**L^p-Boundedness of the Kernels Relative to Generalized Jacobi Weights**

J. J. Guadalupe Hernández(1), M. Pérez Riera(1), F. J. Ruiz Blasco(1) and J. L. Varona Malumbres(2)

(1) Departamento de Matemáticas, Facultad de Ciencias, Universidad de Zaragoza.
(2) Departamento de Matemática Aplicada, Colegio Universitario de La Rioja, Universidad de Zaragoza.

**A.M.S. Classification:** 42C10.

**Resumen**

Sea $w$ un peso de Jacobi generalizado sobre el intervalo $[-1,1]$, es decir, 

$$w(x) = h(x)(1-x)^\alpha(1+x)^\beta \prod_{i=1}^{N} |x-x_i|^{\gamma_i},$$

con $\alpha, \beta, \gamma_i > -1$ y ciertas condiciones de continuidad sobre $h$. Mediante el uso de la teoría de pesos $A_p$, se puede demostrar la convergencia de la serie de Fourier de polinomios ortonormales con respecto a $w$ cuando $\gamma_i \geq 0 \forall i$. En este trabajo obtenemos acotaciones de las normas en $L^p$ de los núcleos relativos a $w$, que permiten extender el resultado de la convergencia en media al caso general.

Let $w$ be a generalized Jacobi weight, that is,

$$w(x) = h(x)(1-x)^\alpha(1+x)^\beta \prod_{i=1}^{N} |x-x_i|^{\gamma_i}, \quad x \in [-1,1],$$

where:

(a) $\alpha, \beta, \gamma_i > -1$, $t_i \in (-1,1)$, $t_i \neq t_j \forall i \neq j$;

(b) $h$ is a positive, continuous function on $[-1,1]$ and $\omega(h,\delta)\delta^{-1} \in L^1(0,2)$, $\omega(h,\delta)$ being the modulus of continuity of $h$.

---

This paper were published in “Polinomios Ortogonales y Aplicaciones” (Actas del VI Symposium, Gijón, 1989), 168–178, Universidad de Oviedo, Gijón, 1990.
Let \( d\mu = w(x) \, dx \) on \([-1, 1]\) and let \( S_n \) \((n \geq 0)\) be the \( n \)-th partial sum of the Fourier series in the orthonormal polynomials with respect to \( d\mu \). The study of the boundedness

\[
\|S_n f\|_{L^p(w \, d\mu)} \leq C \|f\|_{L^p(w \, d\mu)},
\]

where

\[
u(x) = (1 - x)^a(1 + x)^b \prod_{i=1}^N |x - t_i|^\theta_i
\]

and

\[
u(x) = (1 - x)^A(1 + x)^B \prod_{i=1}^N |x - t_i|^G_i
\]

was done by Badkov ([B]) in the case \( u = v \) by means of a direct estimation of the kernels \( K_n(x, y) \) associated with the polynomials orthogonal with respect to \( d\mu \). Later, Varona ([V]) considered the same problem, with \( u \) and \( v \) not necessarily equal; his method consists of an appropriate use of the theory of \( A_p \) weights. He found conditions for (1) which generalized those obtained for \( u = v \) by Badkov. However, this result, which we state below, follows only in the case \( \gamma_i \geq 0 \forall i \):

**Theorem 1.** Let \( \gamma_i \geq 0 \forall i \) and \( 1 < p < \infty \). If the inequalities

\[
\begin{align*}
A + (\alpha + 1) \left( \frac{1}{p} - \frac{1}{2} \right) &< \frac{\alpha + 1}{2}, \\
B + (\beta + 1) \left( \frac{1}{p} - \frac{1}{2} \right) &< \frac{\beta + 1}{2}, \\
G_i + (\gamma_i + 1) \left( \frac{1}{p} - \frac{1}{2} \right) &< \frac{\gamma_i + 1}{2} \forall i,
\end{align*}
\]

(2)

\[
\begin{align*}
A + (\alpha + 1) \left( \frac{1}{p} - \frac{1}{2} \right) &< \frac{\alpha + 1}{2}, \\
B + (\beta + 1) \left( \frac{1}{p} - \frac{1}{2} \right) &< \frac{\beta + 1}{2}, \\
G_i + (\gamma_i + 1) \left( \frac{1}{p} - \frac{1}{2} \right) &< \frac{\gamma_i + 1}{2} \forall i,
\end{align*}
\]

(3)

\[
\begin{align*}
a + (\alpha + 1) \left( \frac{1}{p} - \frac{1}{2} \right) &> -\frac{1}{4}, \\
b + (\beta + 1) \left( \frac{1}{p} - \frac{1}{2} \right) &> -\frac{1}{4}, \\
g_i + (\gamma_i + 1) \left( \frac{1}{p} - \frac{1}{2} \right) &> -\frac{1}{2} \forall i,
\end{align*}
\]

(4)
\[
\begin{aligned}
(a + (\alpha + 1) \left(\frac{1}{p} - \frac{1}{2}\right) > -\frac{\alpha + 1}{2}, \\
(b + (\beta + 1) \left(\frac{1}{p} - \frac{1}{2}\right) > -\frac{\beta + 1}{2}, \\
g_i + (\gamma_i + 1) \left(\frac{1}{p} - \frac{1}{2}\right) > -\frac{\gamma_i + 1}{2} \quad \forall i,
\end{aligned}
\]

(5)

\[
A \leq a, \quad B \leq b, \quad G_i \leq g_i \quad \forall i,
\]

hold, then there exists \( C > 0 \) such that

\[
\|S_n f\|_{L^p(v^p \, d\mu)} \leq C\|f\|_{L^p(v^p \, d\mu)} \quad \forall f \in L^p(v^p \, d\mu), \quad \forall n \in \mathbb{N}.
\]

The objective of this paper is to eliminate the restriction \( \gamma_i \geq 0 \), by studying the norms of the kernels \( K_n(x, y) \). We are going to use the following notation: \( \{P_n(x)\}, \{k_n\} \) and \( \{K_n(x, y)\} \) will be, respectively, the orthonormal polynomials, their leading coefficients and the kernels relatives to \( d\mu \); if \( c \in (-1, 1) \), \( \{P_n(x)\}, \{k_n\} \) and \( \{K_n(x, y)\} \) will be the corresponding to \((x - c)^2 \, d\mu\). Then, it is not difficult to establish the following relations, \( \forall n \geq 1 \):

\[
K_n(x, y) = (x - c)(y - c)K_{n-1}^c(x, y) + \frac{K_n(x, c)K_n(c, y)}{K_n(c, c)},
\]

(7)

\[
K_n(x, c) = \frac{k_n}{k_n^c} P_n(c)P_n^c(x) - \frac{k_{n-1}}{k_{n+1}} P_{n+1}(c)P^c_{n-1}(x).
\]

(8)

It can be also shown (see [MNT], theorems 10 and 11, and [R], page 212) that

\[
\lim_{n \to \infty} \frac{k_n}{k_n^c} = \lim_{n \to \infty} \frac{k_{n-1}}{k_{n+1}} = \frac{1}{2}.
\]

(9)

If we define

\[
d(x, n) = (1 - x + n^{-2})^{-(2\alpha + 1)/4}(1 + x + n^{-2})^{-(2\beta + 1)/4} \prod_{i=1}^{N} (|x - t_i| + n^{-1})^{-\gamma_i/2},
\]

it is known ([B]) that there exists a constant \( C \) such that \( \forall x \in [-1, 1], \forall n \),

\[
|P_n(x)| \leq C d(x, n).
\]

(10)

There are also some well-known estimates for the kernels, one of them being this ([N], page 4 and page 119, theorem 25): if \( c \in (-1, 1) \) and the factor \(|x - c|\) is
present in $w$ with an exponent $\gamma$, there exist some positive constants $C_1$ and $C_2$, depending on $c$, such that, $\forall n$,

$$C_1 n^{\gamma + 1} \leq K_n(c, c) \leq C_2 n^{\gamma + 1}. \quad (11)$$

From now on, all constants will be denoted $C$, so by $C$ we will mean a constant, possibly different in each occurrence. Using (8), (9) and (10), we get the following result:

**Proposition 2.** Let $1 < p < \infty$, $1/p + 1/q = 1$. Suppose the inequalities (4) and (5) hold. Let $-1 < c < 1$ and let $\gamma$ and $g$ be the exponents of $|x-c|$ in $w$ and $u$, respectively. Then, there exists a positive constant $C$ such that, $\forall n \geq 0$,

$$\|K_n(x, c)\|_{L^p(u^p w)} \leq \begin{cases} 
C n^{(\gamma + 1)/q - g}, & \text{if } g < (\gamma + 1)(1/2 - 1/p) + 1/2, \\
C n^{\gamma/2}(\log n)^{1/p}, & \text{if } g = (\gamma + 1)(1/2 - 1/p) + 1/2, \\
C n^{\gamma/2}, & \text{if } (\gamma + 1)(1/2 - 1/p) + 1/2 < g.
\end{cases} \quad (12)$$

**Proof.** From (10) it follows that $|P_n(c)| \leq C n^{\gamma/2}$. Since $\{P_n^c\}$ is the sequence associated with $(x - c)^2 d\mu$, it also follows from (10) that

$$|P_n^c(x)| \leq C (|x - c| + n^{-1})^{-1} d(x, n).$$

Now, from (8) and (9) we get

$$|K_n(x, c)| \leq C n^{\gamma/2}(|x - c| + n^{-1})^{-1} d(x, n). \quad (12)$$

Let us take $\varepsilon > 0$ such that $|t_i - c| > \varepsilon \ \forall i$ for all $t_i \neq c$. We can write

$$\|K_n(x, c)\|_{L^p(u^p w)}^p = \int_{|x-c|>\varepsilon} |K_n(x, c)|^p u(x)^p w(x) \, dx + \int_{|x-c|<\varepsilon} |K_n(x, c)|^p u(x)^p w(x) \, dx.$$ 

Using (12), for the first term we obtain

$$\int_{|x-c|>\varepsilon} |K_n(x, c)|^p u(x)^p w(x) \, dx \leq C n^{p\gamma/2} \int_{|x-c|>\varepsilon} (|x - c| + n^{-1})^{-p} d(x, n)^p u(x)^p w(x) \, dx \leq C_1 n^{p\gamma/2} \int_{-1}^1 d(x, n)^p u(x)^p w(x) \, dx.$$
It is easy to deduce from (4) and (5) that this last integral is bounded by a constant which does not depend on \( n \), so

\[
\int_{|x-c|\leq \varepsilon} |K_n(x,c)|^p u(x)^p w(x) \, dx \leq Cn^{p\gamma/2}.
\]

Let us take now the second term; since for \( |x-c| < \varepsilon \) there exists a constant \( C \) such that, \( \forall n, d(x,n)\leq C(|x-c| + n^{-1})^{-\gamma/2} \), \( u(x) \leq C|x-c|^p \) and \( w(x) \leq C|x-c|^\gamma \), we have

\[
\int_{|x-c|<\varepsilon} |K_n(x,c)|^p u(x)^p w(x) \, dx
\]

\[
\leq Cn^{p\gamma/2} \int_{|x-c|<\varepsilon} (|x-c| + n^{-1})^{-p} d(x,n) u(x)^p w(x) \, dx
\]

\[
\leq C_1 n^{p\gamma/2} \int_{|x-c|<\varepsilon} (|x-c| + n^{-1})^{-p(1+\gamma/2)} |x-c|^{\gamma p+\gamma} \, dx
\]

\[
\leq C_2 n^{p\gamma/2} \int_0^1 (y + n^{-1})^{-p(1+\gamma/2)} y^{\gamma p+\gamma} \, dy
\]

\[
= C_2 n^{p\gamma/2+p(1+\gamma/2) - \gamma p - \gamma - 1} \int_0^1 (ny + 1)^{-p(1+\gamma/2)} (ny)^{\gamma p+\gamma} n \, dy
\]

\[
= C_2 n^{p\gamma/2+p(1+\gamma/2) - \gamma p - \gamma - 1} \int_0^n (r + 1)^{-p(1+\gamma/2)} r^{\gamma p+\gamma} \, dr.
\]

Taking into account that \( p(1+\gamma/2) - \gamma p - \gamma - 1 = p[(\gamma + 1)(1/2 - 1/p) - g + 1/2] \) and there exist some constants \( C_1 \) and \( C_2 \) such that \( C_1 \leq r + 1 \leq C_2 \) on \([0,1]\) and \( C_1 r \leq r + 1 \leq C_2 r \) on \([1,n]\), we finally get the inequality

\[
\int_{|x-c|<\varepsilon} |K_n(x,c)|^p u(x)^p w(x) \, dx
\]

\[
\leq Cn^{p\gamma/2+p(\gamma+1)(1/2-1/p)-g+1/2} \int_0^1 r^{\gamma p+\gamma} \, dr
\]

\[
+ Cn^{p\gamma/2+p(\gamma+1)(1/2-1/p)-g+1/2} \int_1^n r^{-p(\gamma+1)(1/2-1/p)-g+1/2} \, dr.
\]

Since (5) implies \( \gamma p + \gamma > -1 \), the first term can be bounded in this way:

\[
Cn^{p\gamma/2+p(\gamma+1)(1/2-1/p)-g+1/2} \int_0^1 r^{\gamma p+\gamma} \, dr \leq Cn^{p\gamma/2+p(\gamma+1)(1/2-1/p)-g+1/2}.
\]

Let us consider separately the three cases in the statement:
(a) If \( g < (\gamma + 1)(1/2 - 1/p) + 1/2 \), then \(-p[(\gamma + 1)(1/2 - 1/p) - g + 1/2] - 1 < -1\);
so:
\[
\int_1^n r^{-p[(\gamma + 1)(1/2 - 1/p) - g + 1/2] - 1} \, dr \leq C.
\]
In this case, (14) and (15) imply
\[
\int_{|x-c|<\varepsilon} |K_n(x,c)|^p u(x)^p w(x) \, dx \leq C n^{p^q/2 + p[(\gamma + 1)(1/2 - 1/p) - g + 1/2]},
\]
Since \( p[(\gamma + 1)(1/2 - 1/p) - g + 1/2] > 0 \), from this inequality and (13) we obtain
\[
\|K_n(x,c)\|_{L^p(u^p w)}^p \leq C n^{p^q/2 + p[(\gamma + 1)(1/2 - 1/p) - g + 1/2]} = C n^{p[(\gamma + 1)(1 - 1/p) - g]} = C n^{p[(\gamma + 1)/q - g]},
\]
as we had to prove.
(b) If \((\gamma + 1)(1/2 - 1/p) + 1/2 < g\), then \(-p[(\gamma + 1)(1/2 - 1/p) - g + 1/2] - 1 > -1\);
therefore:
\[
\int_1^n r^{-p[(\gamma + 1)(1/2 - 1/p) - g + 1/2] - 1} \, dr \leq C n^{-p[(\gamma + 1)(1/2 - 1/p) - g + 1/2]}.
\]
From (14) and (15), we get now \( \int_{|x-c|<\varepsilon} |K_n(x,c)|^p u(x)^p w(x) \, dx \leq C n^{p^q/2} \) and, finally,
\[
\|K_n(x,c)\|_{L^p(u^p w)}^p \leq C n^{p^q/2}.
\]
(c) If \( g = (\gamma + 1)(1/2 - 1/p) + 1/2 \),
\[
\int_1^n r^{-p[(\gamma + 1)(1/2 - 1/p) - g + 1/2] - 1} \, dr \leq C \log n;
\]
hence,
\[
\int_{|x-c|<\varepsilon} |K_n(x,c)|^p u(x)^p w(x) \, dx \leq C n^{p^q/2} \log n
\]
and
\[
\|K_n(x,c)\|_{L^p(u^p w)}^p \leq C n^{p^q/2} \log n.
\]
This concludes the proof of the proposition. \(\square\)

**Corollary 3.** Let \( 1 < p < \infty, 1/p + 1/q = 1 \). Suppose the inequalities (2) and (3) hold. Let \(-1 < c < 1\) and let \( \gamma \) and \( G \) be the exponents of \(|x-c|\) in \( w \) and \( v \), respectively. Then, there exists a positive constant \( C \) such that, \( \forall n \geq 0 \),
\[
\|K_n(x,c)\|_{L^q(u^{-q} w)} \leq \begin{cases} 
C n^{\gamma/2}, & \text{if } G < (\gamma + 1)(1/2 - 1/p) - 1/2, \\
C n^{\gamma/2}(\log n)^1/q, & \text{if } G = (\gamma + 1)(1/2 - 1/p) - 1/2, \\
C n^{(\gamma + 1)/p + G}, & \text{if } (\gamma + 1)(1/2 - 1/p) - 1/2 < G.
\end{cases}
\]
Proof. Just apply proposition 2 to the weight \( v^{-1} \) and keep in mind the equality \( 1/2 - 1/p = 1/q - 1/2 \).

The following result is just what we need to extend theorem 1 to the general case \( \gamma_i > -1 \):

**Corollary 4.** Let \( 1 < p < \infty \), \( 1/p + 1/q = 1 \). Suppose the inequalities (2), (3), (4), (5) and (6) hold. Let \(-1 < c < 1\). Then, there exists a positive constant \( C \) such that, \( \forall n \geq 0 \),

\[
\|K_n(x, c)\|_{L^p(u^p \mu)} \|K_n(x, c)\|_{L^q(v^{-q} \mu)} \leq CK_n(c, c).
\]

**Proof.** It is a simple consequence of proposition 2, corollary 3 and the estimate (11). The only thing we must do is to consider each case in these results separately.

**Note.** Although it will not be used in what follows, corollary 4 also holds when \( c = \pm 1 \). The proof is similar: starting from other expressions for \( K_n(x, \pm 1) \), analogous results to proposition 2 and corollary 3 can be obtained, and then corollary 4 follows.

We are now ready to extend theorem 1 to the general case \( \gamma_i > -1 \):

**Theorem 5.** Let \( 1 < p < \infty \). If the inequalities

\[
\begin{align*}
A + (\alpha + 1) \left( \frac{1}{p} - \frac{1}{2} \right) &< \frac{1}{4}, \\
B + (\beta + 1) \left( \frac{1}{p} - \frac{1}{2} \right) &< \frac{1}{4}, \\
G_i + (\gamma_i + 1) \left( \frac{1}{p} - \frac{1}{2} \right) &< \frac{1}{2} \quad \forall i,
\end{align*}
\]

\[
\begin{align*}
a + (\alpha + 1) \left( \frac{1}{p} - \frac{1}{2} \right) &> -\frac{1}{4}, \\
b + (\beta + 1) \left( \frac{1}{p} - \frac{1}{2} \right) &> -\frac{1}{4}, \\
g_i + (\gamma_i + 1) \left( \frac{1}{p} - \frac{1}{2} \right) &> -\frac{1}{2} \quad \forall i,
\end{align*}
\]

hold, then there exists \( C > 0 \) such that

\[
\|S_n f\|_{L^p(w^p d\mu)} \leq C\|f\|_{L^p(v^p d\mu)} \quad \forall f \in L^p(v^p d\mu), \quad \forall n \in \mathbb{N}.
\]
Proof. By induction on the number of negative exponents \( \gamma_i \). If \( \gamma_i \geq 0 \) \( \forall i \), the result is true, as we saw before (theorem 1).

Suppose there exist \( k \) negative exponents \( \gamma_i \), with \( k > 0 \), and the result is true for \( k - 1 \). Let \( c \in (-1, 1) \) be a point with a negative exponent \( \gamma \). Let us remember the formula (7):

\[
K_n(x, y) = (x - c)(y - c)K_{n-1}(x, y) + \frac{K_n(x, c)K_n(c, y)}{K_n(c, c)}.
\]

We define the operators

\[
T_n f(x) = \int_{-1}^{1} \frac{K_n(x, c)K_n(c, y)}{K_n(c, c)} f(y)w(y) dy,
\]

\[
R_n f(x) = \int_{-1}^{1} (x - c)(y - c)K_{n-1}^c(x, y) f(y)w(y) dy.
\]

Then, \( S_n = T_n + R_n \). We are going to study firstly the operators \( T_n \):

\[
T_n f(x) = \int \left( K_n(x, c)K_n(c, c) \int_{-1}^{1} K_n(c, y) f(y)w(y) dy \right) dx,
\]

thus

\[
\|T_n f\|_{L^p(u^{p}w)} \leq \frac{\int_{-1}^{1} |K_n(c, y)|v(y)^{-1} |f(y)|v(y)w(y) dy}{K_n(c, c)} \|K_n(x, c)\|_{L^p(u^{p}w)}
\]

\[
\leq \frac{\|K_n(c, y)\|_{L^p(u^{p}w)} \|K_n(x, c)v(x)^{-1}\|_{L^q(u)}}{K_n(c, c)} \|f\|_{L^p(w)}
\]

\[
= \frac{\|K_n(x, c)\|_{L^p(u^{p}w)} \|K_n(x, c)\|_{L^q(u^{-q}w)}}{K_n(c, c)} \|f\|_{L^p(w)}.
\]

From corollary 4 it follows now that \( \|T_n f\|_{L^p(u^{p}d\mu)} \leq C \|f\|_{L^p(\tilde{v}p \, d\mu)} \forall f \in L^p(\tilde{v}p \, d\mu) \), \( \forall n \in \mathbb{N} \). So, we only have to prove the same bound for the operators \( R_n \).

But, if we denote by \( S_n^c \) the partial sums of the Fourier series with respect to the measure \( (x - c)^2 w(x) \, dx \), it turns out that

\[
R_n f(x) = (x - c) \int_{-1}^{1} (y - c)K_{n-1}^c(x, y) f(y)w(y) dy = (x - c)S_n^c \left( \frac{f(y)}{y - c}, x \right),
\]

whence

\[
\|R_n f\|_{L^p(u^{p}w)} \leq C \|f\|_{L^p(u^{p}w)} \forall f \in L^p(\tilde{v}p \, w)
\]

\[
\iff \| (x - c)S_{n-1}^c \left( \frac{f(y)}{y - c}, x \right) \|_{L^p(u^{p}w)} \leq C \|f\|_{L^p(u^{p}w)} \forall f \in L^p(\tilde{v}p \, w)
\]

\[
\iff \| (x - c)S_{n-1} g(x) \|_{L^p(\tilde{v}p \, w)} \leq C \left\| (x - c)g \right\|_{L^p(u^{p}w)} \forall g \in L^p(\tilde{v}p \, w)
\]

\[
\iff \| S_{n-1}^c g \|_{L^p((x - c)^2 w)} \leq C \|g\|_{L^p((x - c)^2 w)} \forall g \in L^p(x - c)^2 \tilde{v}p \, w
\]

\[
\iff \| S_{n-1}^c g \|_{L^p(\tilde{v}p(x - c)^2 w)} \leq C \|g\|_{L^p(\tilde{v}p(x - c)^2 w)} \forall g \in L^p(\tilde{v}p(x - c)^2 w),
\]

8
where \( \tilde{u}(x) = |x - c|^{1-2/p}u(x) \) and \( \tilde{v}(x) = |x - c|^{1-2/p}v(x) \).

Therefore, we must prove the boundedness of the partial sums \( S_n^c \) with the weights \((\tilde{u}, \tilde{v})\). But the Fourier series we are considering now corresponds to the Jacobi generalized weight \((x - c)^2w(x)\), which has only \( k - 1 \) negative exponents \( \gamma_i \), since on the point \( c \) the exponent is \( \gamma + 2 > 1 \). By hypothesis, the theorem holds in this case and we only have to see that the conditions in the statement hold for the weights \((x - c)^2w(x), |x - c|^{1-2/p}u(x)\) and \(|x - c|^{1-2/p}v(x)\).

Except for the point \( c \), these weights have the same exponents as \( w, u \) and \( v \); thus, those conditions are the same and, therefore, they are satisfied. At the point \( c \), the exponents are, respectively,

\[ \gamma + 2, \quad g + 1 - 2/p, \quad G + 1 - 2/p. \]

So, the inequalities we have to check are the following:

\[
\begin{align*}
(G + 1 - 2/p) + (\gamma + 2 + 1)(1/p - 1/2) &< 1/2, \\
(G + 1 - 2/p) + (\gamma + 2 + 1)(1/p - 1/2) &< (\gamma + 2 + 1)/2, \\
(g + 1 - 2/p) + (\gamma + 2 + 1)(1/p - 1/2) &> -1/2, \\
(g + 1 - 2/p) + (\gamma + 2 + 1)(1/p - 1/2) &> -(\gamma + 2 + 1)/2, \\
G + 1 - 2/p &\leq g + 1 - 2/p.
\end{align*}
\]

It is easy to see that all of them are satisfied, from our hypothesis. Consequently, we get

\[
\|S^c_{n-1}g\|_{L^p(\tilde{v}^p(x-c)^2w)} \leq C\|g\|_{L^p(\tilde{v}^p(x-c)^2w)} \quad \forall g \in L^p(\tilde{v}^p(x-c)^2w), \quad \forall n \in \mathbb{N};
\]

thus

\[
\|R_nf\|_{L^p(u^p w)} \leq C\|f\|_{L^p(v^p w)} \quad \forall f \in L^p(v^p w), \quad \forall n \in \mathbb{N}.
\]

And, finally,

\[
\|S_nf\|_{L^p(u^p d\mu)} \leq C\|f\|_{L^p(v^p d\mu)} \quad \forall f \in L^p(v^p \mu), \quad \forall n \in \mathbb{N}.
\]

That is, the result is true for \( k \) negative exponents \( \gamma_i \). By induction, it is true in general and the theorem is proved. \( \Box \)

**Note.** It can be shown that the converse is also valid, that is, the boundedness of the partial sums \( S_n \) implies the five conditions of the theorem.

**References**


