# Fractal dimension of the universal Julia sets for the Chebyshev-Halley family of methods

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## ABSTRACT

The concept of universal Julia set introduced in [5] allows us to conclude that the dynamics of a root-finding algorithm applied to any quadratic polynomial can be understood through the analysis of a particular rational map. In this study we go a step beyond in this direction. In particular, we can define the universal fractal dimension of the aforementioned algorithms as the fractal dimension of they corresponding universal Julia sets.

To compute the fractal dimension of the Julia sets we use an algorithm called box-counting (see [2, 6]), which is a procedure that consists on systematically laying a series of grids of decreasing calibre over a region of the plane that contains the fractal. Next we count the number of boxes in each grid had any part of the fractal.

Here, we consider the well-known Chebyshev-Halley family of iterative methods for solving non-linear equations [1, 8]:

$$z_{n+1} = z_n - \left(1 + \frac{1}{2}\frac{L_f(z_n)}{1-\lambda L_f(z_n)}\right)\frac{f(z_n)}{f'(z_n)}, \quad L_f(z_n) = \frac{f(z_n)f''(z_n)}{f'(z_n)^2}, \tag{1}$$

where $\lambda \in \mathbb{R}$ and $f(z)$ is a function defined on the complex plane. Most of the more famous third-order iterative methods are included in this family, for particular choices of the parameter $\lambda$. For instance, Chebyshev's method, Halley's method or super-Halley method are obtained for $\lambda = 0$, $\lambda = 1/2$ and $\lambda = 1$ respectively. In addition, Newton's method is obtained as a limit case when $\lambda \to \pm\infty$.

We consider the quadratic polynomials $f(z) = (z-a)(z-b)$ with $a, b \in \mathbb{C}$, $a \neq b$. Let $R_\lambda(z)$ be the rational function obtained when the methods of (1) are applied to such polynomials and let $M(z)$ be the Möbius map

$$M(z) = \frac{z-a}{z-b}.$$

We define a new rational map $S_\lambda(z) = M \circ R_\lambda \circ M^{-1}(z)$. Then the basins of attraction and their corresponding boundaries (Julia sets) of both rational functions are conformally equivalent. For our convenience, we can write $S_\mu(z)$ instead of $S_\lambda(z)$, where

$$S_\mu(z) = z^3 \frac{z+\mu}{\mu z + 1}, \quad \mu = 2(1-\lambda). \tag{2}$$

We make a numerical analysis of how the universal fractal dimension of the Julia sets of these methods applied to quadratic polynomial changes with $\mu$. We also make a graphical analysis (following [7]) of these Julia sets that allows us to appreciate some of they properties (if they coincide with the locus of points equidistant from the roots or not, if they are connected or not, etc.).

For instance the universal Julia set associated to $S_\mu(z)$ with $0 \le \mu \le 1$ is the unit circle. Therefore their fractal dimensions are equal to one. However for $\mu < 0$ or $\mu > 1$, the universal Julia sets are more intricated. We make a more detailed analysis in these case and we compare the corresponding fractal dimensions.

Finally we apply the "Gauss-Seidelization" process introduced in [3] to the family of methods (2). In brief, the idea of the Gauss-Seidelization is the following.

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Let us consider a complex function $\phi : \mathbb{C} \to \mathbb{C}$ and an iterative sequence

$$z_{n+1} = \phi(z_n), \qquad z_0 \in \mathbb{C}. \tag{3}$$

If we take $U = \operatorname{Re}\phi$, $V = \operatorname{Im}\phi$, $z_n = x_n + y_n i$ we can write (3) as a system of recurrences

$$\begin{cases} x_{n+1} = U(x_n, y_n), \\ y_{n+1} = V(x_n, y_n), \end{cases} \qquad (x_0, y_0) \in \mathbb{R}^2. \tag{4}$$

Although (4) is, in general, a non-linear recurrence, we can consider the same ideas used to construct the Gauss-Seidel iterative methods in the linear case [4]. In fact, we can define

$$\begin{cases} x_{n+1} = U(x_n, y_n), \\ y_{n+1} = V(x_{n+1}, y_n), \end{cases} \qquad (x_0, y_0) \in \mathbb{R}^2, \tag{5}$$

or

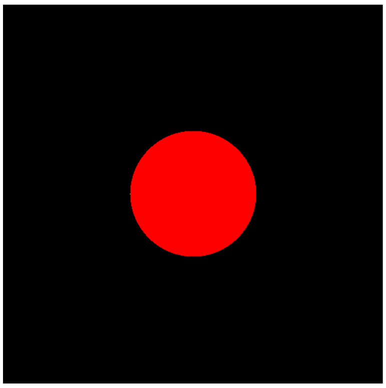
$$\begin{cases} y_{n+1} = V(x_n, y_n), \\ x_{n+1} = U(x_n, y_{n+1}), \end{cases} \qquad (x_0, y_0) \in \mathbb{R}^2. \tag{6}$$

We say that (5) and (6), respectively, are the *xy-Gauss-Seidelization* (*xy*-GS) and the *yx-Gauss-Seidelization* (*yx*-GS) of an iterative method (3).

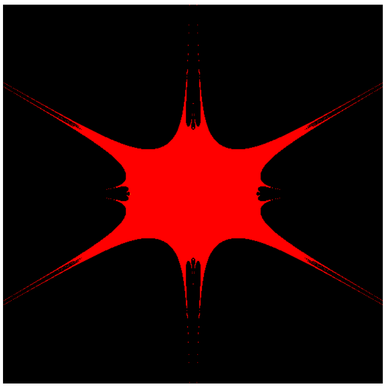
In general, the theoretical study of the dynamics for (5) or (6) can be much more difficult than the study of the dynamics of (3), because complex analysis can no longer be used in the mathematical reasoning. Nevertheless, for the particular choice of the methods defined in (2) we can analyze the influence of the Gauss-Seidelization process in the corresponding universal fractal dimension.

The following figures show the universal Julia sets of some methods together with their corresponding fractal dimensions $d$. So for different values of the parameter $\mu$ we show on the left the universal Julia set of the standard method (2), on the middle the *xy*-Gauss-Seidelization (5) and in the right the *yx*-Gauss-Seidelization (6). All these figures are plotted in the square $\{z = x + yi : -3 \leq x, y \leq 3\}$.

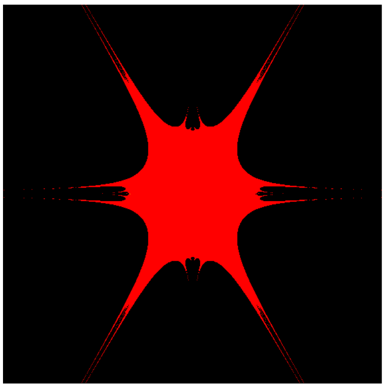
• Figures corresponding to $\mu = 1$ (Halley's method).



$d = 1.00000$

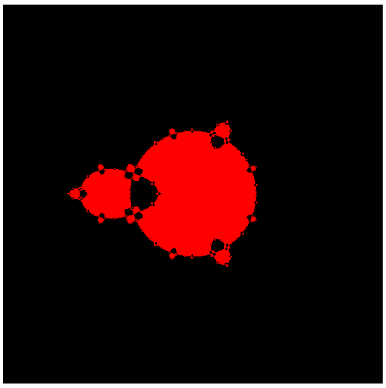


$d = 1.31484$

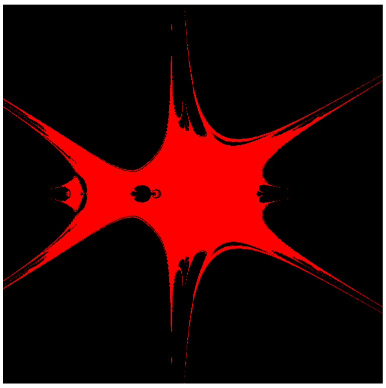


$d = 1.31550$

• Figures corresponding to $\mu = 1.5$.



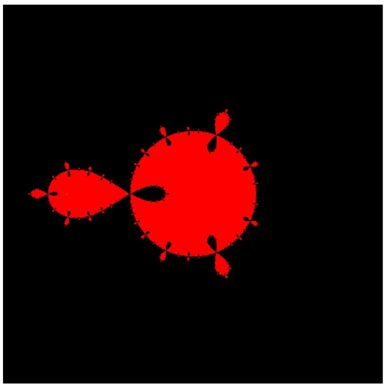
$d = 1.29418$



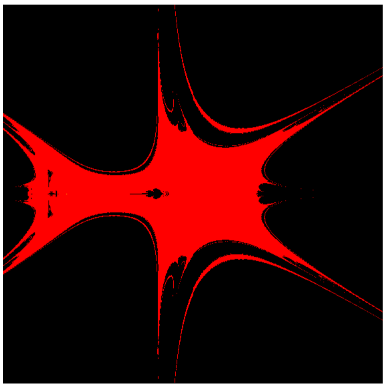
$d = 1.35212$

$d = 1.33573$

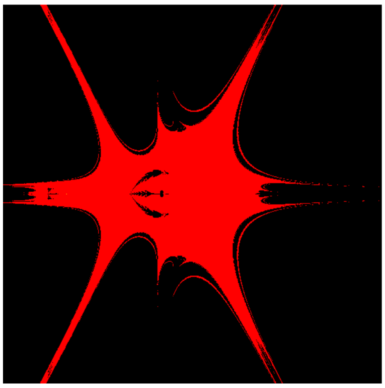
• Figures corresponding to $\mu = 2$ (Chebyshev's method).



$d = 1.31373$

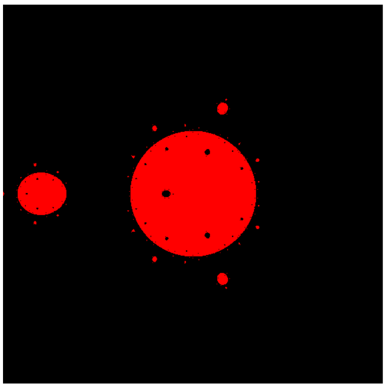


$d = 1.38774$

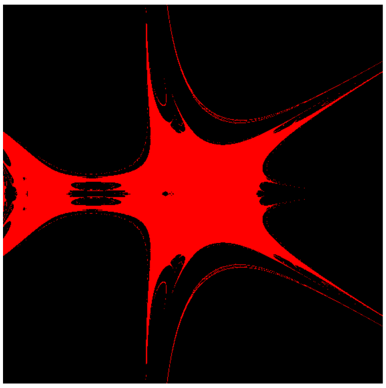


$d = 1.36518$

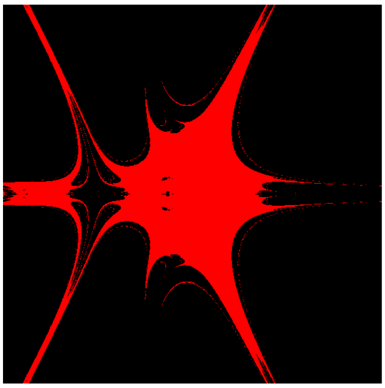
• Figures corresponding to $\mu = 2.5$.



$d = 1.24324$



$d = 1.41929$



$d = 1.37557$

In the following table we show the box-counting dimensions of the standard method (2) and their Gauss-Seidelization (5) and (6) calculated using different values of the parameter $\mu$.

| $\mu$ | Standard | $xy$-GS | $yx$-GS |
|---|---|---|---|
| 1.0 | 1.00000 | 1.31484 | 1.31550 |
| 1.1 | 1.15408 | 1.28180 | 1.30932 |
| 1.2 | 1.25957 | 1.32997 | 1.31045 |
| 1.3 | 1.23461 | 1.38252 | 1.34789 |
| 1.4 | 1.25655 | 1.39499 | 1.33778 |
| 1.5 | 1.29418 | 1.35212 | 1.33573 |
| 1.6 | 1.30250 | 1.33862 | 1.33417 |
| 1.7 | 1.29211 | 1.40527 | 1.34466 |
| 1.8 | 1.29750 | 1.40868 | 1.38288 |
| 1.9 | 1.30773 | 1.42706 | 1.37651 |
| 2.0 | 1.31373 | 1.38774 | 1.36518 |
| 2.1 | 1.28117 | 1.40271 | 1.36322 |
| 2.2 | 1.24746 | 1.40863 | 1.38210 |
| 2.3 | 1.25947 | 1.41262 | 1.37987 |
| 2.4 | 1.23961 | 1.44021 | 1.36299 |
| 2.5 | 1.24324 | 1.41929 | 1.37557 |

As we can see, in all the considered cases the fractal dimensions of the Gauss-Seidelizations are greater than the dimensions corresponding to the standard processes. In some sense, this is surprising, because it is the contrary to the observed in [3] for cubic polynomials.

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