Mean and Weak Convergence of Some Orthogonal Fourier Expansions by Using $A_p$ Theory

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Introduction

Let $d\mu$ be a finite positive Borel measure on $\mathbb{R}$ such that $\text{supp}(d\mu)$ is an infinite set and let $p_n(d\mu)$ denote the corresponding orthonormal polynomials. For $f \in L^1(d\mu)$, $S_n f$ stands for the $n$th partial sum of the orthogonal Fourier expansion of $f$ in $\{p_n(d\mu)\}_{n=0}^{\infty}$, that is,

$$S_n(f, x) = \sum_{k=0}^{n} a_k p_k(x), \quad a_k = \hat{f}(k) = \int_{\mathbb{R}} f p_k d\mu.$$ 

The study of the convergence of $S_n f$ in $L^p(d\mu)$ ($p \neq 2$) has been discussed for several classes of orthogonal polynomials (c.f. Askey-Wainger [1], Badkov [2–4], Muckenhoupt [9–11], Newman-Rudin [13], Pollard [14–16], Wing [19]). For instance, in the case of Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ which are orthogonal in $[-1, 1]$ with respect to the weight $w(x) = (1-x)^{\alpha}(1+x)^{\beta}$, $\alpha, \beta \geq -1/2$, Pollard proved that $|1/p - 1/2| < \min\{1/(4\alpha + 4), 1/(4\beta + 4)\}$ is a sufficient condition for the uniform boundedness $\|S_n f\|_{p, w} \leq C\|f\|_{p, w}$, which is equivalent to the convergence in $L^p(w)$, $1 < p < \infty$. Newman and Rudin showed that the previous condition is also necessary and later Muckenhoupt extended these results to $\alpha, \beta > -1$.

On the other hand, Máté, Nevai and Totik [8] obtained, in a general way, necessary conditions for the mean convergence of Fourier expansions:

**Theorem (Máté–Nevai–Totik).** Let $d\mu$ be such that $\text{supp}(d\mu) = [-1, 1]$, $\mu' > 0$ almost everywhere, $U$ and $V$ nonnegative Borel measurable functions such that neither of them vanishes almost everywhere in $[-1, 1]$ and $V$ is finite.
on a set with positive Lebesgue measure. If \( S_n \) is uniformly bounded from \( L^p(V^p \, d\mu) \) into \( L^p(U^p \, d\mu) \), then

(i) \( U^p \in L^1(d\mu) \), \( V^{-q} \in L^1(d\mu) \), \( q = p/(p-1) \),

(ii) \( \int_{-1}^{1} U(x)^p \mu'(x)^{1-p/2} (1-x^2)^{-p/4} \, dx < \infty \),

(iii) \( \int_{-1}^{1} V(x)^{-q} \mu'(x)^{1-q/2} (1-x^2)^{-q/4} \, dx < \infty \).

Mean convergence

The main subject in this paper is the study of the mean and weak boundedness of the orthogonal Fourier expansion, in some particular cases, by using \( A_p \)-theory, which plays a central role in the weighted norm inequalities for the Hardy-Littlewood maximal operator and the Hilbert transform.

We start with the mean convergence recalling some definitions:

(i) \( (u, v) \in A_p(-1, 1), \quad 1 < p < \infty \), iff there exists a positive constant \( C \) such that

\[
\left( \int_I u(x) \, dx \right) \left( \int_I v(x)^{-1/(p-1)} \, dx \right)^{p-1} \leq C |I|^p
\]

for all intervals \( I \subset [-1, 1] \), where \( |I| \) is the Lebesgue measure of \( I \).

(ii) \( (u, v) \in A_{\delta}^p(-1, 1) \) (\( \delta > 1 \)) iff \( (u^\delta, v^\delta) \in A_p(-1, 1) \).

(iii) Given a sequence \( \{ (u_n, v_n) \}_{n=0}^{\infty} \), we say that \( (u_n, v_n) \in A_p(-1, 1) \) uniformly if there exists a constant \( C \), independent of \( n \), such that

\[
\left( \int_I u_n(x) \, dx \right) \left( \int_I v_n(x)^{-1/(p-1)} \, dx \right)^{p-1} \leq C |I|^p
\]

for all intervals \( I \subset [-1, 1] \).

It is well known that \( (u, v) \in A_p \) is a necessary condition for the boundedness of the Hilbert transform \( H \) from \( L^p(v) \) into \( L^p(u) \) and that \( (u, v) \in A_{\delta}^p \) (for some \( \delta > 1 \)) is a sufficient condition [7], [12]. Analogous conditions work for the uniform boundedness, modifying slightly the arguments in [12].

This is connected with the Fourier expansion, and the idea comes from Pollard: let \( \{ p_n(x) \}_{n=0}^{\infty} \) denote the orthonormal polynomials with respect to \( d\mu = \mu'(x) \, dx \) and \( \{ q_n(x) \}_{n=0}^{\infty} \) the orthonormal polynomials with respect to \( (1-x^2) \, d\mu \). Then

\[
S_n(f, x) = \int_{-1}^{1} f(t) K_n(x, t) \mu'(t) \, dt
\]

and the kernel \( K_n(x, t) \) can be decomposed in the form

\[
K_n(x, t) = r_n T_1(n, x, t) + s_n T_2(n, x, t) + s_n T_3(n, x, t)
\]
Denote:
\[ T_1(n, x, t) = p_{n+1}(x)p_{n+1}(t), \]
\[ T_2(n, x, t) = (1 - x^2)^2 p_{n+1}(x)q_n(t), \]
\[ T_3(n, x, t) = T_2(n, t, x) = (1 - x^2)^2 p_{n+1}(t)q_n(x) x - t. \]

If \( \mu' > 0 \) a.e., then \( \{r_n\} \) and \( \{s_n\} \) are bounded [17]. Let \( U \) and \( V \) be weights, \( 1 < p < \infty \), and
\[
W_i(f, x) = W_{i,n}(f, x) = \int_{-1}^{1} f(t)T_i(n, x, t)\mu'(t) dt \quad (i = 1, 2, 3).
\]

We try to estimate the three terms:
\[
\|\langle W_i,f \rangle U\|_{p,\mu'} \leq C\|fV\|_{p,\mu'}.
\]

Denote:
\[
u_n(x) = |p_{n+1}(x)|^p U(x)^p \mu'(x), \quad v_n(x) = |q_n(x)|^{-p}(1 - x^2)^{-p}V(x)^p \mu'(x)^{1-p},
\]
\[
u_n(x) = |q_n(x)|^p(1 - x^2)^p \mu'(x)^{1-p}, \quad \nu_n(x) = |p_{n+1}(x)|^{-p}V(x)^p \mu'(x)^{-1+p}.
\]

By using Hölder’s inequality and \( A_p \) results we obtain the following sufficient conditions for the boundedness of \( W_i \) \( (i = 1, 2, 3) \):
\[
(u_n, v_n) \in A_p^\delta(-1, 1) \quad \text{uniformly for some } \delta > 1,
\]
\[
(\nu_n, \nu_n) \in A_p^\delta(-1, 1) \quad \text{uniformly for some } \delta > 1.
\]

On the other hand, the conditions
\[
((1 - x^2)^{-p/4}U(x)^p \mu'(x)^{1-p/2}, (1 - x^2)^{-p/4}V(x)^p \mu'(x)^{1-p/2}) \in A_p(-1, 1) \quad (1)
\]
and
\[
((1 - x^2)^{p/4}U(x)^p \mu'(x)^{1-p/2}, (1 - x^2)^{p/4}V(x)^p \mu'(x)^{1-p/2}) \in A_p(-1, 1) \quad (2)
\]
turn out to be necessary for the boundedness of \( W_i \) \( (i = 1, 2, 3) \). From (1), (2) and Th. 2 in [8], Máté-Nevai-Totik’s conditions for the mean convergence of \( S_n f \) can be obtained.

Next, we introduce a particular kind of measures.

**Definition.** We say that \( d\mu = \mu'(x) \, dx \in H \) if \( \mu'(x) = (1 - x)^{\alpha}(1 + x)^{\beta}w(x) \), where:

(i) \( w > 0 \) a.e. and \( C_1 < w(x) < C_2 \) for \( x \in (1 - \varepsilon, 1) \) and \( x \in (-1, -1 + \varepsilon) \).

(ii) \( |p_n(x)| \leq C(1 - x + a_n)^{-\alpha(2+1/4)}(1 + x + b_n)^{-\beta(2+1/4)}w(x)^{-1/2}. \)
(iii) \(|g_n(x)| \leq C(1-x+a_n)^{-(\alpha/2+3/4)}(1+x+b_n)^{-(\beta/2+3/4)}w(x)^{-1/2}\) and \(\{a_n\}, \{b_n\}\) are positive sequences such that \(\lim a_n = \lim b_n = 0\).

There exist particular weights belonging to the class \(H\): the generalized Jacobi weights \((GJ)\) \(d\mu(x) = \mu'(x)\,dx\), being

\[
\mu'(x) = \varphi(x)(1-x)^{\Gamma_1} \prod_{k=2}^{N-1} |x-x_k|^\Gamma k (1+x)^\Gamma N
\]

where \(\Gamma_k \geq 0\) \((k = 1, 2, \ldots, N)\), \(1 > x_2 > \cdots > x_{N-1} > -1\), \(\varphi > 0\) and continuous on \([-1, 1]\) and \(\omega(\delta)/\delta \in L^1(0,1)\), \(\omega\) being the modulus of continuity of \(\varphi\).

**Theorem 1.** Let \(d\mu \in H\), \(U(x) = (1-x)^a(1+x)^b u(x)\), \(V(x) = (1-x)^A(1+x)^B v(x)\), with \(u > 0\) a.e., \(v > 0\) a.e. and such that \(C_1 < u(x), v(x) < C_2\) for \(x \in (1-\varepsilon, 1)\) and \(x \in (-1, -1+\varepsilon)\). If

\[
|(\alpha+1)(1/p-1/2) + (a+A)/2| < (a-A)/2 + \min\{1/4, (\alpha+1)/2\}, \quad A \leq a,
\]

\[
|(\beta+1)(1/p-1/2) + (b+B)/2| < (b-B)/2 + \min\{1/4, (\beta+1)/2\}, \quad B \leq b,
\]

and

\[
(w^{1-p/2}u^p, w^{1-p/2}v^p) \in A^\delta_p(-1,1) \quad \text{for some} \, \delta > 1,
\]

then:

\[
\int_{-1}^1 |S_n(f, x)U(x)|^p \mu'(x)\,dx \leq C \int_{-1}^1 |f(x)V(x)|^p \mu'(x)\,dx.
\]

This theorem is a consequence of the following lemmas:

**Lemma 1.** Let \(\{u_n(x)\}, \{v_n(x)\}, \{U_n(x)\}, \{V_n(x)\}\) be sequences of weights defined on a finite interval \((a, b)\). Let \(c \in (a, b)\) and \(\varepsilon > 0\) be fixed and independent of \(n\). Assume that there exist some positive constants \(\lambda_i\) \((i = 1, 2, 3, 4)\) such that \(\lambda_1 \leq U_n(x), V_n(x) \leq \lambda_2\) on \((a, c+\varepsilon)\) and \(\lambda_3 \leq u_n(x), v_n(x) \leq \lambda_4\) on \((c-\varepsilon, b)\). If \((u_n, v_n) \in A_p(a, c)\) and \((U_n, V_n) \in A_p(c, b)\) uniformly, then \((u_nU_n, v_nV_n) \in A_p(a, b)\) uniformly.

**Lemma 2.** Let \(\{x_n\}\) be a sequence of positive numbers which converges to zero. Then \((x^r(x+x_n)^s, x^R(x+x_n)^S) \in A^\delta_p(0,1)\) uniformly if and only if:

\[
r > -1, \quad R < p - 1, \quad R \leq r, \quad R + S \leq r + s, \quad r + s > -1, \quad R + S < p - 1.
\]

From the above theorem, we have the following result, which was established by Badkov [3] (using other methods and without the restriction \(\Gamma_k \geq 0, 2 \leq k \leq N - 1)\):

\[
\]
Corollary 1. Let \( w \in (GJ) \) and \( U(x) = (1 - x)^a(1 + x)^b \prod_{k=2}^{N-1} |x - x_k|^{c_k} \). If
\[
|\Gamma_1 + 1)(1/2 - 1/p) - a| < \min\{1/4, (\Gamma_1 + 1)/2\},
|\Gamma_N + 1)(1/2 - 1/p) - b| < \min\{1/4, (\Gamma_N + 1)/2\}
\]
and
\[
|\Gamma_k + 1)(1/2 - 1/p) - c_k| < \min\{1/2, (\Gamma_k + 1)/2\} \quad (k = 2, \ldots, N - 1),
\]
then
\[
\int_{-1}^{1} |S_n(f, x)U(x)|^p \mu'(x) \, dx \leq C \int_{-1}^{1} |f(x)U(x)|^p \mu'(x) \, dx.
\]

Weak convergence

Another aim in this paper is to examine the weak behaviour of the orthogonal Fourier expansion, that is to study if there exists a constant \( C \), independent of \( n, y \) and \( f \), such that:
\[
\int_{|S_n(f, x)| > y} d\mu(x) \leq Cy^{-p} \int_{-1}^{1} |f(x)|^p \, d\mu(x), \quad y > 0,
\]
i.e., if \( S_n \) is uniformly bounded from \( L^p(d\mu) \) into \( L^p_y(d\mu) \), \( 1 < p < \infty \).

The previous inequality only can be true, besides the mean convergence interval, in its endpoints. For the Fourier-Legendre expansion \( (d\mu = dx) \), Chanillo [5] proved that the partial sum operator is not weak type \((4, 4)\).

The following result gives necessary conditions for the weak boundedness [6].

Theorem 2. Let \( d\mu \) be such that \( \text{supp}(d\mu) = [-1, 1], \mu' > 0 \, a.e. \), \( U \) and \( V \) be weights, \( 1 < p < \infty \). If there exists a constant \( C \) such that
\[
\|S_nf\|_{L^p_y(U^p \, d\mu)} \leq C\|f\|_{L^p(V^p \, d\mu)}
\]
holds for all integers \( n \geq 0 \) and every \( f \in L^p(V^p \, d\mu) \), then:
(i) \( U^p, V^{-q} \in L^1(d\mu) \),
(ii) \( \mu'(x)^{-1/2}(1 - x^2)^{-1/4} \in L^p_y(U^p \, d\mu) \),
(iii) \( \mu'(x)^{-1/2}(1 - x^2)^{-1/4} \in L^p(V^{-q} \, d\mu) \).

This result is a consequence of the following lemmas:

Lema 3. Let \( U \) and \( V \) be weights and \( 1 < p < \infty \). If there exists a constant \( C \) such that for every \( f \in L^p(V^p \, d\mu) \) the inequality
\[
\|S_nf\|_{L^p_y(U^p \, d\mu)} \leq C\|f\|_{L^p(V^p \, d\mu)}
\]
holds for all integers \( n \geq 0 \), then
\[
\|P_n\|_{L^p_y(V^{-q} \, d\mu)} \|P_n\|_{L^p_y(U^p \, d\mu)} \leq C.
\]
Lema 4 ([8, Th. 2]). Let \( \text{supp}(d\mu) = [-1, 1] \), \( \mu' > 0 \) a.e. in \([-1, 1]\) and \( 0 < p \leq \infty \). There exists a constant \( C \) such that if \( g \) is a Lebesgue-measurable function in \([-1, 1]\), then
\[
\|\mu'(x)^{-1/2}(1 - x^2)^{-1/4}\|_{L^p(|g|^p \, dx)} \leq \liminf_{n \to \infty} \|p_n\|_{L^p(|g|^p \, dx)}.
\]

In particular, if
\[
\liminf_{n \to \infty} \|p_n\|_{L^p(|g|^p \, dx)} = 0
\]
then \( g = 0 \) a.e.

We are going to study the weak boundedness of the Fourier-Jacobi expansion. Since for \(-1 < \alpha, \beta \leq -1/2\) the conditions \(|1/p - 1/2| < \min\{1/(4\alpha + 4), 1/(4\beta + 4)\}\) are trivial for \( p \in (1, \infty)\), we suppose, by symmetry, \( \alpha \geq \beta \) and \( \alpha > -1/2 \). Then, the mean convergence interval is \( 4(\alpha + 1)/(2\alpha + 3) < p < 4(\alpha + 1)/(2\alpha + 1) \).

Remark 1. If \( U(x) = V(x) = 1 \), the inequality \((\alpha + 1)(1/p - 1/2) < 1/4\) is not satisfied for \( p = 4(\alpha + 1)/(2\alpha + 3) \). It implies that \( S_n \) is not weak type \((p, p)\) for the lower endpoint of the interval of mean convergence. The same happens with generalized Jacobi polynomials.

Remark 2. The conditions in Theorem 2 are the same as those of Máté-Nevai-Totik’s theorem. Thus, the conditions obtained by Máté, Nevai and Totik are necessary not only for the mean convergence but also for the weak convergence.

Remark 3. It can be proved that Máté-Nevai-Totik’s conditions are not sufficient for the weak convergence. In order to prove this, consider the Fourier-Legendre expansion \((d\mu = dx)\), \( p = 4 \) and take
\[
U(x) = \left| \log \frac{1 + x}{4} \right|^{-5/8} \left| \log \frac{1 - x}{4} \right|^{-5/8},
\]
\[
V(x) = \left| \log \frac{1 + x}{4} \right|^{-3/8} \left| \log \frac{1 - x}{4} \right|^{-3/8}.
\]

Let \( S_n \) denote the \( n \)th partial sum of the Fourier-Jacobi expansion with respect to \( \mu'(x) = (1 - x)^\alpha (1 + x)^\beta \), being \( \alpha \geq \beta \) and \( \alpha > -1/2 \). Then, the interval of mean convergence is given by \( 4(\alpha + 1)/(2\alpha + 3) < p < 4(\alpha + 1)/(2\alpha + 1) \). Theorem 2 works to prove that \( S_n \) is not weak type on \( L^p(\mu') \) for \( p = 4(\alpha + 1)/(2\alpha + 3) \), but it is not useful to show that \( S_n \) is not weak type for \( p = 4(\alpha + 1)/(2\alpha + 1) \). It leads us to make use of other arguments.

Theorem 3. Let \( r = 4(\alpha + 1)/(2\alpha + 1) \). Then, there exists no constant \( C \), independent of \( n \) and \( f \in L^r(\mu') \), such that
\[
\|S_n f\|_{L^r(\mu')} \leq C \|f\|_{L^r(\mu')}.
\]
Proof. Decompose the kernel $K_n(x, t)$, as before, in the form

$$K_n(x, t) = r_n T_1(n, x, t) + s_n T_2(n, x, t) + s_n T_3(n, x, t).$$

By using the estimates

$$|p_n(x)| \leq C(1 - x)^{-\alpha/2 - 1/4}, \quad |q_n(x)| \leq C(1 - x)^{-\alpha/2 - 3/4}, \quad x \in (0, 1),$$

Hölder’s inequality and standard arguments of $A_p$ theory, the boundedness of $T_1$ and $T_3$ can be proved. Now, it is not difficult to prove that

$$\int |p_n(x)H(f(t)q_{n-1}(1-t^2)\mu(t), x)| > y \mu'(x) dx \leq Cy^{-r} \|f\|_{r, \mu'}$$

is not satisfied for any fixed constant $C$. The proof is by contradiction, constructing a sequence of functions $\{f_{m,n}\}$ such that the constant $C$ appearing in the previous inequality grows with $m$.

References


