THE SPECTRUM OF THE RIGHT INVERSE OF THE DUNKL OPERATOR

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ABSTRACT. From the Dunkl analogue of Gegenbauer's expansion of the plane wave, we derive an explicit closed formula for the spectrum of a right inverse of the Dunkl operator. This is done by stating the problem in such a way it is possible to use the technique due to Ismail and Zhang.

1. Introduction

During the last years, and starting in the seminal paper [14], Mourad Ismail and his collaborators had studied the diagonalization of the inverse of several differential or difference operators. In particular, it has been done in [14, 12] for the differential and the Askey-Wilson divided difference operators in different polynomial bases, in [11] for the full Askey-Wilson weight, in [13] for the q-difference operator, and in [5] for the q^{-1}-Askey-Wilson operator.

This article is devoted to the diagonalization of the right inverse of the Dunkl operator on the real line given by

$$\Lambda_\alpha f(x) = \frac{d}{dx} f(x) + \frac{2\alpha+1}{x}\left(\frac{f(x)-f(-x)}{2}\right).$$

The operator Λ_α is a generalization of the derivative $\frac{d}{dx}$ (which is the case corresponding to $\alpha = -1/2$) whose study, as well as of the so-called Dunkl transform that is closely related to Λ_α, has engender many recent papers (see, for instance, [3, 4, 6, 7, 8, 15, 16, 17]). In the present paper, we are going to find the spectrum of a right inverse for the Dunkl operator Λ_α in spaces weighted by the appropriated weight functions.

In [14], an important tool to study the inverse of the differential operator $\frac{d}{dx}$ is Gegenbauer's expansion of the plane wave in ultraspherical polynomials and

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Bessel functions (see [10, § 4.8, formula (4.8.3), p. 116]):

$$e^{ixt} = \Gamma(\beta) \left(\frac{x}{2}\right)^{-\beta} \sum_{n=0}^{\infty} i^n (\beta+n) J_{\beta+n}(x) C_n^{\beta}(t) \tag{1}$$

(in the particular case $\beta = 0$, this formula is the so-called Jacobi-Anger identity). Instead of this expansion for e^{ixt}, that is the kernel of the Fourier transform, we will require an expansion for $E_\alpha(ixt)$, the kernel of the Dunkl transform on the real line. Thus, we will use that the Dunkl kernel $E_\alpha(ixt)$ can be expanded as

$$E_\alpha(ixt) = \Gamma(\alpha+\beta+1) \left(\frac{x}{2}\right)^{-\alpha-\beta-1} \sum_{n=0}^{\infty} i^n (\alpha+\beta+n+1) J_{\alpha+\beta+n+1}(x) C_n^{(\beta+1/2,\alpha+1/2)}(t), \tag{2}$$

where $C_n^{(\beta+1/2,\alpha+1/2)}$ are the so-called generalized Gegenbauer polynomials and J_a denotes the Bessel function of order a. The identity (2) holds for $\alpha, \beta > -1$ and $\alpha + \beta > -1$. It was proved by M. Rösler in [17] for $\alpha \geq -1/2$, and by the authors in [2] for $\alpha > -1$. By using (2) in the place of (1), a part of the technique for $\frac{d}{dx}$ can be followed, and the results obtained are very satisfactory, finally getting a nice extension of the result of Ismail and Zhang in [14], in the same sense as the Bessel functions generalize the trigonometric functions, the Dunkl operator has the derivative as a particular case, or the Dunkl transform is an extension of the Fourier transform.

The structure of the paper is as follows. In Section 2 we state the precise notation and give some preliminaries related to the Dunkl operator on the real line and to the generalized Gegenbauer polynomials, as well as the identity (2), that is the main tool to solve our problem. The essential part of the paper is included in Section 3. We define the right inverse of the Dunkl operator in spaces weighted by the weight function of the generalized Gegenbauer polynomials and use this definition to find explicit formulas for the spectrum of this right inverse that we will denote $T_{\beta,\alpha}$. It turns out that the eigenvalues of the resulting integral operator $T_{\beta,\alpha}$ are defined in terms of zeros of Bessel functions $\{\pm i/j_{\alpha+\beta+1,k}\}_{k\geq 1}$ and the eigenfunctions are

$$g_{\pm i/j_{\alpha+\beta+1,k}}(t) = \mp i \left(\frac{j_{\alpha+\beta+1,k}}{2}\right)^{\alpha+\beta+1} \frac{E_\alpha(\mp t j_{\alpha+\beta+1,k})}{\Gamma(\alpha+\beta+1)(\alpha+\beta+2) J_{\alpha+\beta}(j_{\alpha+\beta+1,k})};$$

actually, this is the main result of the paper, whose complete details can be seen in Theorem 3.

2. Notation and preliminaries

2.1. **The Dunkl operator on the real line.** For $\alpha > -1$, let J_α denote the Bessel function of order α and, for complex values of the variable z, let

$$\mathcal{I}_\alpha(z) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(iz)}{(iz)^\alpha} = \Gamma(\alpha+1) \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{k!\,\Gamma(k+\alpha+1)}$$

($\mathcal{I}_\alpha$ is a small variation of the so-called modified Bessel function of the first kind and order α, usually denoted by I_α; see [18]). Moreover, let us take

$$E_\alpha(z) = \mathcal{I}_\alpha(z) + \frac{z}{2(\alpha+1)}\,\mathcal{I}_{\alpha+1}(z), \qquad z \in \mathbb{C};$$

this function is called the Dunkl kernel.

The Dunkl operators on $\mathbb{R}^n$ are differential-difference operators associated with some finite reflection groups (see [7]). We consider the Dunkl operator Λ_α, $\alpha \geq -1/2$, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ given by

$$\Lambda_\alpha f(x) = \frac{d}{dx} f(x) + \frac{2\alpha+1}{x}\left(\frac{f(x)-f(-x)}{2}\right). \tag{3}$$

For $\alpha \geq -1/2$ and $\lambda \in \mathbb{C}$, the initial value problem

$$\begin{cases} \Lambda_\alpha f(x) = \lambda f(x), & x \in \mathbb{R}, \\ f(0) = 1 \end{cases} \tag{4}$$

has $E_\alpha(\lambda x)$ as its unique solution (see [8] and [15]). For $\alpha = -1/2$, it is clear that $\Lambda_{-1/2} = d/dx$, and $E_{-1/2}(\lambda x) = e^{\lambda x}$.

Along this paper we will usually employ E_α in the form

$$E_\alpha(ix) = 2^\alpha \Gamma(\alpha+1)\left(\frac{J_\alpha(x)}{x^\alpha} + \frac{J_{\alpha+1}(x)}{x^{\alpha+1}}\, xi\right). \tag{5}$$

If, moreover, we take

$$d\mu_\alpha(x) = (2^{\alpha+1}\Gamma(\alpha+1))^{-1}|x|^{2\alpha+1}\,dx,$$

the name Dunkl kernel for E_α is justified because it is the kernel of the Dunkl transform of order $\alpha \geq -1/2$ given by

$$\mathcal{F}_\alpha f(x) = \int_{\mathbb{R}} f(t) E_\alpha(-ixt)\, d\mu_\alpha(t), \quad x \in \mathbb{R}, \tag{6}$$

for $f \in L^1(\mathbb{R}, d\mu_\alpha)$. In a similar way to the Fourier transform (which is the particular case $\alpha = -1/2$), the definition is extended $L^2(\mathbb{R}, d\mu_\alpha)$ and $\mathcal{F}_\alpha$ becomes to be an isometric isomorphism on $L^2(\mathbb{R}, d\mu_\alpha)$. Yet more, the Dunkl transform $\mathcal{F}_\alpha$ can also be defined in $L^2(\mathbb{R}, d\mu_\alpha)$ for $-1 < \alpha < -1/2$; although in this case the expression (6) is no longer valid for $f \in L^1(\mathbb{R}, d\mu_\alpha)$ in general, it preserves the same properties in $L^2(\mathbb{R}, d\mu_\alpha)$ (see [16] for details). Following this spirit, we will do our study not only for $\alpha \geq -1/2$, but for the whole range $\alpha > -1$.

2.2. **Generalized Gegenbauer polynomials.** Following [9, Definition 1.5.5, p. 27], let us introduce the generalized Gegenbauer polynomials $C^{(\lambda,\nu)}(t)$ for $\lambda > -1/2$, $\nu \geq 0$ and $n \geq 0$ (the case $\nu = 0$ corresponding with the ordinary Gegenbauer polynomials); actually, for convenience with the notation of this paper, we are going to use $C_n^{(\beta+1/2,\alpha+1/2)}(x)$. In this way, for $\beta > -1$ and

$\alpha \geq -1/2$, the generalized Gegenbauer polynomials are defined by

$$C_{2n}^{(\beta+1/2,\alpha+1/2)}(t) = (-1)^n \frac{(\alpha+\beta+1)_n}{(\alpha+1)_n} P_n^{(\alpha,\beta)}(1-2t^2), \tag{7}$$

$$C_{2n+1}^{(\beta+1/2,\alpha+1/2)}(t) = (-1)^n \frac{(\alpha+\beta+1)_{n+1}}{(\alpha+1)_{n+1}} t P_n^{(\alpha+1,\beta)}(1-2t^2), \tag{8}$$

where in the coefficients we are using the Pochhammer symbol $(a)_n = a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$. Note that there is no problem in extending the definition of the generalized Gegenbauer polynomials taking $\alpha > -1$, so we will assume this situation.

The definitions (7) and (8) easily allow to translate the well known orthogonality of the Jacobi polynomials on $(-1,1)$ to the orthogonality of the generalized Gegenbauer polynomials on $(-1,1)$ with a related weight, which is more useful in the context of this paper. In particular, the generalized Gegenbauer polynomials $\{C_k^{(\beta+1/2,\alpha+1/2)}(t)\}_{k\geq 0}$ form a complete orthogonal system on $L^2((-1,1),(1-t^2)^\beta\,d\mu_\alpha(t))$ for $\alpha,\beta > -1$. Moreover, from the L^2-norm of the Jacobi polynomials (see [1, formula 22.2.1, p. 774]), it is easy to find

$$\begin{aligned} h_{2n}^{(\beta,\alpha)} &= \int_{-1}^{1} \left[C_{2n}^{(\beta+1/2,\alpha+1/2)}(t)\right]^2 (1-t^2)^\beta\,d\mu_\alpha(t) \\ &= \frac{1}{2^{\alpha+1}} \frac{\Gamma(\alpha+1)\Gamma(\beta+n+1)\Gamma(\alpha+\beta+n+1)}{(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+1)^2\Gamma(\alpha+n+1)\,n!}, \end{aligned} \tag{9}$$

$$\begin{aligned} h_{2n+1}^{(\beta,\alpha)} &= \int_{-1}^{1} \left[C_{2n+1}^{(\beta+1/2,\alpha+1/2)}(t)\right]^2 (1-t^2)^\beta\,d\mu_\alpha(t) \\ &= \frac{1}{2^{\alpha+1}} \frac{\Gamma(\alpha+1)\Gamma(\beta+n+1)\Gamma(\alpha+\beta+n+2)}{(\alpha+\beta+2n+2)\Gamma(\alpha+\beta+1)^2\Gamma(\alpha+n+2)\,n!}. \end{aligned} \tag{10}$$

2.3. **The expansion of the Dunkl kernel.** The last ingredient in this section with preliminaries is to explain the (2) that we give in the introduction. Firstly, let us note that, when you use something like $J_{a+n}(x)/x^a$, we mean

$$\frac{J_{a+n}(x)}{x^a} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k!\,\Gamma(k+a+n+1)} \left(\frac{x}{2}\right)^{2k+n},$$

that is an analytic function on the complex plane for every $a > -1$ and every integer $n \geq 0$.

Then, the precise statement of the Dunkl analogue of Gegenbauer's expansion of the plane wave is that, for $\alpha,\beta > -1$ and $\alpha+\beta > -1$, the Dunkl kernel can be expanded, for $x \in \mathbb{R}$, as

$$E_\alpha(ixt) = 2^{\alpha+\beta+1}\Gamma(\alpha+\beta+1) \sum_{n=0}^{\infty} i^n(\alpha+\beta+n+1) \frac{J_{\alpha+\beta+n+1}(x)}{x^{\alpha+\beta+1}} C_n^{(\beta+1/2,\alpha+1/2)}(t), \tag{11}$$

and this expansion holds in $L^2((-1,1),d\mu_\alpha(t))$.

This important identity has been proved in [17] for $\alpha \geq -1/2$ by showing that, under this restriction, formulas (1) and (11) are equivalent. The proof in one direction is clear, just by specializing the parameters. To obtain (11) from (1), we can use the intertwining operator

$$V_\alpha g(t) = \frac{\Gamma(2\alpha+2)}{2^{2\alpha+1}\Gamma(\alpha+1/2)\Gamma(\alpha+3/2)} \int_{-1}^{1} g(st)(1-s)^{\alpha-1/2}(1+s)^{\alpha+1/2}\, ds$$

(see [9, Definition 1.5.1, p. 24], we change the parameter μ in the definition given in [9] by $\alpha+1/2$), defined for $\alpha \geq -1/2$. With this notation we have

$$V_\alpha C_n^{\alpha+\beta+1}(t) = C_n^{(\beta+1/2,\alpha+1/2)}(t)$$

and

$$V_\alpha e^{i\cdot}(t) = E_\alpha(it).$$

In this way, applying V_α to (1) (with $\alpha+\beta+1$ instead of β) we get (11).

This idea cannot be used in the whole range $\alpha > -1$, and the extension of (11) to this case can be found in [2].

3. Diagonalization of the right inverse of Dunkl operator

As stated in the introduction, Ismail and Zhang study in [14] the eigenfunctions and the eigenvalues of the right inverse of the derivative operator, and the main tool is a suitable expansion of the corresponding plane wave. The aim of this section is the analysis of this question for the Dunkl operator (3) using the expansion (11). This is a very natural extension, because the ordinary derivative corresponds to the Dunkl operator Λ_α with $\alpha = -1/2$.

To simplify the notation, let us introduce the measure

$$d\mu_{\beta,\alpha}(t) = (1-t^2)^\beta\, d\mu_\alpha(t) = \tfrac{1}{2^{\alpha+1}\Gamma(\alpha+1)}(1-t^2)^\beta |t|^{2\alpha+1}\, dt.$$

It is not difficult to prove, using the appropriate recurrence relations for the Jacobi polynomials, that the Dunkl operator over the generalized Gegenbauer operator satisfies

$$\Lambda_\alpha C_n^{(\beta+1/2,\alpha+1/2)}(t) = 2(\alpha+\beta+1) C_{n-1}^{(\beta+3/2,\alpha+1/2)}(t). \tag{12}$$

Motivated by the identity (12), we define the right inverse operator of Λ_α over $L^2((-1,1), d\mu_{\beta+1,\alpha})$ by

$$T_{\beta,\alpha} g(t) = \sum_{n=1}^{\infty} \frac{g_{n-1}}{2(\alpha+\beta+1)}\, C_n^{(\beta+1/2,\alpha+1/2)}(t),$$

whenever g has the expansion

$$g(t) = \sum_{n=0}^{\infty} g_n C_n^{(\beta+3/2,\alpha+1/2)}(t),$$

with

$$g_n = \left(h_n^{(\beta+1,\alpha)}\right)^{-1} \int_{-1}^{1} g(r) C_n^{(\beta+3/2,\alpha+1/2)}(r)\, d\mu_{\beta+1,\alpha}(r)$$

(the norms $h_n^{(\beta+1,\alpha)}$ are given in (9) and (10)). Then, we can write

$$T_{\beta,\alpha}g(t) = \int_{-1}^{1} g(r) K_{\beta,\alpha}(t,r)\, d\mu_{\beta+1,\alpha}(r),$$

with

$$K_{\beta,\alpha}(t,r) = \frac{1}{2(\alpha+\beta+1)} \sum_{n=1}^{\infty} \frac{C_n^{(\beta+1/2,\alpha+1/2)}(t) C_{n-1}^{(\beta+3/2,\alpha+1/2)}(r)}{h_{n-1}^{(\beta+1,\alpha)}}.$$

We want to diagonalize $T_{\beta,\alpha}$. To this end we have to find values $\lambda \in \mathbb{C}$ and functions $g_\lambda \in L^2((-1,1), d\mu_{\beta,\alpha}) \cap L^2((-1,1), d\mu_{\beta+1,\alpha})$ such that

$$\lambda g_\lambda(t) = T_{\beta,\alpha} g_\lambda(t), \tag{13}$$

where the expansion of g in $\{C_n^{(\beta+1/2,\alpha+1/2)}\}_{n\geq 0}$ is

$$g_\lambda(t) \sim \sum_{n=1}^{\infty} a_n(\lambda) C_n^{(\beta+1/2,\alpha+1/2)}(t). \tag{14}$$

To find the eigenvalues and the eigenfunctions we start obtaining a recurrence relation for the coefficients $\{a_n(\lambda)\}_{n\geq 1}$. This is done by using an expression to write $(1-r^2)C_{n-1}^{(\beta+3/2,\alpha+1/2)}(r)$ in terms of $C_{n-1}^{(\beta+1/2,\alpha+1/2)}(r)$ and $C_{n+1}^{(\beta+1/2,\alpha+1/2)}(r)$. This relation can be deduced from the identity for the Jacobi polynomials (see [1, formula 22.7.16, p. 782])

$$P_n^{(a,b+1)}(z) = \frac{2(n+b+1)P_n^{(a,b)}(z) + 2(n+1)P_{n+1}^{(a,b)}(z)}{(2n+a+b+2)(1+z)}. \tag{15}$$

Indeed, taking $z = 1 - 2t^2$ in (15) and using the definition of the generalized Gegenbauer polynomials, we have

$$(\alpha+\beta+1)(1-r^2)C_{n-1}^{(\beta+3/2,\alpha+1/2)}(r) = A_n C_{n-1}^{(\beta+1/2,\alpha+1/2)}(r) - B_n C_{n+1}^{(\beta+1/2,\alpha+1/2)}(r), \tag{16}$$

with

$$A_n = \begin{cases} \dfrac{(\beta+k+1)(\alpha+\beta+k+1)}{\alpha+\beta+2k+2}, & \text{if } n = 2k+1, \\[2mm] \dfrac{(\beta+k)(\alpha+\beta+k+1)}{\alpha+\beta+2k+1}, & \text{if } n = 2k, \end{cases}$$

$$B_n = \begin{cases} \dfrac{(k+1)(\alpha+k+1)}{\alpha+\beta+2k+2}, & \text{if } n = 2k+1, \\[2mm] \dfrac{k(\alpha+k+1)}{\alpha+\beta+2k+1}, & \text{if } n = 2k. \end{cases}$$

From (16) we have the decomposition

$$(1-r^2)K_{\beta,\alpha}(t,r) = \frac{1}{2(\alpha+\beta+1)^2}\sum_{n=1}^{\infty} A_n \frac{C_n^{(\beta+1/2,\alpha+1/2)}(t)C_{n-1}^{(\beta+1/2,\alpha+1/2)}(r)}{h_{n-1}^{(\beta+1,\alpha)}} - \frac{1}{2(\alpha+\beta+1)^2}\sum_{n=1}^{\infty} B_n \frac{C_n^{(\beta+1/2,\alpha+1/2)}(t)C_{n+1}^{(\beta+1/2,\alpha+1/2)}(r)}{h_{n-1}^{(\beta+1,\alpha)}}.$$

With the previous identity, we get

$$T_{\beta,\alpha}g_\lambda(t) = \frac{1}{2(\alpha+\beta+1)^2}\sum_{n=2}^{\infty} a_{n-1}(\lambda)A_n \frac{h_{n-1}^{(\beta,\alpha)}}{h_{n-1}^{(\beta+1,\alpha)}} C_n^{(\beta+1/2,\alpha+1/2)}(t) - \frac{1}{2(\alpha+\beta+1)^2}\sum_{n=1}^{\infty} a_{n+1}(\lambda)B_n \frac{h_{n+1}^{(\beta,\alpha)}}{h_{n-1}^{(\beta+1,\alpha)}} C_n^{(\beta+1/2,\alpha+1/2)}(t).$$

In this way, identifying the coefficients in both sides of (13), we obtain the recurrence relation

$$\lambda a_n(\lambda) = \frac{1}{2(\alpha+\beta+1)^2}\left(a_{n-1}(\lambda)A_n \frac{h_{n-1}^{(\beta,\alpha)}}{h_{n-1}^{(\beta+1,\alpha)}} - a_{n+1}(\lambda)B_n \frac{h_{n+1}^{(\beta,\alpha)}}{h_{n-1}^{(\beta+1,\alpha)}}\right), \qquad n>1,$$

and

$$\lambda a_1(\lambda) = -a_2(\lambda)\frac{B_1}{2(\alpha+\beta+1)^2}\frac{h_2^{(\beta,\alpha)}}{h_0^{(\beta+1,\alpha)}},$$

which, applying (9) and (10), becomes

$$\lambda a_n(\lambda) = \frac{a_{n-1}(\lambda)}{2(\alpha+\beta+n)} - \frac{a_{n+1}(\lambda)}{2(\alpha+\beta+n+2)}, \qquad n>1, \tag{17}$$

and

$$\lambda a_1(\lambda) = -\frac{a_2(\lambda)}{2(\alpha+\beta+3)}. \tag{18}$$

Now, we can prove the following:

Theorem 1. *For $\alpha,\beta>-1$, and $\alpha+\beta>-1$, let $R_{\beta+1/2,\alpha+1/2}$ be the closure of the span of $\{C_n^{(\beta+1/2,\alpha+1/2)}\}_{n\geq 1}$ in $L^2((-1,1),d\mu_{\beta+1,\alpha})$; then*

$$L^2((-1,1),d\mu_{\beta+1,\alpha}) = R_{\beta+1/2,\alpha+1/2} \oplus R^{\perp}_{\beta+1/2,\alpha+1/2}, \tag{19}$$

where

$$R^{\perp}_{\beta+1/2,\alpha+1/2} = \begin{cases} \operatorname{span}\{(1-t^2)^{-1}\}, & \text{for } \beta>0, \\ \{0\}, & \text{for } 0\geq\beta>-1. \end{cases} \tag{20}$$

Furthermore, if we let $g_\lambda(x)\in R_{\beta+1/2,\alpha+1/2}$ have the orthogonal expansion (14), then the eigenvalue equation (13) holds if and only if

$$\sum_{n=1}^{\infty} |a_n(\lambda)|^2 n^{2\beta-1} < \infty. \tag{21}$$

Proof. It is clear that $L^2((-1,1), d\mu_{\beta,\alpha}) \subset L^2((-1,1), d\mu_{\beta+1,\alpha})$ and $T_{\beta,\alpha}$ maps $L^2((-1,1), d\mu_{\beta+1,\alpha})$ into $L^2((-1,1), d\mu_{\beta,\alpha})$. Moreover, $T_{\beta,\alpha}$ is a bounded operator and its norm is controlled by a constant M given by

$$M^2 = \frac{1}{4(\alpha+\beta+1)^2} \sup_{n\geq 0} \frac{h_{n+1}^{(\beta,\alpha)}}{h_n^{(\beta+1,\alpha)}}.$$

Indeed,

$$\begin{aligned}\|T_{\beta,\alpha} g\|^2_{L^2((-1,1),d\mu_{\beta,\alpha})} &= \sum_{n=1}^{\infty} \frac{|g_{n-1}|^2}{4(\alpha+\beta+1)^2} h_n^{(\beta,\alpha)} \\ &\leq M^2 \sum_{n=1}^{\infty} |g_n|^2 h_n^{(\beta+1,\alpha)} = M^2 \|g\|_{L^2((-1,1),d\mu_{\beta+1,\alpha})}.\end{aligned}$$

Note that M is finite because

$$\frac{h_{2k+1}^{(\beta,\alpha)}}{h_{2k}^{(\beta+1,\alpha)}} = \frac{(\alpha+\beta+1)^2}{(\beta+k+1)(\alpha+k+1)} \qquad \text{and} \qquad \frac{h_{2k}^{(\beta,\alpha)}}{h_{2k-1}^{(\beta+1,\alpha)}} = \frac{(\alpha+\beta+1)^2}{k(\alpha+\beta+k+1)}.$$

In this way, we can deduce that $R_{\beta+1/2,\alpha+1/2}$ is an invariant subspace for $T_{\beta,\alpha}$ and the decomposition (19) holds. Now, each function $f \in R^{\perp}_{\beta+1/2,\alpha+1/2}$ will satisfy the conditions $\|f\|_{L^2((-1,1),d\mu_{\beta+1,\alpha})} < \infty$ and

$$\int_{-1}^{1} f(r) C_n^{(\beta+1/2,\alpha+1/2)}(r)\, d\mu_{\beta+1,\alpha}(r) = 0, \qquad n = 1, 2, \dots. \tag{22}$$

Using that $\{C_n^{(\beta+1/2,\alpha+1/2)}\}_{n\geq 0}$ is a complete orthogonal system in $L^2((-1,1), d\mu_{\beta,\alpha})$ and the relation $d\mu_{\beta+1,\alpha}(r) = (1-r^2)\, d\mu_{\beta,\alpha}(r)$, from (22) we get

$$(1-r^2) f(r) = K$$

for a certain constant K. So, taking into account that $f \in L^2((-1,1), d\mu_{\beta+1,\alpha})$, we conclude (20).

Let g_λ be a function having the expansion (14) and verifying (13). Then, using that $g_\lambda \in L^2((-1,1), d\mu_{\beta,\alpha})$, it follows that

$$\|g_\lambda\|^2_{L^2((-1,1),d\mu_{\beta,\alpha})} = \sum_{n=1}^{\infty} |a_n(\lambda)|^2 h_n^{(\beta,\alpha)} < \infty$$

and this imply (21) because $h_n^{(\beta,\alpha)} \approx n^{2\beta-1}$.

On other hand, let us suppose that (21) and (14) hold. To prove that g_λ is a solution of (13), we need rewrite the function g_λ in terms of $C_n^{(\beta+3/2,\alpha+1/2)}$. To this end, we use that the Jacobi polynomials satisfy the identity [1, formula 22.7.19, p. 782]

$$(2n+a+b) P_n^{(a,b-1)}(z) = (n+a+b) P_n^{(a,b)}(z) + (n+a) P_{n-1}^{(a,b)}(z)$$

to produce

$$C_n^{(\beta+1/2,\alpha+1/2)}(t) = \frac{\alpha+\beta+1}{\alpha+\beta+n+1} (C_n^{(\beta+3/2,\alpha+1/2)}(t) - C_{n-2}^{(\beta+3/2,\alpha+1/2)}(t)).$$

So, we find

$$g_\lambda(t) = \sum_{n=1}^{\infty} \frac{\alpha+\beta+1}{\alpha+\beta+n+1} a_n(\lambda) C_n^{(\beta+3/2,\alpha+1/2)}(t) - \sum_{n=1}^{\infty} \frac{\alpha+\beta+1}{\alpha+\beta+n+1} a_n(\lambda) C_{n-2}^{(\beta+3/2,\alpha+1/2)}(t). \tag{23}$$

Thus,

$$\|g_\lambda\|^2_{L^2((-1,1),d\mu_{\beta+1,\alpha})} \sim \sum_{n=1}^{\infty} |a_n(\lambda)|^2 n^{2\beta-1} < \infty$$

and $g_\lambda \in L^2((-1,1), d\mu_{\beta+1,\alpha})$. Moreover, from (23), with (17) and (18), we can check that g_λ is an eigenfunction of $T_{\beta,\alpha}$. □

It is clear, from (17) and (18), that $a_1(\lambda) \neq 0$ (in other case $a_n(\lambda) = 0$ for $n > 1$). So this is a multiplicative factor and can be factored out. To verify the condition (21), we have to renormalize the sequence $\{a_n(\lambda)\}_{n\geq 1}$. Set

$$a_n(\lambda) = i^{n-1} \frac{(\alpha+\beta+n+1)}{(\alpha+\beta+2)} b_{n-1}(i\lambda) a_1(\lambda). \tag{24}$$

Then, using the relations (17) and (18), we can check that

$$2w(\alpha+\beta+n+2) b_n(w) = b_{n-1}(w) + b_{n+1}(w), \qquad n \geq 1, \tag{25}$$

where $w = i\lambda$, $b_{-1}(w) = 0$, and $b_0(w) = 1$. If $R_{n,a}(z)$ denotes the Lommel polynomials, we define $h_{n,a}(z) = R_{n,a}(1/z)$, which are known as modified Lommel polynomials. Lommel polynomials satisfy the three terms recurrence relation (see [18, § 9.63, formula (2), p. 299])

$$\frac{2(n+a)}{z} R_{n,a}(z) = R_{n-1,a}(z) + R_{n+1,a}(z).$$

In this way, we deduce that $b_n(w) = h_{n,\alpha+\beta+2}(w)$.

Let us identify the values verifying (21).

Theorem 2. *The convergence condition* (21) *holds if and only if* λ *is purely imaginary,* $\lambda \neq 0$*, and* $J_{\alpha+\beta+1}(i/\lambda) = 0$*.*

Proof. By Hurwitz' theorem [18, § 9.65, formula (1), p. 302], we have the asymptotic relation

$$h_{n,a}(z) \sim \Gamma(n+a)(2z)^{n+a-1} J_{a-1}(1/z), \qquad n \to \infty.$$

This fact, taking into account (24) and that $b_n(i\lambda) = h_{n-1,\alpha+\beta+2}(i\lambda)$, shows that in order for (21) to hold it is necessary that $J_{\alpha+\beta+1}(i/\lambda) = 0$ or possibly $\lambda = 0$. For $\lambda = 0$ we can deduce, by (25), that $b_{2n+1}(0) = 0$ and $b_{2n}(0) = (-1)^n$, then (21) does not hold.

To deduce the sufficiency of $J_{\alpha+\beta+1}(i/\lambda) = 0$ we need the identity [18, § 9.6, formula (1), p. 295]

$$J_{a+n}(z) = R_{n,a}(z) J_a(z) - R_{n-1,a+1}(z) J_{a-1}(z).$$

From this relation we have

$$-J_{\alpha+\beta+n+1}(i/\lambda) = h_{n-1,\alpha+\beta+2}(\lambda/i) J_{\alpha+\beta}(i/\lambda) \tag{26}$$

when $J_{\alpha+\beta+1}(i/\lambda) = 0$. In this way, by the asymptotic

$$J_{a+n}(z) \sim \frac{(2z)^{-n-a}}{\Gamma(a+n+1)}, \qquad n \to \infty,$$

we can conclude that (21) is satisfied. □

Finally, we can obtain the eigenvalues and the eigenfunctions of $T_{\beta,\alpha}$ (remember that the Bessel function J_a has an increasing sequence of positive zeros, and that we are denoting them by $j_{a,k}$, for $k \geq 1$).

Theorem 3. *Let $\alpha, \beta > -1$, and $\alpha + \beta > -1$. Then the eigenvalues of the integral operator $T_{\beta,\alpha}$ are $\{\pm i/j_{\alpha+\beta+1,k}\}_{k\geq 1}$ and the eigenfunctions $g_{\pm i/j_{\alpha+\beta+1,k}}(t)$ have the series expansion in terms of the generalized Gegenbauer polynomials*

$$\sum_{n=1}^{\infty} (\mp i)^{n-1} \frac{(\alpha+\beta+n+1)}{(\alpha+\beta+2)} h_{n-1,\alpha+\beta+2}\left(\frac{1}{j_{\alpha+\beta+1,k}}\right) C_n^{(\beta+1/2,\alpha+1/2)}(t). \tag{27}$$

Moreover,

$$g_{\pm i/j_{\alpha+\beta+1,k}}(t) = \mp i \left(\frac{j_{\alpha+\beta+1,k}}{2}\right)^{\alpha+\beta+1} \frac{E_\alpha(\mp t j_{\alpha+\beta+1,k})}{\Gamma(\alpha+\beta+1)(\alpha+\beta+2) J_{\alpha+\beta}(j_{\alpha+\beta+1,k})}. \tag{28}$$

Proof. The eigenvalues follow immediately from Theorem 2. The expression (27) for the eigenfunctions is a consequence of (14), (24), the fact $b_n(i\lambda) = h_{n-1,\alpha+\beta+2}(i\lambda)$ and the identity

$$h_{n-1,\alpha+\beta+2}\left(i \frac{\pm i}{j_{\alpha+\beta+1,k}}\right) = (\mp 1)^{n-1} h_{n-1,\alpha+\beta+2}\left(\frac{1}{j_{\alpha+\beta+1,k}}\right),$$

which is obtained using that $R_{n,a}(-z) = (-1)^n R_{n,a}(a)$.

Let us prove (28). Taking $\lambda = \pm i/j_{\alpha+\beta+1,k}$ in (26), we obtain that

$$J_{\alpha+\beta+n+1}(j_{\alpha+\beta+1,k}) = -h_{n-1,\alpha+\beta+2}\left(\frac{1}{j_{\alpha+\beta+1,k}}\right) J_{\alpha+\beta}(j_{\alpha+\beta+1,k}).$$

From the previous identity, (27) becomes

$$\begin{aligned}
g_{\pm i/j_{\alpha+\beta+1,k}}(t) &= \frac{-1}{(\alpha+\beta+2)J_{\alpha+\beta}(j_{\alpha+\beta+1,k})} \\
&\quad \times \sum_{n=1}^{\infty} (\mp i)^{n-1}(\alpha+\beta+n+1)J_{\alpha+\beta+n+1}(j_{\alpha+\beta+1,k})C_n^{(\beta+1/2,\alpha+1/2)}(t) \\
&= \frac{\mp i\, j_{\alpha+\beta+1,k}^{\alpha+\beta+1}}{(\alpha+\beta+2)J_{\alpha+\beta}(j_{\alpha+\beta+1,k})} \\
&\quad \times \sum_{n=0}^{\infty} i^n(\alpha+\beta+n+1)\frac{J_{\alpha+\beta+n+1}(\mp j_{\alpha+\beta+1,k})}{(\mp j_{\alpha+\beta+1,k})^{\alpha+\beta+1}}C_n^{(\beta+1/2,\alpha+1/2)}(t) \\
&= \mp i \left(\frac{j_{\alpha+\beta+1,k}}{2}\right)^{\alpha+\beta+1} \frac{E_\alpha(\mp t j_{\alpha+\beta+1,k})}{\Gamma(\alpha+\beta+1)(\alpha+\beta+2)J_{\alpha+\beta}(j_{\alpha+\beta+1,k})}
\end{aligned}$$

where in the last step we have used (11). □

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