WEIGHTED $L^p$-BOUNDEDNESS OF FOURIER SERIES RELATIVE TO GENERALIZED JACOBI WEIGHTS.

José J. Guadalupe 1, Mario Pérez 1, Francisco J. Ruiz 1 and Juan L. Varona 2

Abstract. Let $w$ be a generalized Jacobi weight on the interval $[-1,1]$ and, for each function $f$, let $S_n f$ denote the $n$-th partial sum of the Fourier series of $f$ with respect to the orthogonal polynomials relative to $w$. We prove a result about uniform boundedness of the operators $S_n$ in some weighted $L^p$ spaces. The study of the norms of the kernels $K_n$ associated with the operators $S_n$ allows us to obtain a relation between the Fourier series relative to different generalized Jacobi weights.

Let $w$ be a generalized Jacobi weight, that is,

$$w(x) = h(x)(1 - x)^a(1 + x)^b \prod_{i=1}^{N} |x - t_i|^{\gamma_i}, \quad x \in [-1,1]$$

where

a) $\alpha, \beta, \gamma_i > -1, \ t_i \in (-1,1), \ t_i \neq t_j \ \forall i \neq j$;

b) $h$ is a positive, continuous function on $[-1,1]$ and $w(h, \delta)\delta^{-1} \in L^1(0,1)$, $w(h, \delta)$ being the modulus of continuity of $h$.

Let $d\mu = w(x) \, dx$ on $[-1,1]$ and let $S_n (n \geq 0)$ be the $n$-th partial sum of the Fourier series in the orthonormal polynomials with respect to $d\mu$. The study of the boundedness

$$\|S_n f\|_{L^p(u^p \, d\mu)} \leq C \|f\|_{L^p(v^p \, d\mu)}, \quad (1)$$

was done by Badkov ([1]) in the case $u = v$ by means of a direct estimation of the kernels $K_n (x,y)$ associated with the polynomials orthogonal with respect to $d\mu$. Later, one of us ([10]) considered the same problem, with $u$ and $v$ not necessarily equal; his method consists of an appropriate use of the theory of $A_p$ weights. He found conditions for (1) which generalized those obtained for $u = v$ by Badkov. However, this result, which we state below, follows only in the case $\gamma_i \geq 0, \ i = 1, \ldots, N$.

---

1 Dpto. de Matemáticas. Universidad de Zaragoza. 50009 Zaragoza. Spain.

Theorem 1. Let $\gamma_i \geq 0$, $i = 1, \ldots, N$ and $1 < p < \infty$. If the inequalities

$$
\begin{align*}
A + (\alpha + 1)\left(\frac{1}{p} - \frac{1}{2}\right) < & \min\left\{\frac{1}{4}, \frac{\alpha + 1}{2}\right\} \\
B + (\beta + 1)\left(\frac{1}{p} - \frac{1}{2}\right) < & \min\left\{\frac{1}{4}, \frac{\beta + 1}{2}\right\} \\
G_i + (\gamma_i + 1)\left(\frac{1}{p} - \frac{1}{2}\right) < & \min\left\{\frac{1}{2}, \frac{\gamma_i + 1}{2}\right\} \\
(i = 1, \ldots, N)
\end{align*}
$$

(2)

and

$$
\begin{align*}
a + (\alpha + 1)\left(\frac{1}{p} - \frac{1}{2}\right) > & -\min\left\{\frac{1}{4}, \frac{\alpha + 1}{2}\right\} \\
b + (\beta + 1)\left(\frac{1}{p} - \frac{1}{2}\right) > & -\min\left\{\frac{1}{4}, \frac{\beta + 1}{2}\right\} \\
g_i + (\gamma_i + 1)\left(\frac{1}{p} - \frac{1}{2}\right) > & -\min\left\{\frac{1}{2}, \frac{\gamma_i + 1}{2}\right\} \\
(i = 1, \ldots, N)
\end{align*}
$$

(3)

and

$$
A \leq a, \quad B \leq b, \quad G_i \leq g_i
$$

(4)

hold, then

$$\exists C > 0 \text{ such that } \|S_n f\|_{L^p(u^p d\mu)} \leq C \|f\|_{L^p(v^p d\mu)} \forall f \in L^p(v^p d\mu), \forall n \in \mathbb{N}.$$

The objective of this paper is to show that the result remains true without the restriction $\gamma_i \geq 0$ and that conditions (2), (3) and (4) are also necessary for the uniform boundedness:

Theorem 2. Let $1 < p < \infty$. Then, there exists $C > 0$ such that

$$\|S_n f\|_{L^p(u^p d\mu)} \leq C \|f\|_{L^p(v^p d\mu)} \forall f \in L^p(v^p d\mu), \forall n \in \mathbb{N},$$

if and only if the inequalities (2), (3) and (4) are satisfied.

For the sake of completeness, we give a brief sketch of the proof of theorem 1 (see also [10]). By using Pollard’s decomposition of the kernels $K_n(x, y)$ (see [8], [5]), the uniform boundedness of $S_n$ can be reduced to that of the Hilbert transform with pairs of weights

$$
(|P_{n+1}(x)|^p u(x)^p w(x), |Q_n(x)|^{-p} (1 - x^2)^{-p} v(x)^p w(x)^{1-p})
$$

and

$$
(|Q_n(x)|^p (1 - x^2)^p u(x)^p w(x), |P_{n+1}(x)|^{-p} v(x)^p w(x)^{1-p}),
$$

$Q_n$ being the $n$-th orthonormal polynomial relative to the measure $(1 - x^2)d\mu$. Using now Hunt-Muckenhoupt-Wheeden and Neugebauer results (see [2], [6]), together with some known estimates for generalized Jacobi polynomials (see (8) below), for the above uniform boundedness the following conditions turn out to be sufficient:

$$(u_n^\delta, v_n^\delta) \in A_p((-1, 1))$$

and

$$(\bar{u}_n^\delta, \bar{v}_n^\delta) \in A_p((-1, 1))$$
for some $\delta > 1$, with $A_p$ constants independent of $n$, where
\[
    u_n(x) = (1 - x)^{ap+\alpha}(1 - x + n^{-2})^{-p(2\alpha+1)/4}
        \times (1 + x)^{bp+\beta}(1 + x + n^{-2})^{-p(2\beta+1)/4}
        \times \prod_{i=1}^N |x - t_i|^{g_i p + \gamma_i(\{x - t_i\} + n^{-1})^{-p\gamma_i/2}},
\]
and similar expressions for $\bar{u}_n$.

Lemma 3. Let $\{x_n\}_{n \geq 0}$ be a sequence of positive numbers converging to 0. Let $r, s, R, S \in \mathbb{R}$. Then,
\[
    (|x|^r(|x| + x_n)^s, |x|^r(|x| + x_n)^S) \in A_p((-1, 1))
\]
with a constant independent of $n$ if and only if the following inequalities hold:
\[
    r > -1; \quad R < p - 1; \quad R \leq r; \\
    r + s > -1; \quad R + S < p - 1; \quad R + S \leq r + s.
\]

At least in the case $u = v$ (thus $g_i = G_i$, $\forall i$), inequality $R \leq r$ requires $\gamma_i \geq 0 \ \forall i$. But, with this assumption, theorem 1 follows.

Let us introduce now some notation: $\{P_n(x)\}$, $\{k_n\}$ and $\{K_n(x, y)\}$ will be, respectively, the orthonormal polynomials, their leading coefficients and the kernels relatives to $d\mu$; if $c \in (-1, 1)$, $\{P_n^c(x)\}$, $\{k_n^c\}$ and $\{K_n^c(x, y)\}$ will be the corresponding to $(x - c)^2d\mu$. Then, it is not difficult to establish $\forall n \in \mathbb{N}$ the relations
\[
    K_n(x, y) = (x - c)(y - c)K_{n-1}(x, y) + \frac{K_n(x, c)K_n(c, y)}{K_n(c, c)}; \quad (5)
\]
\[
    K_n(x, c) = \frac{k_n}{k_n^c} P_n(c)P_n^c(x) - \frac{k_{n-1}^c}{k_{n+1}} P_{n+1}(c)P_{n-1}(x). \quad (6)
\]

It can be also shown (see [4], theorems 10 and 11, and [9], pag. 212) that
\[
    \lim_{n \to \infty} \frac{k_n}{k_n^c} = \lim_{n \to \infty} \frac{k_{n-1}^c}{k_{n+1}} = \frac{1}{2}. \quad (7)
\]

If we define
\[
    d(x, n) = (1 - x + n^{-2})^{-(2\alpha+1)/4}(1 + x + n^{-2})^{-(2\beta+1)/4}\prod_{i=1}^N (|x - t_i| + n^{-1})^{-\gamma_i/2},
\]

3
it is known ([1]) that there exists a constant $C$ such that $\forall x \in [-1,1], \forall n \in \mathbb{N}$

$$|P_n(x)| \leq Cd(x,n).$$ \hspace{1cm} (8)

There are also some well-known estimates for the kernels, one of them being this ([7], pag. 4 and pag. 119, theorem 25): if $c \in (-1,1)$ and the factor $|x - c|$ occurs in $w$ with an exponent $\gamma$, there exist some positive constants $C_1$ and $C_2$, depending on $c$, such that $\forall n \in \mathbb{N}$

$$C_1 n^{\gamma+1} \leq K_n(c,c) \leq C_2 n^{\gamma+1}. \hspace{1cm} (9)$$

From now on, all constants will be denoted $C$, so by $C$ we will mean a constant, possibly different in each occurrence. Using (6), (7) and (8) we obtain the following result:

**Proposition 4.** Let $1 < p < \infty$, $1/p + 1/q = 1$ and suppose the inequality (3) holds. Let $-1 < c < 1$ and let $\gamma$ and $g$ be the exponents of $|x - c|$ in $w$ and $u$, respectively. Then, there exists a positive constant $C$ such that $\forall n \geq 0$:

$$\parallel K_n(x,c) \parallel_{L^p(u^p w)} \leq \begin{cases} 
C n^{(\gamma+1)/q - g} & \text{if } g < (\gamma + 1)(1/2 - 1/p) + 1/2 \\
C n^{\gamma/2} (\log n)^{1/p} & \text{if } g = (\gamma + 1)(1/2 - 1/p) + 1/2 \\
C n^{\gamma/2} & \text{if } (\gamma + 1)(1/2 - 1/p) + 1/2 < g
\end{cases}$$ \hspace{1cm} (10)

**Proof.** From (8) it follows that $|P_n(c)| \leq C_n^{\gamma/2}$. Since $\{P_n\}$ is the sequence associated with $(x - c)^2 d\mu$, it also follows from (8) that

$$|P_n^c(x)| \leq C(|x - c| + n^{-1})^{-1} d(x,n).$$

Now, from (6) and (7) we get:

$$|K_n(x,c)| \leq C n^{\gamma/2} (|x - c| + n^{-1})^{-1} d(x,n). \hspace{1cm} (10)$$

Let us take $\varepsilon > 0$ such that $|t_i - c| > \varepsilon$ for all $t_i \neq c$. We can write:

$$\parallel K_n(x,c) \parallel_{L^p(u^p w)}^p$$

$$= \int_{|x-c| \geq \varepsilon} |K_n(x,c)|^p u(x)^p w(x) dx + \int_{|x-c| < \varepsilon} |K_n(x,c)|^p u(x)^p w(x) dx$$

Using (10), we obtain for the first term

$$\int_{|x-c| \geq \varepsilon} |K_n(x,c)|^p u(x)^p w(x) dx \leq C n^{p\gamma/2} \int_{|x-c| \geq \varepsilon} (|x - c| + n^{-1})^{-p} d(x,n)^p u(x)^p w(x) dx$$

$$\leq C n^{p\gamma/2} \int_{-1}^1 d(x,n)^p u(x)^p w(x) dx.$$
It is easy to deduce from (3) that this last integral is bounded by a constant which does not depend on \( n \), so
\[
\int_{|x-c| \geq \varepsilon} |K_n(x,c)|^p u(x)^p w(x) dx \leq C n^{p\gamma/2}. \tag{11}
\]

Let us take now the second term; since for \( |x-c| < \varepsilon \) there exists a constant \( C \) such that \( \forall n \; d(x,n) \leq C(|x-c| + n^{-1})^{-\gamma/2} \), \( u(x) \leq C|x-c|^p \) and \( w(x) \leq C|x-c|^\gamma \), we have
\[
\int_{|x-c| < \varepsilon} |K_n(x,c)|^p u(x)^p w(x) dx \leq C n^{p\gamma/2} \int_{|x-c| < \varepsilon} (|x-c| + n^{-1})^{-p(1+\gamma/2)} |x-c|^{gp+\gamma} dx
\]
\[
\leq C n^{p\gamma/2} \int_{|x-c| < \varepsilon} (|x-c| + n^{-1})^{-p(1+\gamma/2)} \cdot |x-c|^{gp+\gamma} dy
\]
\[
= C n^{p\gamma/2 + p(1+\gamma/2) - gp - \gamma - 1} \int_0^1 (ny + 1)^{-p(1+\gamma/2)} (ny)^{gp+\gamma} n dy
\]
\[
= C n^{p\gamma/2 + p(1+\gamma/2) - gp - \gamma - 1} \int_0^n (r + 1)^{-p(1+\gamma/2)} (\gamma + 1) r^{gp+\gamma} dr.
\]

Taking into account that \( p(1+\gamma/2) - gp - \gamma - 1 = p[(\gamma + 1)(1/2 - 1/p) - g + 1/2] \) and there exist some constants \( C_1 \) and \( C_2 \) such that \( C_1 \leq r + 1 \leq C_2 \) on \([0, 1]\) and \( C_1 r \leq r + 1 \leq C_2 r \) on \([1, n]\), we finally get the inequality
\[
\int_{|x-c| < \varepsilon} |K_n(x,c)|^p u(x)^p w(x) dx \leq C n^{p\gamma/2 + p[(\gamma + 1)(1/2 - 1/p) - g + 1/2]} \int_0^1 r^{gp+\gamma} dr
\]
\[
+ C n^{p\gamma/2 + p[(\gamma + 1)(1/2 - 1/p) - g + 1/2]} \int_1^n r^{-p[(\gamma + 1)(1/2 - 1/p) - g + 1/2] - 1} dr. \tag{12}
\]

Since (3) implies \( gp + \gamma > -1 \), the first term is bounded by
\[
C n^{p\gamma/2 + p[(\gamma + 1)(1/2 - 1/p) - g + 1/2]} \int_0^1 r^{gp+\gamma} dr \leq C n^{p\gamma/2 + p[(\gamma + 1)(1/2 - 1/p) - g + 1/2]} . \tag{13}
\]

For the second term, let us consider separately the three cases in the statement.

a) If \( g < (\gamma + 1)(1/2 - 1/p) + 1/2 \), then \( -p[(\gamma + 1)(1/2 - 1/p) - g + 1/2] - 1 < -1 \). Thus
\[
\int_1^n r^{-p[(\gamma + 1)(1/2 - 1/p) - g + 1/2] - 1} dr \leq C.
\]

In this case, (12) and (13) imply:
\[
\int_{|x-c| < \varepsilon} |K_n(x,c)|^p u(x)^p w(x) dx \leq C n^{p\gamma/2 + p[(\gamma + 1)(1/2 - 1/p) - g + 1/2]}.
\]

5
Since \( p[(\gamma + 1)(1/2 - 1/p) - g + 1/2] > 0 \), from this inequality and (11) we obtain
\[
\|K_n(x, c)\|_{L^p(u^p w)}^p \leq C n^{p[(\gamma + 1)(1/2 - 1/p) - g + 1/2]}
= C n^{p[(\gamma + 1)(1-1/p) - g]} = C n^{p[(\gamma + 1)/q - g]},
\]
as we had to prove.

b) If \( (\gamma + 1)(1/2 - 1/p) + 1/2 < g \), then
\[
- p[(g + 1)(1/2 - 1/p) - g + 1/2] - 1 > -1.
\]
Therefore
\[
\int_1^n r^{-p[(\gamma + 1)(1/2 - 1/p) - g + 1/2] - 1} dr \leq C n^{-p[(\gamma + 1)(1/2 - 1/p) - g + 1/2]}.\]

By (12) and (13), it follows
\[
\int_{|x - c| < \varepsilon} |K_n(x, c)|^p u(x)^p w(x) dx \leq C n^{p\gamma/2}
\]
and
\[
\|K_n(x, c)\|_{L^p(u^p w)}^p \leq C n^{p\gamma/2}.
\]

c) If \( g = (\gamma + 1)(1/2 - 1/p) + 1/2 \)
\[
\int_1^n r^{-p[(\gamma + 1)(1/2 - 1/p) - g + 1/2] - 1} dr = \log n;
\]
hence,
\[
\int_{|x - c| < \varepsilon} |K_n(x, c)|^p u(x)^p w(x) dx \leq C n^{p\gamma/2} \log n
\]
and
\[
\|K_n(x, c)\|_{L^p(u^p w)}^p \leq C n^{p\gamma/2} \log n.
\]

This concludes the proof of the proposition.

**Corollary 5.** Let \( 1 < p < \infty \), \( 1/p + 1/q = 1 \) and suppose the inequality (2) holds. Let \(-1 < c < 1 \) and \( \gamma \) and \( G \) be the exponents of \( |x - c| \) in \( w \) and \( v \), respectively. Then, there exists a positive constant \( C \) such that \( \forall n \in \mathbb{N} \)
\[
\|K_n(x, c)\|_{L^q(v^{-q} w)} \leq \begin{cases} 
C n^{\gamma/2} & \text{if } G < (\gamma + 1)(1/2 - 1/p) + 1/2 \\
C n^{\gamma/2}(\log n)^{1/q} & \text{if } G = (\gamma + 1)(1/2 - 1/p) + 1/2 \\
C n^{(\gamma + 1)/p + G} & \text{if } (\gamma + 1)(1/2 - 1/p) + 1/2 < G
\end{cases}
\]

**Proof.** Just apply proposition 4 to the weight \( v^{-1} \) and keep in mind the equality \( 1/2 - 1/p = 1/q - 1/2 \).

The following result is just what we need to extend theorem 1 to the general case \( \gamma_i > -1 \).
Corollary 6. Let $1 < p < \infty$, $1/p + 1/q = 1$. Suppose the inequalities (2), (3) and (4) hold. Let $-1 < c < 1$. Then, there exists a positive constant $C$ such that $\forall n \geq 0$:

$$\|K_n(x, c)\|_{L^p(u pw)} \|K_n(x, c)\|_{L^q(v^{-q}w)} \leq C K_n(c, c).$$

Proof. It is a simple consequence of proposition 4, corollary 5 and the estimate (9). The only thing we must do is to consider each case in these results separately.

Note. Although it will not be used in what follows, corollary 6 also holds when $c = \pm 1$. The proof is similar: starting from other expressions for $K_n(x, \pm 1)$, analogous results to proposition 4 and corollary 5 can be obtained, and then corollary 6 follows.

We are now ready to prove our main result:

Proof of theorem 2. a) Let us assume first that the inequalities (2), (3) and (4) hold. We prove that the operators $S_n$ are uniformly bounded by induction on the number of negative exponents $\gamma_i$. If $\gamma_i \geq 0 \ \forall i$, the result is true, as we saw before (theorem 1). Now, suppose there exist $k$ negative exponents $\gamma_i$, with $k > 0$, and the result is true for $k - 1$. Let $c \in (-1, 1)$ be a point with a negative exponent $\gamma$. Let us remember the formula (5):

$$K_n(x, y) = (x - c)(y - c)K_{n-1}^c(x, y) + \frac{K_n(x, c)K_n(c, y)}{K_n(c, c)}.$$

We define the operators:

$$T_n f(x) = \int_{-1}^{1} K_n(x, c)K_n(c, y) f(y)w(y)dy,$$

$$R_n f(x) = \int_{-1}^{1} (x - c)(y - c)K_{n-1}^c(x, y)f(y)w(y)dy.$$

Then, $S_n = T_n + R_n$. We are going to study firstly the operators $T_n$:

$$T_n f(x) = \frac{K_n(x, c)}{K_n(c, c)} \int_{-1}^{1} K_n(c, y)f(y)w(y)dy;$$

thus

$$\|T_n f\|_{L^p(u pw)} \leq \int_{-1}^{1} |K_n(c, y)|v(y)^{-1}|f(y)|v(y)w(y)dy \frac{\|K_n(x, c)\|_{L^p(u pw)}}{K_n(c, c)}$$

$$\leq \frac{\|K_n(x, c)\|_{L^p(u pw)} \|K_n(x, c)\|_{L^q(w)}}{K_n(c, c)} \frac{\|f\|_{L^p(w)}}{v}$$

$$= \frac{\|K_n(x, c)\|_{L^p(u pw)} \|K_n(x, c)\|_{L^q(v^{-q}w)}}{K_n(c, c)} \frac{\|f\|_{L^p(v^{-q}w)}}{\|f\|_{L^p(u pw)}}.$$

From corollary 6 it follows

$$\|T_n f\|_{L^p(u pw)} \leq C \|f\|_{L^p(v^{-q}w)} \|f\|_{L^p(u pw)} \forall f \in L^p(v^p d\mu), \forall n \in \mathbb{N}.$$
So, we only need to prove the same bound for the operators $R_n$. But, if we denote by $S_n^c$ the partial sums of the Fourier series with respect to the measure $(x - c)^2 w(x)dx$, it turns out that

$$R_n f(x) = (x - c) \int_{-1}^{1} (y - c) K_{n-1}^c(x, y) f(y) w(y) dy = (x - c) S_n^c(y - c, x),$$

whence

$$\|R_n f\|_{L^p(u^p w)} \leq C \|f\|_{L^p(v^p w)} \forall f \in L^p(v^p w), \forall n \in \mathbb{N}$$


\[
\iff \| (x - c) S_n^c(y - c, x) \|_{L^p(u^p w)} \leq C \| f \|_{L^p(v^p w)} \forall f \in L^p(v^p w), \forall n \in \mathbb{N}
\]


\[
\iff \| (x - c) S_n^c(y - c, x) \|_{L^p(u^p w)} \leq C \| (x - c) g \|_{L^p(v^p w)} \forall g \in L^p(|x - c|^p v^p w), \forall n \in \mathbb{N}
\]


\[
\iff \| S_n^c(y - c, x) \|_{L^p(|x - c|^p u^p w)} \leq C \| g \|_{L^p(|x - c|^p v^p w)} \forall g \in L^p(|x - c|^p v^p w), \forall n \in \mathbb{N}
\]

$$\iff \| S_n^c(y - c, x) \|_{L^p(\tilde{v}^p(x - c)^2 w)} \leq C \| g \|_{L^p(\tilde{v}^p(x - c)^2 w)} \forall g \in L^p(\tilde{v}^p(x - c)^2 w), \forall n \in \mathbb{N},$$

where $\tilde{u}(x) = |x - c|^{-1/p} u(x)$ and $\tilde{v}(x) = |x - c|^{-1/p} v(x)$.

Therefore, we must prove the boundedness of the partial sums $S_n^c$ with the pair of weights $(\tilde{u}, \tilde{v})$. But the Fourier series we are considering now corresponds to the Jacobi generalized weight $(x - c)^2 w(x)$, which has only $k - 1$ negative exponents $\gamma_i$, since on the point $c$ the exponent is $\gamma + 2 > 1$. By hypothesis, the theorem holds in this case and we only have to see that the conditions in the statement hold for the weights $(x - c)^2 w(x)$, $|x - c|^{-1/p} u(x)$ and $|x - c|^{-1/p} v(x)$.

Except for the point $c$, these weights have the same exponents as $w$, $u$ and $v$. Thus, those conditions are the same and therefore they are satisfied. At the point $c$, the exponents are, respectively: $\gamma + 2$, $g + 1 - 2/p$, $G + 1 - 2/p$.

So, we have to check the inequalities

\[
(G + 1 - \frac{2}{p}) + (\gamma + 2 + 1)(\frac{1}{p} - \frac{1}{\gamma + 2 + 1}) < \min\{\frac{1}{2}, \frac{\gamma + 2 + 1}{2}\},
\]

\[
(g + 1 - \frac{2}{p}) + (\gamma + 2 + 1)(\frac{1}{p} - \frac{1}{\gamma + 2 + 1}) > -\min\{\frac{1}{2}, \frac{\gamma + 2 + 1}{2}\}
\]

and

$$G + 1 - \frac{2}{p} \leq g + 1 - \frac{2}{p}.$$ 

It is clear, from our hypothesis, that they are satisfied. Consequently, we have

$$\| S_n^c(y - c, x) \|_{L^p(\tilde{v}^p(x - c)^2 w)} \leq C \| g \|_{L^p(\tilde{v}^p(x - c)^2 w)} \forall g \in L^p(\tilde{v}^p(x - c)^2 w), \forall n \in \mathbb{N}.$$ 

Thus,

$$\| R_n f \|_{L^p(u^p w)} \leq C \| f \|_{L^p(v^p w)} \forall f \in L^p(v^p w), \forall n \in \mathbb{N}$$

and

$$\| S_n f \|_{L^p(v^p w)} \leq C \| f \|_{L^p(u^p w)} \forall f \in L^p(v^p w), \forall n \in \mathbb{N}.$$
Therefore, the result is true for \( k \) negative exponents \( \gamma_i \). By induction, it is true in general and the first part of the theorem is proved.

b) Now, assume that the operators \( S_n \) are uniformly bounded. Let us prove that (2), (3) and (4) are satisfied.

From a result of Máté, Nevai and Totik ([3], theorem 1), it follows

\[
\begin{align*}
&u \in L^p(d\mu); \\
v^{-1} \in L^q(d\mu); \\
w(x)^{-1/2}(1 - x^2)^{-1/4}u(x) \in L^p(w(x)dx); \\
w(x)^{-1/2}(1 - x^2)^{-1/4}v(x)^{-1} \in L^q(w(x)dx).
\end{align*}
\]

These conditions are equivalent to (2) and (3). Thus, we only need to prove (4), that is:

\[\exists C > 0 \text{ such that } u \leq Cv \mu - a.e.\]

In fact, we are going to show that the same \( C \) of the hypothesis works. First of all, let us note that from the hypothesis it follows

\[\|R\|_{L^p(u^p d\mu)} \leq C\|R\|_{L^p(v^p d\mu)} \tag{14}\]

for every polynomial \( R \), since \( S_n R = R \) if \( n \) is big enough.

It is clear that there exists a polynomial \( Q \) such that both \( |Q|^p u^p \) and \( |Q|^p v^p \) are \( \mu \)-integrable. Let us denote \( u' = |Q|^p u^p \) and \( v' = |Q|^p v^p \). Then, for every \( f \in L^p(u'd\mu) \cap L^p(v'd\mu) \) there exists a sequence of polynomials \( R_n \) such that

\[
\lim_{n \to \infty} \int_{-1}^{1} |f - R_n|^p (u' + v')d\mu = 0.
\]

From this and (14) we obtain

\[
\int_{-1}^{1} |f|^p u'd\mu = \lim_{n \to \infty} \int_{-1}^{1} |R_n Q|^p u^p d\mu \leq C^p \lim_{n \to \infty} \int_{-1}^{1} |R_n Q|^p v^p d\mu = C^p \int_{-1}^{1} |f|^p v'd\mu.
\]

Taking now \( E = \{x \in [-1, 1]; u(x) > Cv(x)\} \) and \( f \) the characteristic function on \( E \), we deduce \( \mu(E) = 0 \).

REFERENCES


