 72/25 \end{smallmatrix}\right)$, whose rank is 1, so the geometric multiplicity of $12/5$ is 1. Then, the number of real roots of $W_F(x)$, according with the geometric multiplicity, is 3, as predicted by conjecture 2.16.

In the case of Hermite polynomials, $H_{2n+1}(x)$ is odd and $H_{2n}(x)$ is even for every n, so it is essential to take into account the geometric multiplicity of the root $x_0 = 0$. This applies also to Jacobi polynomials with parameters $\alpha = \beta$ (that is, Gegenbauer polynomials). Multiplicity bigger than 1 at $x_0 = 0$ also appears when α and β differ by an even integer. In those cases, according to Definition 2.3, we have effectively computed the rank of the matrix $\mathcal{W}_F(0)$ so as to find the geometric multiplicity. In each case, the numerical experiments agree with our conjecture.

4.3 Computational evidences for the conjectures on Casorati determinants

Regarding Casorati determinants, we have again done a huge number of symbolic experiments with Maple, Mathematica and Sage.

Now, to produce a measure μ as in (3.1), we only need to generate finite sequences of pseudorandom numbers a_n and α_n. To avoid the usual problems of float numbers and their algorithms (rounding, truncating, instability), we take the points a_n and the mass points α_n of the measure to be integers (this is not an essential restriction). Then, we compute the moments of the measure by

$$\mu_k = \sum_{n=0}^{\Upsilon} \alpha_n a_n^k, \qquad k = 0, \dots, \Upsilon,$$

and use the moments to compute the orthogonal polynomials p_n (with positive leading coefficient). These are well known standard procedures. With an appropriate normalization of the orthogonal polynomials, only integer numbers are involved.

Given a set F (which can be built through pseudorandom procedures, as well), we can generate many measures μ, compute the Casorati determinant $C_F^{G,\mu}$ for many sets G, and check whether $C_F^{G,\mu}$ is positive or not. In all the cases, the numerical experiments agree with our conjecture.

In the process of stating conjecture 3.2, we have performed millions of these experiments, with Υ up to 150, and Casorati determinants up to 15×15. We have thus observed that the result fails if G is an admissible set other than a segment. It is worth while remarking that in the course of these experiments we discovered a relevant bug of Mathematica when computing determinants of big integers (see [Durán et al. 14]).

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